SOME PROPERTIES OF CONTINUOUS K-G-FRAMES IN HILBERT SPACES

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The present paper studies some properties of c-K-g-frames in separable Hilbert spaces, which are extensions of K-g-frames and c-g-frames. In addition, necessary and sufficient conditions for constructing c-K-g-frames are given by using bounded operators. The notion of c-K-g-duals for c-K-g-frames are introduced and such duals are characterized. Moreover, we also discuss on operators preserving c-K-g-frames. Finally, some equalities and inequalities about c-K-g-frames are investigated.

Keywords: c-K-g-frame, dual c-K-g-frame, K-g frame.

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1. Introduction and Preliminaries

Frames (discrete frames) were introduced by Duffin and Schaeffer in 1952 [12] for studying some profound problems in nonharmonic Fourier series. Discrete and continuous frames arise in many applications in both pure and applied mathematics. After the fundamental paper by Daubechies, Grossmann and Meyer (2017 Abel Prize winer) in 1986 [10], the theory of frames found more attention and a lot of papers and books were published in this area. They took the key step of connecting frames with wavelets and Gabor systems in that paper. Particularly, frame theory has been extensively used in many fields such as filter bank theory, signal and image processing, coding and communications [25] and other areas. For a more complete treatment of frame theory we recommend the excellent book of Christensen [9], and the tutorials of Casazza [7, 8] and the Memoir of Han and Larson [17].

Over the years, various extensions of the frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames which were introduced by W. Sun in [26]. For example, one can consider: bounded quasi-projectors, fusion frames, pseudo-frames, oblique frames, outer frames and etc.

Frames and their relatives are most often considered in the discrete case, for instance in signal processing [12]. However, continuous frames have also been studied and offer interesting mathematical problems. They have been introduced originally by Ali, Gazeau and Antoine [2] and also, independently, by Kaiser [19]. Since then, several papers dealt with various aspects of the concept, see for instance [13, 14] or [6, 22, 23, 24]. By combining the above mentioned extensions of frames, the new and more general notion called *continuous g-frame* has been introduced in [1].

Traditionally, frames were studied for the whole space or for the closed subspaces. Gavruta in [15] gave another generalization of frames namely K-frames, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space.

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In general, the range of an operator is not a closed subspace. K-frames allow us in a stable way, to reconstruct elements from the range of a linear and bounded operator in a Hilbert space.

K-g-frames have been introduced in [5, 18] and some properties and characterizations of K-g-frames has been obtained, for more information on K-g-frames, see [18, 29]. Extending the above mentioned notions, the new concept namely c-K-g-frames was introduced by authors [3] and this paper will provide new results concerning it .

Throughout this paper, (Ω, μ) is a measure space with positive measure μ, H, H_1, H_2 and H_ω are separable Hilbert spaces and $\mathcal{B}(H, H_\omega)$ is the collection of all bounded linear operators of H into H_ω where $\omega \in \Omega$. If $H_\omega = H$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$ and $K \in \mathcal{B}(H)$. For an operator U, the range is denoted by $\mathcal{R}(U)$ and the null space with $\mathcal{N}(U)$.

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following sense (see [12]).

Proposition 1.1. *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

Proposition 1.2. *Let $U \in \mathcal{B}(H_1, H_2)$. Then the following assertions hold:*

(1) *$\mathcal{R}(U)$ is closed in H_2 if and only if $\mathcal{R}(U^*)$ is closed in H_1.*
(2) *$(U^*)^\dagger = (U^\dagger)^*$.*
(3) *The orthogonal projection of H_2 onto $\mathcal{R}(U)$ is given by $UU^\dagger$.*
(4) *The orthogonal projection of H_1 onto $\mathcal{R}(U^\dagger)$ is given by $U^\dagger U$.*
(5) *$\mathcal{N}(U^\dagger) = \mathcal{R}^\perp(U)$ and $\mathcal{R}(U^\dagger) = \mathcal{N}^\perp(U)$.*

Theorem 1.1. [11] *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(1) *$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;*
(2) *$L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some$\lambda > 0$;*
(3) *there exists a mapping $U \in \mathcal{B}(H_1, H_2)$ such that $L_1 = L_2 U$.*

Moreover, if these conditions are valid, then there exists a unique operator U so that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\mathcal{N}(L_1) = \mathcal{N}(U)$;
(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

Theorem 1.2. [1] *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with bound B. Then the mapping T on $\Big(\oplus_{\omega\in\omega} H_\omega, \mu\Big)_{L^2}$ to H defined by*

$$\langle TF, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu(\omega), \quad F \in \Big(\oplus_{\omega\in\Omega} H_\omega, \mu\Big)_{L^2}, \ g \in H,$$

is linear and bounded with $\|T\| \leq \sqrt{B}$. Furthermore for each $g \in H$ and $\omega \in \Omega$,

$$T^*(g)(\omega) = \Lambda_\omega g.$$

The continuous version of K-g-frames have been introduced in [3] in the following way. Some properties of K-g-frames have been studied in [31, 32].

Definition 1.1. *Let $K \in \mathcal{B}(H)$. A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \ \omega \in \Omega\}$ is called a continuous K-g-frame or c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(i) *$\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable for each $f \in H$;*

(ii) *there exist constants* $0 < A \leq B < \infty$ *such that*

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \leq B\|f\|^2, \;\; f \in H. \tag{1}$$

The constants A, B are called lower and upper c-K-g-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called Parseval c-K-g-frame. The family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality in (1) holds. In this case, B is called the Bessel constant.

Now, suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. The c-K-g-frame operator is defined by

$$S : H \longrightarrow H$$

$$\langle Sf, g\rangle = \int_\Omega \langle f, \Lambda_\omega^*\Lambda_\omega g\rangle \, d\mu(\omega), \qquad \forall f, g \in H.$$

Therefore,

$$AKK^* \leq S \leq BI.$$

Remark 1.1. *Like K-frame operator, the c-K-g-frame operator is not invertible. In general if K has closed range, then S is invertible on $\mathcal{R}(K)$ and we have (see* [30]*)*

$$B^{-1}\|f\|^2 \leq \langle (S|_{\mathcal{R}(K)})^{-1}f, f\rangle \leq A^{-1}\|K^\dagger\|^2\|f\|^2, \quad f \in H.$$

2. Construction of c-K-g-Frames with Bounded Operators

For a given c-K-g-frame $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ of H, we will obtain another c-K-g-frame for the space. One approach is to construct a family $\{\Lambda_\omega U^* \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega}$, where $U \in \mathcal{B}(H)$. The following theorem gives us necessary and sufficient conditions for $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ to be a c-K-g-frame of H.

Theorem 2.1. *Let $K \in \mathcal{B}(H)$ be closed range, $\{\Lambda_\omega\}_{\omega\in\Omega}$ a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $U \in \mathcal{B}(H)$. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if there exists a constant $\delta > 0$ such that for all $f \in H$*

$$\|U^*f\| \geq \delta\|K^*f\|.$$

Proof. Suppose that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with the lower bound C. If B is the upper bound of $\{\Lambda_\omega\}_{\omega\in\Omega}$, then for each $f \in H$,

$$C\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U^*f\|^2 \, d\mu \leq B\|U^*f\|^2.$$

Hence $\|U^*f\| \geq \sqrt{\frac{C}{B}}\|K^*f\|$. For the opposite implication, for every $f \in H$ we have

$$\|U^*f\| = \|(K^\dagger)^*K^*U^*f\| \leq \|K^\dagger\|.\|K^*U^*f\|.$$

Therefore,

$$A\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 \leq A\|K^\dagger\|^{-2}\|U^*f\|^2 \leq A\|K^*U^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U^*f\|^2 d\,\mu.$$

For the upper bound, it is clear that

$$\int_\Omega \|\Lambda_\omega U^*f\|^2 d\,\mu \leq B\|U\|^2\|f\|^2, \quad f \in H.$$

□

Corollary 2.1. *Let $K \in \mathcal{B}(H)$ be closed range, $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a tight c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bound A and $U \in \mathcal{B}(H)$. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a tight c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bound C if and only if*

$$\|U^*f\| = \frac{C}{A}\|K^*f\|, \quad f \in H.$$

Corollary 2.2. *Let $K \in \mathcal{B}(H)$ be closed range, $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ and $U \in \mathcal{B}(H)$. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ if and only if U is surjective.*

Proof. By Theorem 2.1, we get for any $f \in H$

$$\delta\|K^\dagger\|^{-1}\|f\| \leq \delta\|K^*f\| \leq \|U^*f\|$$

and this completes the proof. □

3. Duals of c-K-g-frames

One of the most important challenges in the research of frames is the characterization of duals. Recall that for a given frame $\{f_k\}_{k=1}^\infty$, the Bessel sequence $\{g_k\}_{k=1}^\infty$ is a dual of $\{f_k\}_{k=1}^\infty$ which provides $f = \sum_{k=1}^\infty \langle f, g_k\rangle f_k$ for any $f \in H$.

Duals of frames for the first time have been characterized by Li [20, 21]. One can find that for a given frame $\{f_k\}_{k=1}^\infty$, its dual frames are precisely the families

$$\{g_k\}_{k=1}^\infty = \left\{S^{-1}f_k + h_k - \sum_{j=1}^\infty \langle S^{-1}f_k, f_j\rangle h_j\right\}_{k=1}^\infty,$$

where $\{h_k\}_{k=1}^\infty$ is a Bessel sequence in H, [9].
In this section, we introduce the notation of c-K-g-duals for c-K-g-frames and characterize such duals.

Definition 3.1. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. A c-g-Bessel family $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ for H is called a c-K-g-dual of Λ if for every $f, h \in H$*

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda_\omega^*\Gamma_\omega f, h\rangle \, d\mu(\omega).$$

In this case, we say that Λ and Γ are pair duals.
By Theorem 1.2, the proof of the following is straightforward.

Theorem 3.1. *Let $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ be a c-K-g-dual for Λ. Then the following conditions are equivalent.*

(I) $T_\Lambda T_\Gamma^* = K$;
(II) $T_\Gamma T_\Lambda^* = K^*$;
(III) $\langle Kf, f\rangle = \langle T_\Gamma^* f, T_\Lambda^* f\rangle$.

Theorem 3.2. *If Λ and Γ are pair duals, then Γ is a c-K^*-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Let $f \in H$, we have by Theorem 3.1 item (I)

$$\begin{aligned}
\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2 \\
&= |\langle T_\Gamma^* f, T_\Lambda^* Kf\rangle|^2 \\
&\leq \|T_\Gamma^* f\|^2 \|T_\Lambda\|^2 \|Kf\|^2 \\
&\leq B\|Kf\|^2 \|T_\Gamma^* f\|^2 \\
&\leq B\|Kf\|^2 \int_\Omega \|\Gamma_\omega(f)\|^2 d\mu.
\end{aligned}$$

and the proof is completed. □

We are ready to introduce an explicit c-K-g-dual for each c-K-g-frame. This helps us to characterize all c-K-g-duals of these frames.

Theorem 3.3. *Let $K \in \mathcal{B}(H)$ be closed range and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H. Also, suppose that for each $f \in H$, the mapping $\omega \longrightarrow \Lambda_\omega(f)$ is Bochner integrable. Then $\widetilde{\Lambda} = \Big\{\Lambda_\omega\big((S|_{\mathcal{R}(K)})^{-1}\big)^* K\Big\}_{\omega\in\Omega}$ is a c-K-g-dual of $\Lambda\pi_{\mathcal{R}(K)}$ with bounds B^{-1} and $A^{-1}\|K\|^2\|K^\dagger\|^2$, respectively.*

Proof. Since $\widetilde{\Lambda}$ is a c-g-Bessel sequence and S is not a self-adjoint on $\mathcal{R}(K)$, but by Lemma 3.1 and Corollary 3.2 in [22], we can show $(S|_{\mathcal{R}(K)})^* = \pi_{\mathcal{R}(K)}S|_{S(\mathcal{R}(K))}$, so for every $f, h \in H$ we have

$$\begin{aligned}
\langle Kf, h\rangle &= \Big\langle \big((S|_{\mathcal{R}(K)})^{-1}S|_{\mathcal{R}(K)}\big)^* Kf, h\Big\rangle \\
&= \Big\langle \pi_{\mathcal{R}(K)}S|_{S(\mathcal{R}(K))}\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, h\Big\rangle \\
&= \Big\langle S|_{S(\mathcal{R}(K))}\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, \pi_{\mathcal{R}(K)}h\Big\rangle \\
&= \int_\Omega \big\langle \big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, \Lambda_\omega^*\Lambda_\omega\pi_{\mathcal{R}(K)}h\big\rangle \, d\mu \\
&= \int_\Omega \big\langle \pi_{\mathcal{R}(K)}\Lambda_\omega^*\Lambda_\omega\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, h\big\rangle \, d\mu.
\end{aligned}$$

Therefore, $\widetilde{\Lambda}$ is a c-K-g-dual for $\Lambda\pi_{\mathcal{R}(K)}$ with the lower bound B^{-1} by theorem 3.2. Furthermore, by Remark 1.1, for each $f \in \mathcal{R}(K)$ we obtain

$$\begin{aligned}
\big\|\big((S|_{\mathcal{R}(K)})^{-1}\big)^* f\big\|^2 &= \big\langle (S|_{\mathcal{R}(K)})^{-1}\big((S|_{\mathcal{R}(K)})^{-1}\big)^* f, f\big\rangle \\
&\le A^{-1}\|K^\dagger\|^2\big\|\big((S|_{\mathcal{R}(K)})^{-1}\big)^* f\big\|\|f\|.
\end{aligned}$$

So,

$$\big\|\big((S|_{\mathcal{R}(K)})^{-1}\big)^* f\big\| \le A^{-1}\|K^\dagger\|^2\|f\|.$$

Thus, for any $f \in H$

$$\begin{aligned}
\int_\Omega \|\Lambda_\omega\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf\|^2 \, d\mu &= \int_\Omega \big\langle \big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, \Lambda_\omega^*\Lambda_\omega\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf\big\rangle \, d\mu \\
&= \big\langle S|_{\mathcal{R}(K)}\big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf, \big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf\big\rangle \\
&= \big\langle Kf, \big((S|_{\mathcal{R}(K)})^{-1}\big)^* Kf\big\rangle \\
&\le A^{-1}\|K^\dagger\|^2\|K\|^2\|f\|^2.
\end{aligned}$$

The proof is completed. □

4. Operators preserving c-K-g-frames

Throughout this section, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$..

Theorem 4.1. *Suppose that Λ is a c-K-g-frame for H with the lower bound A and $U \in \mathcal{B}(H)$ with $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. Then Λ is a c-U-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$.*

Proof. Via Theorem 1.1, there exists $\alpha > 0$ such that $UU^* \le \alpha^2 KK^*$. Hence, for each $f \in H$ we have

$$A\alpha^{-2}\|U^*f\|^2 \le A\|K^*f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 d\mu.$$

□

Theorem 4.2. *Suppose that $K \in \mathcal{B}(H)$ is with dense range, Λ is a c-K-g-frame for H and $U \in \mathcal{B}(H)$ is with closed range. If $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H then U is surjective.*

Proof. Assume that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H with frame bounds A and B. Then for any $f \in H$, we have

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu. \tag{2}$$

Since K has dense range, so K^* is injective. By (2), $\mathcal{N}(U^*) \subseteq \mathcal{N}(K^*)$ then U^* is injective. Moreover $\mathcal{R}(U) = \mathcal{N}(U^*)^\perp = H$. Thus, U is surjective. □

Theorem 4.3. *Let $K \in \mathcal{B}(H)$ and Λ be a c-K-g-frame for H with the frame bounds A, B. If $U \in \mathcal{B}(H)$ has closed range with $UK = KU$, then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{R}(U)$.*

Proof. It is obvious that $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-g-Bessel sequence with the bound $B\|U\|^2$. Since U has closed range, so it has the pseudo-inverse $U^\dagger$ such that $UU^\dagger = I$. Now, for each $f \in \mathcal{R}(U)$, we get $K^*f = (U^\dagger)^* U^* K^* f$. So, we obtain $\|K^*f\| \leq \|U^\dagger\|.\|U^*K^*f\|$. Therefore, $\|U^\dagger\|^{-1}\|K^*f\| \leq \|U^*K^*f\|$ and we have

$$\begin{aligned}\int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu &\geq A\|K^*U^*f\|^2\\ &= A\|U^*K^*f\|^2\\ &\geq A\|U^\dagger\|^{-2}\|K^*f\|^2.\end{aligned}$$

This completes the proof. □

Corollary 4.1. *Let $K \in \mathcal{B}(H)$ be dense range, Λ be a c-K-g-frame for H and $U \in \mathcal{B}(H)$ is with closed range. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H if and only if U is surjective.*

Theorem 4.4. *Let $K \in \mathcal{B}(H)$ be dense range, Λ be a c-K-g-frame for H and $U \in \mathcal{B}(H)$ be closed range. If $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ and $\{\Lambda_\omega U\}_{\omega\in\Omega}$ are c-K-g-frames for H, then U is invertible.*

Proof. Suppose that A_1, B_1 are the frame bounds of $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ and A_2, B_2 are the frame bounds of $\{\Lambda_\omega U\}_{\omega\in\Omega}$. Since for any $f \in H$

$$A_1\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu \leq B_1\|f\|^2 \tag{3}$$

and $\mathcal{N}(U^*) \subseteq \mathcal{N}(K^*)$, therefore K^* is injective. Moreover, $\mathcal{R}(U)$=$\mathcal{N}(U^*)^\perp = H$, then U is surjective. Also, by

$$A_2\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega U f\|^2 d\,\mu \leq B_2\|f\|^2, \tag{4}$$

we get $\mathcal{N}(U) \subset \mathcal{N}(K^*)$, so U is injective. Thus U is invertible. □

Theorem 4.5. *Let Λ be a c-K-g-frame for H and $U \in \mathcal{B}(H)$ be co-isometry (i.e. $UU^* = 1$) with $UK = KU$. Then $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-K-g-frame for H.*

Proof. Let A, B be the frame bounds of Λ. By Theorem 4.3, $\{\Lambda_\omega U^*\}_{\omega\in\Omega}$ is a c-g-Bessel sequence. Since U is co-isometry, we have for each $f \in H$

$$\begin{aligned}\int_\Omega \|\Lambda_\omega U^* f\|^2 d\,\mu &\geq A\|K^*U^*f\|^2\\ &= A\|U^*K^*f\|^2\\ &= A\|K^*f\|^2\end{aligned}$$

and the proof is completed. □

5. Some Equalities and Inequalities for C-K-g-Frames

Some equalities and inequalities have been established for ordinary frames and their duals in [16, 33] and have been extended to obtain several important equalities and inequalities for K-frames and continuous g-frames in [4, 28]. In this Section, we will generalize some equalities and inequalities about c-K-g-frames.
Let $K \in \mathcal{B}(H)$, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with respect to $\{H_\omega\}$ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ be a c-K-g dual of Λ. For any measurable subspace $\Omega_1 \subset \Omega$, we define

$$M_{\Omega_1} : H \longrightarrow H,$$
$$\langle M_{\Omega_1} f, g\rangle = \int_{\Omega_1} \langle \Lambda_\omega^* \Gamma_\omega f, g\rangle \, d\mu$$

for any $f, g \in H$. It is clear that M_{Ω_1} is well-defined and bounded operator, moreover $M_{\Omega_1} + M_{\Omega_1^c} = K$.

Theorem 5.1. *Let Λ be a c-K-g-frame for H with the dual Γ. Then for each $f \in H$,*

$$\int_{\Omega_1} \langle \Lambda_\omega^* \Gamma_\omega f(\omega), Kf\rangle \, d\mu - \|M_{\Omega_1} f\|^2 = \int_{\Omega_1} \overline{\langle \Lambda_\omega^* \Gamma_\omega f(\omega), Kf\rangle} \, d\mu - \|M_{\Omega_1^c} f\|^2.$$

Proof. For each $f \in H$, we have

$$\begin{aligned}
\langle M_{\Omega_1} f, Kf\rangle - \|M_{\Omega_1} f\|^2 &= \langle K^* M_{\Omega_1} f, f\rangle - \langle M_{\Omega_1}^* M_{\Omega_1} f, f\rangle \\
&= \langle (K - M_{\Omega_1})^* M_{\Omega_1} f, f\rangle \\
&= \langle M_{\Omega_1^c}^* (K - M_{\Omega_1^c}) f, f\rangle \\
&= \langle M_{\Omega_1^c}^* K f, f\rangle - \langle M_{\Omega_1^c}^* M_{\Omega_1^c} f, f\rangle \\
&= \langle Kf, M_{\Omega_1^c} f\rangle - \langle M_{\Omega_1^c} f, M_{\Omega_1^c} f\rangle \\
&= \overline{\langle M_{\Omega_1^c} f, Kf\rangle} - \|M_{\Omega_1^c} f\|^2
\end{aligned}$$

and the proof is completed. □

Definition 5.1. *Suppose that $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for H with respect to $\{H_\omega\}$. We define*

$$S_{\Omega_1} : H \longrightarrow H,$$
$$\langle S_{\Omega_1} f, g\rangle = \int_{\Omega_1} \langle \Lambda_\omega f, \Lambda_\omega g\rangle \, d\mu$$

for every $f, g \in H$.

It is clear that S_{Ω_1} and $S_{\Omega_1^c}$ are positive operator and since $0 \leq S_{\Omega_1^c} = S - S_{\Omega_1}$, hence $\|S_{\Omega_1}\| \leq \|S\|$

Theorem 5.2. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval c-K-g-frame for H. For each $f \in H$, $\Omega_1 \subset \Omega$ and $\Delta \subseteq \Omega_1^c$, we get*

$$\begin{aligned}
&\|S_{\Omega_1\cup\Delta} f\|^2 - \|S_{\Omega_1^c\setminus\Delta} f\|^2 \\
&= \int_{\Omega_1} \langle \Lambda_\omega f, \Lambda_\omega K K^* f\rangle \, d\mu - \overline{\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega K K^* f\rangle \, d\mu} + 2Re \int_{\Delta} \langle \Lambda_\omega f, \Lambda_\omega K K^* f\rangle \, d\mu.
\end{aligned}$$

Proof. Let $\Omega_1 \subset \Omega$. So, $S_{\Omega_1} + S_{\Omega_1^c} = KK^*$. Hence,

$$\begin{aligned}
S_{\Omega_1}^2 - S_{\Omega_1^c}^2 &= S_{\Omega_1}^2 - (KK^* - S_{\Omega_1})^2 \\
&= KK^* S_{\Omega_1} + S_{\Omega_1} KK^* - (KK^*)^2 \\
&= KK^* S_{\Omega_1} - S_{\Omega_1^c} KK^*.
\end{aligned}$$

Now, for each $f \in H$ we have

$$\|S_{\Omega_1} f\|^2 - \|S_{\Omega_1^c} f\|^2 = \langle KK^* S_{\Omega_1} f, f\rangle - \langle S_{\Omega_1^c} KK^* f, f\rangle.$$

Consequently, for $\Omega_1 \cup \Delta$ instead of Ω_1:

$$\begin{aligned}
\|S_{\Omega_1\cup\Delta} f\|^2 - \|S_{\Omega_1^c\setminus\Delta} f\|^2 &= \int_{\Omega_1\cup\Delta} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu - \overline{\langle S_{\Omega_1^c\setminus\Delta} f, KK^* f\rangle} \\
&= \int_{\Omega_1\cup\Delta} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu - \overline{\int_{\Omega_1^c\setminus\Delta} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu} \\
&= \int_{\Omega_1} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu - \overline{\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu} \\
&+ 2Re\Big(\int_{\Delta} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu\Big).
\end{aligned}$$

□

Theorem 5.3. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval c-K-g-frame for H. For each $f \in H$ and $\Omega_1 \subset \Omega$, we get*

$$\begin{aligned}
&Re\Big(\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu\Big) + \|S_{\Omega_1} f\|^2 \\
&= Re\Big(\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu\Big) + \|S_{\Omega_1^c} f\|^2 \\
&\geq \frac{3}{4}\|KK^* f\|^2.
\end{aligned}$$

Proof. Since $S_{\Omega_1}^2 - S_{\Omega_1^c}^2 = KK^* S_{\Omega_1} - S_{\Omega_1^c} KK^*$, then

$$S_{\Omega_1}^2 + S_{\Omega_1^c}^2 = 2(\frac{KK^*}{2} - S_{\Omega_1})^2 + \frac{(KK^*)^2}{2} \geq \frac{(KK^*)^2}{2}.$$

Therefore,

$$\begin{aligned}
KK^* S_{\Omega_1} + S_{\Omega_1^c}^2 + (KK^* S_{\Omega_1} + S_{\Omega_1^c}^2)^* &= KK^* S_{\Omega_1} + S_{\Omega_1^c}^2 + S_{\Omega_1} KK^* + S_{\Omega_1^c}^2 \\
&= KK^*(S_{\Omega_1} + S_{\Omega_1^c}) + S_{\Omega_1}^2 + S_{\Omega_1^c}^2 \\
&= (S_{\Omega_1} + S_{\Omega_1^c})KK^* + S_{\Omega_1}^2 + S_{\Omega_1^c}^2 \geq \frac{3}{2}(KK^*)^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&Re\Big(\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu\Big) + \|S_{\Omega_1} f\|^2 \\
&= Re\Big(\int_{\Omega_1} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle \, d\mu\Big) + \|S_{\Omega_1^c} f\|^2 \\
&= \frac{1}{2}\big(\langle KK^* S_{\Omega_1} h, h\rangle + \langle S_{\Omega_1^c}^2 h, h\rangle + \langle h, KK^* S_{\Omega_1} h\rangle + \langle h, S_{\Omega_1^c}^2 h\rangle\big) \geq \frac{3}{4}\|KK^* f\|^2.
\end{aligned}$$

□

In this section, we introduce some notations $v_+(\Lambda, \Omega_1)$ and $v_-(\Lambda, \Omega_1)$ to characterize more properties of Parseval c-K-g-frames, these notations have been introduced for K-frames and c-g-frames in [4, 28].

Assume that $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-K-g-frame for H and $\Omega_1 \subset \Omega$ and $f \in H$. We consider

$$v_+(\Lambda,\Omega_1) := \sup_{f\neq 0} \frac{Re\big(\int_{\Omega_1^c}\langle \Lambda_\omega KK^*f, \Lambda_\omega f\rangle\, d\mu\big) + \|S_{\Omega_1}f\|^2}{\|KK^*f\|^2}$$

and

$$v_-(\Lambda,\Omega_1) := \inf_{f\neq 0} \frac{Re\big(\int_{\Omega_1^c}\langle \Lambda_\omega KK^*f, \Lambda_\omega f\rangle\, d\mu\big) + \|S_{\Omega_1}f\|^2}{\|KK^*f\|^2}.$$

We aim to show some properties of these numbers by Theorem 5.2. For this, the following Lemma will be useful.

Lemma 5.1. *Let K be a closed range operator and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-K-g-frame for H with the lower bound A. Then,*

(I) *For each $f \in H$, $\|S_\Omega f\|^2 \leq \|S_\Omega\| \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu$;*

(II) *For any $f \in \mathcal{R}(K)$, $\int_\Omega \|\Lambda_\omega f\|^2 d\,\mu \leq \frac{1}{A}\|K^\dagger\|^2 \|S_\Omega f\|^2$.*

Proof. (I) We can write for any $f \in H$, $\|S_\Omega f\|^2 \leq \|S_\Omega\| |\langle S_\Omega f, f\rangle| = \|S_\Omega\| \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu$. (II) For each $f \in \mathcal{R}(k)$, we can write

$$\Big(\int_\Omega \|\Lambda_\omega f\|^2 d\,\mu\Big)^2 = |\langle S_\Omega f, f\rangle|^2 \leq \|S_\Omega f\|^2 \|f\|^2$$
$$\leq \|S_\Omega f\|^2 \|(K^\dagger)^* K^* f\|^2 \leq \|S_\Omega f\|^2 \|(K^\dagger)\|^2 \|K^* f\|^2 \leq \frac{1}{A}\|S_\Omega f\|^2 \|(K^\dagger)\|^2 \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu,$$

and the proof is completed. □

Theorem 5.4. *Let $\Lambda := \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval c-K-g-frame for H. Then the following assertions hold.*

(I) $\frac{3}{4} \leq v_-(\Lambda,\Omega_1) \leq v_+(\Lambda,\Omega_1) \leq \|K\|\|K^\dagger\|(1 + \|K\|\|K^\dagger\|)$

(II) $v_+(\Lambda,\Omega_1) = v_+(\Lambda,\Omega_1^c)$ *and* $v_-(\Lambda,\Omega_1) = v_-(\Lambda,\Omega_1^c)$.

Proof. Assume that $f \in H$. Since Λ is a c-g-Bessel sequence, so by applying Lemma 5.1, item (I), we have

$$\|S_{\Omega_1} f\|^2 \leq \|S_{\Omega_1}\| \int_{\Omega_1} \|\Lambda_\omega f\|^2 d\,\mu \leq \|S_{\Omega_1}\| \int_\Omega \|\Lambda_\omega f\|^2 d\,\mu$$
$$\leq \|K\|^2 \|K^* f\|^2 = \|K\|^2 \|K^\dagger KK^* f\|^2 \leq \|K\|^2 \|K^\dagger\|^2 \|KK^* f\|^2.$$

Moreover,

$$Re\Big(\int_{\Omega_1^c} \langle \Lambda_\omega f, \Lambda_\omega KK^* f\rangle\, d\mu\Big) \leq \Big(\int_\Omega \|\Lambda_\omega f\|^2\, d\mu\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Lambda_\omega KK^* f\|^2\, d\mu\Big)^{\frac{1}{2}}$$
$$= \|K^* f\| \|K^* KK^* f\| = \|K^\dagger KK^* f\| \|K^* KK^* f\| \leq \|K\| \|K^\dagger\| \|KK^* f\|^2.$$

Therefore,

$$v_-(\Lambda,\Omega_1) \leq v_+(\Lambda,\Omega_1) \leq \|K\|\|K^\dagger\|(1 + \|K\|\|K^\dagger\|).$$

(II). Via the proof of Theorem 5.2, we have $S^2_{\Omega_1} + S_{\Omega_1^c} KK^* = KK^* S_{\Omega_1} + S^2_{\Omega_1^c}$. Hence, for any $f \in H$, $\langle S^2_{\Omega_1} f, f\rangle + \langle S_{\Omega_1^c} KK^* f, f\rangle = \langle S^2_{\Omega_1^c} f, f\rangle + \langle KK^* S_{\Omega_1} f, f\rangle$. So,

$$\|S_{\Omega_1} f\|^2 + \Big(\int_{\Omega_1^c} \langle \Lambda_\omega KK^* f, \Lambda_\omega f\rangle\, d\mu\Big) = \|S_{\Omega_1^c} f\|^2 + \Big(\int_{\Omega_1} \overline{\langle \Lambda_\omega KK^* f, \Lambda_\omega f\rangle}\, d\mu\Big),$$

and this shows item (II). □

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REFERENCES

[1] *M. R. Abdollahpour and M. H. Faroughi*, Continuous G-frames in Hilbert spaces, Southeast Asian Bull. Math., **32** (2008), 1-19.

[2] *S. T. Ali, J. P. Antoine and J. P. Gazeau*, Continuous frames in Hilbert spaces, Annals of Phy., **222** (1993), 1-37.

[3] *E. Alizadeh, A. Rahimi, E. Osgooei and M. Rahmani*, Continuous K- G-frames in Hilbert spaces, Bull. Iran. Math. Soc., (2018), (DOI: 10.1007/s41980-018-0186-7).

[4] *F. Arabyani Neyshaburi, G. Mohajeri Minaei and E. Anjidani*, On some equalities and inequalities for K-frames , Indian. J. Pure. Appl. Math.,(2018).

[5] *M. S. Asgari and H. Rahimi*, Generalized frames for operators in Hilbert spaces, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **17** (2014), No. 2, 1450013 (20 p).

[6] *P. Balazs, D. Bayer and A. Rahimi*, Multipliers for countinuous frames in Hilbert spaces, J. Phys. A: Math. Theor., **45** (2012), 2240023(20 p).

[7] *P. G. Casazza*, The Art of Frame Theory, Taiwanese Journal of Math., **4**(2000), No. 2, 1-127.

[8] *P.G. Casazza*, Modern tools for Weyl-Heisenberg (Gabor) frame theory, Advances in Imaging and Electron Physics., **115** (2000), 1-127.

[9] *O. Christensen*, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2016.

[10] *I. Daubechies, A. Grossman and Y. Meyer*, Painless nonorthogonal expansions, J. Math. Phys., **27**(5) (1986), 1271-1283.

[11] *R. G. Douglas*, No majorization, factorization and range inclusion of operators on Hilbert space, Pro. Amer. Math. Sco., **17** (2) (1966), 413-415.

[12] *R. J. Duffin and A. C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72** (1952), 341-366.

[13] *M. Fornasier and H. Rauhut*, Continuous frames, function spaces, and the discretization problem, J. Fourier Anal. Appl., **11**(3) (2015), 245-287.

[14] *J. P. Gabardo and D. Han*, Frames associated with measurable space, Adv. Comp. Math., **18**(3) (2003), 127-147.

[15] *L. Gavruta*, Frames for operators, Appl. Comput. Harmon. Anal., **32** (2012), 139-144.

[16] *P. Gavruta*, On some identities and inequalities for frames in Hilbert spaces , J. Math. Anal. Appl., **321** (2006), 61-73.

[17] *D. Han and D.R. Larson*, Frames, bases and group representations, Memoirs AMS697 (2000).

[18] *D. Hua and Y. Hung*, K-g-frames and stability of K-g-frames in Hilbert spaces, J. Korean Math. Soc., **53** (6) (2016), 1331-1345.

[19] *G. Kaiser*, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.

[20] *S. Li*, On general frame decompositions, Numer. Funct. Anal. Optim., **16** (1995), No. 9-10, 1181-1191.

[21] *S. Li and H. Ogawa*, Pseudo-duals of frames with applications, Appl. Comput. Harmon. Anal., **11** (2001), 289-304.

[22] *M. Rahmani*, On some properties of c-frames, J. Math. Res. Appl., Vol. **37** (4) (2017), 466-476.

[23] *M. Rahmani*, Sum of c-frames, c-Riesz Bases and orthonormal mappings, U. P. B. Sci. Bull, Series A., Vol. **77** (3) (2015), 3-14

[24] *A. Rahimi, A. Najati and Y. N. Dehghan*, Continuous frames in Hilbert spaces, Method. Funct. Anal. Topology., Vol. **12** (2) (2006), 170-182.

[25] *T. Strohmer and R. Heath. Jr*, Grassmannian frames with applications to conding and Communications, Apple. Comput. Harmon. Anal., **14** (2003), 257-275.

[26] *W. C. Sun*, G-frames and g-Riesz, J. Math. Anal. Appl., **322** (1) (2006), 437- 452.

[27] *Zhong-Qi Xiang*, Canonical Dual K-g-Bessel Sequences and K-g-Frame Sequences, Results Math., Vol. **73** (1)(2018).

[28] *X. Xiang Chun and Z. Xiao Ming*, Some equalities and inequalities of g-continuous frames, SCI. China. Math., Vol. **53** (10) (2010), 2621-2632.

[29] *X. Xiao and Y. Zhu*, Exact K-g-frames in Hilbert spaces, Results Math., Vol. **72** (3) (2017), 1329-1339.

[30] *X. Xiao, Y. Zhu and L. Găvruta*, Some properties of K-frames in Hilbert spaces, Results. Math., Vol. **63** (3-4)(2013), 1243-1255.

[31] *Y. Zhou and Y. Zhu*, K-g-frames and dual g-frames for closed subspaces, Acta Math. Sinica (Chin. Ser)., **56** (5) (2013), 799-806.

[32] *Y. Zhou and Y. Zhu*, Characterization of K-g-frames in Hilbert spaces, Acta Math. Sinica (Chin. Ser)., Vol. **57** (5) (2014), 1031-1040.

[33] *X. Zhu, G. Wu*, A note on some equalities for frames in Hilbert spaces , Appl. Math. Lett., Vol. **23** (7) (2010), 788-790.