3-HOMOMORPHISM AMENABILITY OF BANACH ALGEBRAS

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In the current work, the notion of 3*-homomorphism amenability of Banach algebras is introduced. The hereditary properties of this new concept for Banach algebras are investigated and the relations between* 3*-homomorphism amenability of a Banach algebra and its ideals are found. The equivalence of* 3*-homomorphism amenability and the concept of the (right) character amenability is proved. As a consequence,* 3*-homomorphism amenability for either of the group algebra* $L^1(G)$ *or the Fourier algebra* $A(G)$ *is equivalent to the amenability of the underlying group* G*.*

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1. Introduction

The concept of amenability for Banach algebras was initiated by Johnson in [11]. One of the fundamental results was that $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable locally compact group. Since then, several modifications of this notion, through using homomorphisms between Banach algebras, have been introduced by others (for instance, see [2] and [15]).

In [12], Kaniuth, Lau and Pym studied the concept of φ-amenability of a Banach algebra A, where φ is a character on A. They also gave two characterizations of φ-amenability in terms of cohomology groups and Hahn-Banach type extension property in [13]. Monfared [14] introduced the notion of character amenability for Banach algebras. He used [14] in characterizing the structure of left (right) character amenable Banach algebras in several ways, and also showed that for any locally compact group G, left (right) character amenability of the generalized Fourier algebra $A_p(G)$ $(1 < p < \infty)$, is equivalent to the amenability of G. This result also holds for the group algebra $L^1(G)$. Characterizations of character amenability of the Fourier-Stieltjes algebra $B(G)$ and natural uniform algebra on a compact space are obtained in [10]. Recently, the concept of character contractibility for Banach algebras was introduced in [16].

Let A and B be (Banach) algebras. A linear map $h : A \longrightarrow B$ is called 3-homomorphism if $h(a_1a_2a_3) = h(a_1)h(a_2)h(a_3)$ for all $a_1, a_2, a_3 \in A$. Obviously, every homomorphism is a 3-homomorphism, but the converse is false, in general. If $h : A \longrightarrow B$ is a homomorphism, then $g := -h$ is a 3-homomorphism that is

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not a homomorphism [3]. However, each 3-homomorphism $h : A \longrightarrow \mathbb{C}$ induces a homomorphism on A which is not necessary unique (see Theorem 2.4 for more details).

For any Banach space X and Banach algebra A, each 3-homomorphism $\varphi : A \longrightarrow \mathbb{C}$ induces a (innumerable) module structure on X. In the second section, using this structure(s), we introduce the concept of 3-φ-amenability for A and characterize it in term of first Hochschild cohomology group of A with coefficients in X^* (for general form, see [1]). We also define the notion of 3-homomorphism amenability for Banach algebras and find the relationship between this new definition and character amenability. This relation will show that for a locally compact group G, the group algebra $L^1(G)$ or the generalized Fourier $A_p(G)$ $(1 < p < \infty)$ are 3-homomorphism amenable if and only if G is amenable. We investigate the hereditary properties of 3-homomorphism amenability for Banach algebras. Among many other things, we also study the relations between 3-homomorphism amenability of a Banach algebra and its ideals. Finally, in section 3, we introduce and investigate 3-homomorphism contractibility of Banach algebras.

2. 3-homomorphism amenability

We first bring some definitions in the Banach algebras setting. Let A be a Banach algebra, and let X be a Banach A-bimodule. A bounded linear map $D : A \longrightarrow X$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

For each $x \in X$, we define a map $D_x : A \longrightarrow X$ by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in A).$$

It is easily seen that D_x is a derivation. Derivations of this form are called *inner derivations*. $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first Hochschild cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X).$$

Let X be a A-bimodule. Then the dual space X^* of X is also a Banach A-bimodule by the following module actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \qquad (a \in \mathcal{A},\ x \in X,\ f \in X^*).$$

With the above notations, a Banach algebra A is called *amenable* (*contractible*) if $\mathcal{H}^1(A, X^*) = \{0\}$ ($\mathcal{H}^1(A, X) = \{0\}$) for every Banach A-bimodule X.

Throughout this paper, A is a Banach algebra and $\sigma^{(3)}(A)$ is the set of all non-zero 3-homomorphisms from A to $\mathbb{C}$. We denote the set of all non-zero characters on A by $\sigma(A)$.

Let $\varphi \in \sigma^{(3)}(A)$ and choose $u \in A$ such that $\varphi(u) = 1$. If X is a Banach space, then X can be viewed as a Banach left A-module by the following action:

$$a \cdot x = \varphi(u^3\ a)x, \quad (a \in A,\ x \in X). \tag{1}$$

In fact, for each $a, b \in A,\ x \in X$ we have

$$\begin{aligned} a \cdot (b \cdot x) &= \varphi(u^3 a)\varphi(u^3 b)x = \varphi(u^2)\varphi(u)\varphi(a)\varphi(u^2)\varphi(u)\varphi(b)x \\ &= \varphi(u^2)\varphi(a)\varphi(u^2)\varphi(b)x = \varphi(u^2)\varphi(au^2 b)x \\ &= \varphi(u^2)\varphi(au)\varphi(u)\varphi(b)x = \varphi(u^2)\varphi(au)\varphi(b)x \\ &= \varphi(au)\varphi(u^2)\varphi(b)x = \varphi(au^3 b)x \\ &= \varphi(u^3)\varphi(a)\varphi(b)x = \varphi(u^3 ab)x = ab \cdot x. \end{aligned} \tag{2}$$

Note that $a\cdot(b\cdot x) = ab\cdot x = \varphi(a)\varphi(b)x$ which is independent from the choice of u. If the left action of A on X is given by (1), then it is easy to see that $a\cdot(x\cdot b) = (a\cdot x)\cdot b$ for all $a, b \in A$ and $x \in X$. Therefore, the Banach space X admits a A-bimodule structure dependent to an element $u \in A$ with $\varphi(u) = 1$. One can also verify that in this case the right action of A on the dual A-bimodule X^* will be $f \cdot a = \varphi(u^3\ a)f$ for all $a \in A$ and $f \in X^*$.

Let A^{**} be the second dual of a Banach algebra A. For any m and n in A^{**}, we denote by mn, *the first Arens product* of m and n in A^{**}. This product is defined in stages as follows: For every $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$, we define $f \cdot a$ and $n \cdot f$ in A^*; mn in A^{**} by

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle$$
$$\langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle$$
$$\langle mn, f\rangle = \langle m, n \cdot f\rangle.$$

With this multiplication, A^{**} is a Banach algebra and A is a subalgebra of A^{**}. The image of $a \in A$ in A^{**} under the canonical embedding is denoted by $\widehat{a}$. However, for each $m, n \in A^{**}$ we have

$$mn = w^* - \lim_{\alpha} w^* - \lim_{\beta} \widehat{a}_\alpha \widehat{b}_\beta,$$

where (a_α) and (b_β) are bounded nets in A such that $m = w^* - \lim_\alpha \widehat{a}_\alpha$ and $n = w^* - \lim_\beta \widehat{b}_\beta$. For fixed $n \in A^{**}$, $m \mapsto mn$ is w^*-w^*-continuous on A^{**}, and for fixed $a \in A$, $m \mapsto am$ is w^*-w^*-continuous on A^{**} (for more details refer to [5]). Moreover, if $\varphi \in \sigma^{(3)}(A)$, then $mnp(\varphi) = m(\varphi)n(\varphi)p(\varphi)$. It follows from [3, Proposition 3.1] that $\varphi^{**} \in \sigma^{(3)}(A^{**})$, and so each $\varphi \in \sigma^{(3)}(A)$ extends uniquely to some elements $\widetilde{\varphi} = \varphi^{**}$ of $\sigma^{(3)}(A^{**})$. Now, the theory of second dual shows that kerφ is weak*-dense in kerφ^{**} and that kerφ^{**}=(ker$\varphi)^{**}$. One should remember that the second dual space ker φ^{**} contains ker φ in the same sense that A^{**} naturally contains A.

Let A be Banach algebra, $m \in A^{**}$, and let $\varphi \in \sigma^{(3)}(A)$ such that $\varphi(u) = 1$ for some $u \in A$. Then m is said to be 3-φ-*mean* on A^* (at u) if $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(u^3 a)m(f)$ for all $f \in A^*$ and $a \in A$. Also, A is called 3-φ-*amenable* if there exists a 3-φ-mean m on A^*.

Note that if φ is a non-zero multiplicative linear functional on A, then the left module structure (1) and the definition of 3-φ-mean will absolutely overlap with φ-amenability of A (character amenability) which has been introduced in [12] ([14]).

In the next result, we characterize 3-φ-amenability of a Banach algebra in term of cohomology groups.

Theorem 2.1. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A)$ such that $\varphi(u) = 1$. Then the following are equivalent:*

(i) *A is 3-φ-amenable (at u);*
(ii) *If X is a Banach A-bimodules X in which the left action is given by $a \cdot x = \varphi(u^3\ a)x$, for all $a \in A$ and $x \in X$, then $\mathcal{H}^1(A, X^*) = \{0\}$;*
(iii) *If A acts on $K = ker\varphi^{**}$ from left as $a \cdot m = \varphi(t^3 a)m$ for all $m \in A^{**}$ and naturally from right, then every continuous derivation from A into K is inner.*

Proof. (i) $\Rightarrow$ (ii) Suppose that m is a 3-φ-mean on A^* and $D : A \longrightarrow X^*$ is a module (3-φ-)derivation. Set $f = (D')^*(m) \in X^*$, where D' is the restriction of D^* to X. For each $x \in X, a, b \in A$

$$\begin{aligned}\langle D'(x \cdot a), b\rangle &= \langle D(b), x \cdot a\rangle = \langle a \cdot D(b), x\rangle \\ &= \langle D(ab), x\rangle - \langle D(a) \cdot b, x\rangle \\ &= \langle D'(x), ab\rangle - \langle D(a), b \cdot x\rangle \\ &= \langle D'(x) \cdot a, b\rangle - \varphi(u^3 b)\langle D(a), x\rangle.\end{aligned}$$

Hence

$$D'(x \cdot a) = D'(x) \cdot a - \varphi(u^2)\langle D(a), x\rangle\varphi \tag{3}$$

for all $x \in X, a \in A$. If $\varphi(u^2) = 0$, then $\varphi(u^3 a) = 0$ for all $a \in A$. This implies that left action A on X is zero which is a contradiction. Thus $\varphi(u^2) \neq 0$. Now, it follows from the definition of D' and (3) that

$$\begin{aligned}\langle a \cdot f, x\rangle &= \langle f, x \cdot a\rangle = \langle (D')^*(m), x \cdot a\rangle = \langle m, D'(x \cdot a)\rangle \\ &= \langle m, D'(x) \cdot a\rangle - \langle D(a), x\rangle m(\varphi)\varphi(u^2) \\ &= \langle m, D'(x)\rangle\varphi(u^3 a) - \langle D(a), x\rangle\varphi(u^2) \\ &= \langle f, x\rangle\varphi(u^3 a) - \langle D(a), x\rangle\varphi(u^2).\end{aligned}$$

Thus

$$D(a)\varphi(u^2) = \varphi(u^3 a)f - a \cdot f. \tag{4}$$

On the other hand, we have $D'(a \cdot x) = \varphi(u^3 a)D'(x)$ because

$$\langle D'(a \cdot x), b\rangle = \langle D(b), a \cdot x\rangle = \varphi(u^3 a)\langle D(b), x\rangle = \varphi(u^3 a)\langle D'(x), b\rangle,$$

for all $x \in X, a, b \in \mathcal{A}$. This conclude that

$$\begin{aligned}\langle f \cdot a, x\rangle &= \langle f, a \cdot x\rangle = \langle (D')^*(m), a \cdot x\rangle \\ &= \langle m, D'(a \cdot x)\rangle = \varphi(u^3 a)\langle m, D'(x)\rangle \\ &= \varphi(u^3 a)\langle f, x\rangle,\end{aligned}$$

and so $f \cdot a = \varphi(u^3 a)f$. Combining this equality with (4), we get $D(a) = a \cdot g - g \cdot a$, where $g = -\frac{1}{\varphi(u^2)}f$. Therefore D is an inner derivation.

(ii) $\Rightarrow$ (iii) That is trivial.

(iii) $\Rightarrow$ (i) Take $b \in A$ such that $\varphi(b) = 1$. For each $a \in A$, we have

$$\varphi(ab^3 - b^3 a) = \varphi(a)\varphi(b)\varphi(b^2) - \varphi(b^2)\varphi(b)\varphi(a) = 0.$$

The above equality shows that the map $D : A \longrightarrow K$ defined by $D(a) = ab^3 - b^3 a$ ($a \in A$) is a derivation, and so there exists $n \in K$ such that $D(a) = a \cdot n - n \cdot a$ for all

$a \in A$. Put $m = b^3 + n$. Then $a \cdot m = m \cdot a = \varphi(u^3a)m$ for all $a \in A$. Also $m(\varphi) = \varphi(b^3) + n(\varphi) = \varphi(b)^3 = 1$. Consequently, m is a 3-φ-mean on A (at u). □

The proving process of the above Theorem shows that if we replace u by another element v in A such that $\varphi(v) = 1$, although the left module structure on X will be different, still all assertions are equivalent. Hence 3-φ-amenability of A is independent from the choice of u. So if we put an element u with $\varphi(u) = 1$, there is no risk of confusion. Now, Theorem 2.1 leads us to the following definition:

Definition 2.1. *A Banach algebra A is said to be 3-homomorphism amenable if* Theorem 2.1 *holds for all $\varphi \in \sigma^{(3)}(A) \cup \{0\}$.*

Theorem 2.2. *Let A and B be Banach algebras and $\theta : A \longrightarrow B$ be a continuous homomorphism with dense range. If $\varphi \in \sigma^{(3)}(A)$ and A is 3-$\varphi \circ \theta$-amenable (at u), then B is 3-φ-amenable (at $\theta(u)$).*

Proof. Assume that $m \in A^{**}$ is 3-$\varphi \circ \theta$-amenable (at u), and so $m(\varphi \circ \theta) = 1$ and $m(f \cdot a) = \varphi \circ \theta(u^3a)m(f) = \varphi(\theta(u)^3\theta(a))m(f)$ for all $a \in A$ and $f \in A^*$. Consider $n \in B^{**}$ defined by $n(g) = m(g \circ \theta)$ for each $g \in B^*$. Obviously, $n(\varphi) = 1$. It is easy to see that $(g \cdot \theta(a)) \circ \theta = (g \circ \theta) \cdot a$ for all $a \in A$. Hence, for each $a \in A$ and $g \in B^*$, we have

$$\begin{aligned} n(g \cdot \theta(a)) &= m((g \cdot \theta(a)) \circ \theta) = m((g \circ \theta) \cdot a) \\ &= \varphi \circ \theta(u^3a)m(g \circ \theta) = \varphi(\theta(u)^3\theta(a))n(g). \end{aligned}$$

By density of the range of θ and continuity of n, B is 3-φ-amenable at $\theta(u)$. □

We need the following lemma to characterize 3-homomorphism amenable ideals of a 3-homomorphism amenable Banach algebra.

Lemma 2.1. *Let A be a Banach algebra and I be a closed ideal of A. If $\varphi \in \sigma^{(3)}(A)$ such that $\overline{\varphi} = \varphi\,|_I \neq 0$, then 3-$\varphi$-amenability of A implies 3-$\overline{\varphi}$-amenability of I.*

Proof. Assume that $m \in A^{**}$ is a 3-φ-mean on A^* (at u). It is proved in [12, Lemma 3.1] that $m(f \cdot a) = 0$ for all $f \in I^{\perp}$ and $a \in I$. Since $\varphi(u^2) \neq 0$ (see the proof of Theorem 2.1), $m(f) = 0$ for all $f \in I^{\perp}$, and so $m \in I^{\perp\perp}$. Thus m gives rise to a bounded linear functional $\widetilde{m}$ on $A^*/I^{\perp} = I^*$ defined by $\widetilde{m}(g) = m(f)$, (for $g \in I^*$), where f is an arbitrary element of A^* extending g. For any $g \in I^*$ and $a \in I$, we have $\widetilde{m}(\overline{\varphi}) = m(\varphi) = 1$ and

$$\widetilde{m}(g \cdot a) = m(f \cdot a) = \varphi(u^3a)m(f) = \varphi(u^3a)\widetilde{m}(g).$$

Note that in the above inequalities, $(f \cdot a)\,|_I = g \cdot a$ and $u^3a \in I$. □

Recall that a left (right) bounded approximate identity for a normed algebra A is a bounded net $(e_j)_j$ in A such that $\lim_j e_ja = a$ ($\lim_j ae_j = a$) for each $a \in A$.

Theorem 2.3. *Suppose that A is a 3-homomorphism amenable Banach algebra and I is a closed ideal of A. Then, I is a 3-homomorphism amenable if and only if I has a bounded right approximate identity.*

Proof. It follows from [11, Proposition 1.5 and 1.6] that the existence of a bounded right approximate identity for A is equivalent to $\mathcal{H}^1(A, X^*) = 0$ for every Banach A-bimodule X for which that left module action of A is $a \cdot x = 0$.

For the converse, in view of Lemma 2.1 it is sufficient to show that every $\psi \in \sigma^3(I)$ extends to some $\tilde{\psi} \in \sigma^3(A)$. Suppose that (a_j) is a bounded right approximate identity for I. The kernel of ψ, say J, is a closed ideal in A. Indeed,

$$\|aa_j^2 - a\| \leq \|aa_j - a\|\|a_j\| + \|aa_j - a\|$$

for all $a \in I$. Thus $\psi(ab) = \lim_j \psi(aba_j^2) = \lim_j \psi(a_j)\psi(aa_j)\psi(b) = 0$ for all $a \in A$ and $b \in J$. Similarly, $\psi(ba) = 0$. Choose $v \in I$ such that $\psi(v) = 1$. Then v^2 is an identity of I modulo J and so $\widetilde{\psi}(x) = \psi(xv^2)$ $(x \in A)$ define an element of $\sigma^3(A)$ extending ψ. In fact, for each $x, y, z \in A$

$$\widetilde{\psi}(xyz) = \psi(xyzv^2) = \psi(xv^2yv^2zv^2) = \psi(xv^2)\psi(yv^2)\psi(zv^2) = \widetilde{\psi}(x)\widetilde{\psi}(y)\widetilde{\psi}(z).$$

In the above inequalities, we use from this fact that

$$xv^2yv^2zv^2 - xyzv^2 = x(v^2yv^2zv^2 - yv^2zv^2) + xy(v^2zv^2 - zv^2) \in J. \quad \square$$

Let A be a Banach algebra and let $\varphi \in \sigma^3(A)$. From now on, we denote the kernel of φ by $I(\varphi)$ which is a ideal of A. In other words, if $u_0 \in A$ such that $\varphi(u_0) = 1$, then

$$\varphi(ab) = \varphi(u_0)^3\varphi(ab) = \varphi(u_0)\varphi(u_0^2ab) = \varphi(u_0^2)\varphi(a)\varphi(b) = 0,$$

for all $a \in A$ and $b \in I(\varphi)$. Hence, $I(\varphi)$ is a left ideal of A. Similarly, $I(\varphi)$ is a right ideal of A.

Proposition 2.1. *Suppose that A is a 3-φ-amenable Banach algebra and that A has a bounded right approximate identity .Then $I(\varphi)$ has a bounded right approximate identity.*

Proof. Let $\varphi(u) = 1$ and let $J(\varphi) = \{n \in A^{**} : \langle n, \varphi\rangle = 0\}$. Take $p \in A^{**}$ and $n \in J(\varphi)$, then $\langle np, \varphi\rangle = \lim_\alpha \lim_\beta \langle \varphi, a_\alpha b_\beta\rangle$, where (a_α) and (b_β) are bounded nets in A such that $\widehat{a}_\alpha \xrightarrow{w^*} n, \widehat{b}_\beta \xrightarrow{w^*} p$. So,

$$np(\varphi) = \lim_\alpha \lim_\beta \varphi(u^3)\varphi(a_\alpha b_\beta) = \lim_\alpha \lim_\beta \varphi(u^2)\varphi(a_\alpha)\varphi(b_\beta) = \varphi(u^3)n(\varphi)p(\varphi).$$

Hence, $J(\varphi)$ is w^*- closed ideal of A^{**} and it can be canonically identified with the second dual $I(\varphi)^{**}$. Now, suppose that $m_\varphi \in A^{**}$ such that $m_\varphi(\varphi) = 1$ and $m_\varphi(f \cdot a) = \varphi(u^3a)m_\varphi(f)$ for all $f \in A^*$ and $a \in A$. Putting $N = \{f \in A^* : f|_{I(\varphi)} = 0\}$, we have

$$I(\varphi)^* = A^*/N = (\mathbb{C}\varphi + N)/N \text{ and } I(\varphi)^{**} = \{n \in A^{**} : n|_N = 0\}.$$

Let (e_α) be a bounded right approximate identity of A. Since $\lim_\alpha \varphi(xe_\alpha^3) = \lim_\alpha \varphi(xe_\alpha) = \varphi(x)$ for all $x \in A$, we get

$$\lim_\alpha \varphi(x)\varphi(e_\alpha)^2 = \lim_\alpha \varphi(xe_\alpha)\varphi(e_\alpha)^2 = \lim_\alpha \varphi(xe_\alpha^3) = \varphi(x),$$

and so $\lim_\alpha \varphi(e_\alpha)^2 = 1$ or equivalently, $\lim_\alpha(\widehat{e}_\alpha(\varphi))^2 = 1$. This shows that $\lim_\alpha(\varphi(e_\alpha^2)) = 1$. Now, let E be a w^*-limit point of (e_α^2) in A^{**} and $m = E - m_\varphi$. Then $\langle m, \varphi\rangle = \lim_\alpha \varphi(e_\alpha^2) - m_\varphi(\varphi) = 0$, and thus $m \in J(\varphi)$. We next claim m is right identity for $J(\varphi)$. Indeed, for $a \in I(\varphi)$ and $f \in A^*$

$$\begin{aligned}\langle a \cdot m, f\rangle = \langle m, f \cdot a\rangle &= \lim_\alpha \langle e_\alpha^2, f \cdot a\rangle - m_\varphi(f \cdot a) \\ &= \lim_\alpha \langle f, ae_\alpha^2\rangle - \varphi(u^3a)m_\varphi(f) = \langle f, a\rangle\end{aligned}$$

The rest of the proof is similar to [12, Proposition 2.2]. □

Proposition 2.2. *Let A be a Banach algebra and let $\varphi \in \sigma^3(A)$. If the ideal $I(\varphi)$ has a bounded right approximate identity, then A is 3-φ-amenable.*

Proof. Suppose that $I = I(\varphi)$ has a bounded right approximate identity $(s_\beta)_\beta$. Since I is an ideal of codimension one in A (refer also to the proof of Theorem 2.3), A also has a bounded right approximate identity $(e_\alpha)_\alpha$ [6, Proposition 7.1]. Cohen factorization theorem shows that $A^* \cdot A$ and $A^* \cdot I$ are closed linear subspace of A^*. Let $u \in A$ such that $\varphi(u) = 1$. Then $au^2 - a \in I$, for all $a \in A$ and $a = au^2 + i$ for some $i \in I$. Hence, $A = A \cdot u^2 + I$ and

$$A^* \cdot A = A^* \cdot (A \cdot u^2) + A^* \cdot I \subseteq A^* \cdot u^2 + A^* \cdot I$$

However, $A^* \cdot u^2$ and $A^* \cdot I$ are linear subspace of $A^* \cdot A$, and so $A^* \cdot u^2 + A^* \cdot I \subseteq A^* A$. Thus $A^* \cdot u^2 + A^* \cdot I = A^* \cdot A$. Also, we have $\varphi \cdot u^2(a) = \varphi(u^2 a) = \varphi(u)^2 \varphi(a) = \varphi(a)$ for all $a \in A$, and thus $\varphi \cdot u^2 = \varphi$. Note that $\varphi \cdot u^2 \notin A^* \cdot I$. To see this, assume that $\varphi \cdot u^2 = f \cdot b$ for some $b \in I$ and $f \in A^*$. Then

$$\langle f, b \rangle = \lim_\beta \langle f, bs_\beta \rangle = \lim_\beta \langle f \cdot b, s_\beta \rangle = \lim_\beta \varphi(s_\beta) = 0.$$

On the other hand,

$$\langle f, b \rangle = \lim_\alpha \langle f, be_\alpha \rangle = \lim_\alpha \varphi(e_\alpha) \neq 0,$$

If $\lim_\alpha \varphi(e_\alpha) = 0$, then for all $x \in A$ we have

$$\varphi(x) = \lim_\alpha \varphi({e_\alpha}^2 x) = \lim_\alpha \varphi(e_\alpha)\varphi(e_\alpha)\varphi(x) = 0$$

Since $\varphi \cdot u^2 \notin A^* \cdot I$, by Hahn-Banach theorem there exists $p \in (A^* \cdot A)^*$ such that $p(A^* \cdot I) = \{0\}$ and $p(\varphi \cdot u^2) = 1$. Now, we define $m(f) = p(f \cdot u^2)$ for all $f \in A^*$. Then m is a bounded linear functional on A^* and $m(\varphi) = 1$. Let $a = b + \mu u^2$, $b \in I$, $\mu \in \mathbb{C}$ and $\varphi(v) = 1$ for some $v \in A$. Clearly, $u^4 - u^2 \in I$, and so

$$\begin{aligned} m(f \cdot a) = p((f \cdot a) \cdot u^2) &= p(f \cdot bu^2) + \mu p(f \cdot u^4) \\ &= p(f \cdot bu^2) + \mu p(f \cdot (u^4 - u^2)) + \mu p(f \cdot u^2) \\ &= \varphi(\mu v^3 u^2) m(f) = \varphi(bv^3 + \mu v^3 u^2) m(f) \\ &= \varphi(v^3 a) m(f) \end{aligned}$$

for all $f \in A^*$ and $a \in A$. Therefore, A is a 3-φ amenable (at v). □

According to Propositions 2.1 and 2.2 and this fact that 3-0-amenability of a Banach algebra A is equivalent to having a bounded right approximate identity, we have the following corollary.

Corollary 2.1. *A Banach algebra A is 3-homomorphism amenable if and only if for all $\varphi \in \sigma^3(A)$, $I(\varphi)$ has a bounded right approximate identity.*

Example 2.1. (i) Let A be a C^*- algebra. Then, by Corollary 2.1 we can observe that every C^*- algebra is 3-homomorphism amenable. This arises from the fact that every closed ideal of a C^*-algebra has a bounded approximate identity. On the other hand, a C^*- algebra is amenable if and only if is nuclear ([4] and [9]).
(ii) Let A be a (commutative) Banach algebra and let $\varphi \in \sigma^{(3)}(A)$ be such that

$\varphi(z_0) = 1$ for some $z_0 \in A$. Consider actions A on $\mathbb{C}$ by $a \cdot z = z \cdot a = \varphi(z_0^3 a)z; a \in A, z_0 \in \mathbb{C}$. If the non-trivial map $d : A \longrightarrow \mathbb{C}^* = \mathbb{C}$ is a continuous 3-φ-derivation, then it can not be inner because all inner derivation from A into $\mathbb{C}$ are zero. This example portrays that there exist semisimple commutative Banach algebras A which fail to be 3-homomorphism amenable.

Recall that a Banach algebra A is right (left) character amenable if for all $\varphi \in \sigma(A) \cup \{0\}$ and all Banach A-bimodule X for which the left (right) module action is given by $a \cdot x = \varphi(a)x$ $(x \cdot a = \varphi(a)x)$; $a \in A, x \in X$, every continuous derivation $D : A \longrightarrow X^*$ is inner.

Theorem 2.4. *Let A be a Banach algebra. Then A is a right character amenable if and only if it is 3-homomorphism amenable.*

Proof. Let A be 3-homomorphism amenable and φ be a character on A. Then φ is a 3-homomorphism on A. Assume that A is 3-φ-amenable (at u). We consider A-bimodules structure on a Banach space X by taking the left action to be $a \cdot x = \varphi(a)x; a \in A,\ x \in X$ and taking the right action to be the natural one. Define a left action A on X as $a \bullet x = u^3 a \cdot x$. Since φ is a character, $a \bullet x = \varphi(u^3 a)x = \varphi(u)^3\varphi(a)x = \varphi(a)x$. These equalities show that two left actions over X are equal. Thus A is φ-amenable, and so it is right character amenable.
Conversely, suppose that A is a right character amenable and φ belongs to $\sigma^{(3)}(A) \cup \{0\}$. Define $\widetilde{\varphi}(a) := \varphi(u^3 a); a \in A$ with $\varphi(u) = 1$. It follows from (2) that $\widetilde{\varphi}$ is a character on A. Since A is $\widetilde{\varphi}$-amenable, every derivation $D : A \longrightarrow X^*$ in which $a \cdot x = \widetilde{\varphi}(a)x$ is inner. Therefore, A is 3-φ-amenable. □

Let A be a non-unital Banach algebra. Then $A^\# = A \oplus \mathbb{C}$, the unitization of A, is a unital Banach algebra which contains A as a closed ideal. We denote the identity of $A^\#$ by e. For any $\varphi \in \sigma^3(A^\#)$, $\varphi(e) = 1$ or $\varphi(e) = -1$. If $\varphi(e) = 1$ then φ is a character, and if $\varphi(e) = -1$ then φ is a anti-character of A, that is $\varphi(xy) = -\varphi(x)\varphi(y)$ for all $x, y \in A$. So $\sigma^3(A^\#) = \sigma(A^\#) \bigcup -\sigma(A^\#)$. Using the fact that $A^\#$ is right (left) character amenable if and only if A is right (left) character amenable [14, Theorem 2.6] and since anti character amenability of A is equivalent to its character amenability (one can verify easily) we get to the following result.

Proposition 2.3. *The unitization algebra $A^\#$ is a 3-homomorphism amenable if and only if A is a right character amenable Banach algebra.*

Using Theorem 2.4 and Proposition 2.3 we have the following corollary. The corresponding result for character amenability has been obtained in [14, Theroen 2.6].

Corollary 2.2. *The unitization algebra $A^\#$ is 3-homomorphism amenable if and only if A is 3-homomorphism amenable.*

Let G be locally compact group. The Fourier algebra $A(G)$ and the generalized Fourier algebra $A_p(G)$ $(p \in (1, +\infty))$ which were introduced in [7] and [8], respectively, are commutative Banach algebras with pointwise operations of addition and multiplication. It is proved in [14, Corollary 2.4] that $A_p(G)$ is character amenable if and only if G is an amenable locally compact group; by Theorem 2.4 we have the following consequence.

Corollary 2.3. *Let $1 < p < \infty$, G be a locally compact group, and A be either of the Banach algebras $L^1(G)$ or $A_p(G)$. Then the following are equivalent:*

(i) *A is right character amenable;*
(ii) *A is 3-homomorphism amenable;*
(iii) *G is amenable*

Proof. The equivalence of (i) and (ii) follows from Theorem 2.4 and the equivalence of (i) and (iii) has been shown in [14, Corollary 2.4]. □

Let G be locally compact group. The measure algebra on G is denoted by $M(G)$. Thus $M(G)$ is the Banach algebra of complex-valued, regular Borel measures on G; the total variation of $\mu \in M(G)$ is denoted by $|\mu|$, and $\|\mu\| = |\mu|(G)$. Also, the convolution multiplication $\star$ on $M(G)$ is defined by setting

$$\langle \mu \star \nu, f\rangle = \int_G \int_G f(st)d\mu(s)d\nu(t) \quad (\mu, \nu \in M(G), f \in C_0(G))$$

where $C_0(G)$ is the Banach algebra of all continuous linear functionals on G vanishing at infinity.

Corollary 2.4. *The measure algebra $M(G)$ is 3-homomorphism amenable if and only if G is a discrete amenable group.*

Proof. The result follows immediately from Theorem 2.4 and [14, Corollary 2.5]. □

Proposition 2.4. *Let A be a Banach algebras and $\varphi \in \sigma^{(3)}(A) \cup \{0\}$. Then A is 3-φ-amenable if and only if A^{**} is 3-$\widetilde{\varphi}$-amenable.*

Proof. Let m be a 3-φ-mean on A^* (at u). For each $n \in A^{**}$ and $\lambda \in A^{***}$, take bounded nets $(a_j) \in A$ and $(f_k) \in A^*$ with $\widehat{a}_j \xrightarrow{w^*} n$ and $\widehat{f}_k \xrightarrow{w^*} \lambda$. We identify m as an element $\widehat{m}$ of A^{***}. Thus $\widehat{m}(\varphi) = 1$ and

$$\begin{aligned}
\langle \widehat{m}, \lambda \cdot n\rangle &= \langle \lambda, n \cdot m\rangle = \lim_k \langle n \cdot m, f_k\rangle \\
&= \lim_k \langle n, m \cdot f_k\rangle = \lim_k \lim_j \langle m \cdot f_k, a_j\rangle \\
&= \lim_k \lim_j \langle m, f_k \cdot a_j\rangle = \lim_k \lim_j \varphi(u^3 a_j)\langle m, f_k\rangle \\
&= \lim_j \varphi(u^3 a_j) \lim_k \langle m, f_k\rangle = \widetilde{\varphi}(u^3 n)\langle \widehat{m}, \lambda\rangle.
\end{aligned}$$

Consequently, A^{**} is 3-$\widetilde{\varphi}$-amenable.

Conversely, suppose that $\Phi \in A^{****}$ satisfies $\Phi(\widetilde{\varphi}) = \Phi(\varphi) = 1$ and $\Phi(\lambda \cdot n) = \widetilde{\varphi}(u^3 n)\Phi(\lambda)$ for all $n \in A^{**}$ and $\lambda \in A^{***}$. Then, the restriction of Φ to A^* is a 3-φ-mean on A^*. □

Let A and B be Banach algebras and $A\widehat{\otimes}B$ be the projective tensor product of A and B. For $\varphi \in \sigma^{(3)}(A)$ and $\psi \in \sigma^{(3)}(B)$, consider $\varphi \otimes \psi$ by $(\varphi \otimes \psi)(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A$ and $b \in B$. This clearly forces that $\varphi \otimes \psi \in \sigma^{(3)}(A\widehat{\otimes}B)$.

Theorem 2.5. *Let A and B be Banach algebras, let $\varphi \in \sigma^{(3)}(A)$ and $\psi \in \sigma^{(3)}(B)$ such that $\varphi(u) = 1$ and $\psi(v) = 1$ for some $u \in A$ and $v \in B$. If $A\widehat{\otimes}B$ is 3-$\varphi \otimes \psi$-amenable (at $u \otimes v$), then A is 3-φ-amenable (at u) and B is 3-ψ-amenable (at v).*

Proof. Assume that there exists $m \in (A\widehat{\otimes}B)^{**}$ such that

$$m((f \otimes \psi) \cdot (a \otimes b)) = (\varphi \otimes \psi)(u^3 a \otimes v^3 b)m(f \otimes \psi)$$

for all $a \in A, b \in B$ and $f \in A^*$. Take $a_0, a_1 \in A$ and $b_0, b_1 \in B$ such that $\varphi(a_0) = \varphi(a_1) = \psi(b_0) = \psi(b_1) = 1$. Define $\overline{m} \in A^{**}$ by $\overline{m}(f) = m(f \otimes \psi)$, where $f \in A^*$. It is easy to check that $(\varphi \otimes \psi)(u^3 a_0 a_1 \otimes v^3 b_0 b_1) = 1$. Now, for any $a \in A$ and $f \in A^*$, we have

$$\begin{aligned}
\langle \overline{m}, f \cdot a\rangle &= (\varphi \otimes \psi)(u^3 a_0 a_1 \otimes v^3 b_0 b_1)\langle m, (f \cdot a) \otimes \psi)\rangle \\
&= \langle m, (f \cdot a) \otimes \psi) \cdot (a_0 a_1 \otimes b_0 b_1)\rangle \\
&= \langle m, (f \cdot (a a_0 a_1)) \otimes (\psi \cdot b_0 b_1))\rangle \\
&= \langle m, ((f \otimes \psi) \cdot (a a_0 a_1 \otimes b_0 b_1))\rangle \\
&= (\varphi \otimes \psi)(u^3 a a_0 a_1 \otimes v^3 b_0 b_1)\langle m, f \otimes \psi\rangle \\
&= \varphi(u^3 a)\langle \overline{m}, f\rangle.
\end{aligned}$$

Also $\overline{m}(\varphi) = m(\varphi \otimes \psi) = 1$. Therefore A is 3-φ-amenable. Similarly, B is 3-ψ-amenable. □

3. 3-homomorphism contractibility

We start this section with a definition.

Definition 3.1. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A) \cup \{0\}$. Then A is said to be 3-φ-contractible, if every continuous derivation $D : A \to X$ is inner, whenever left action A over X is given by $a \cdot x = \varphi(u^3 a)x$ with $\varphi(u) = 1$.*

Theorem 3.1. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A) \cup \{0\}$ such that $\varphi(u) = 1$ for some $u \in A$. Then the following are equivalent:*

(i) *A is 3-φ-contractible (at u);*
(ii) *There exists $m \in A$ such that $\varphi(m) = 1$ and $ma = \varphi(u^3 a)m$ for all $a \in A$.*

Proof. (i) ⇒ (ii) Take $b_0 \in A$ such that $\varphi(b_0) = 1$. Define a A-bimodule structure on $X = A$ by $a \cdot x = \varphi(u^3 a)x, x \cdot a = xa$ for all $a \in A$ and $x \in X$. Then $D(a) = \varphi(u^3 a)b_0^3 - b_0^3 a, a \in A$ defines a continuous derivation from A into $I(\varphi)$ since

$$\begin{aligned}
\varphi(D(a)) &= \varphi(u^3 a)\varphi(b_0^3) - \varphi(b_0^3 a) \\
&= \varphi(u^2)\varphi(u)\varphi(a)\varphi(b_0^3) - \varphi(b_0^3 a) \\
&= \varphi(u^3 b_0^3)\varphi(a) - \varphi(b_0^3 a) \\
&= \varphi(u^3)\varphi(b_0)\varphi(b_0^2)\varphi(a) - \varphi(u^3)\varphi(b_0)\varphi(b_0^2)\varphi(a) \\
&= 0.
\end{aligned}$$

Due to 3-φ-contractibility of A, there exists $n \in I(\varphi)$ such that $D(a) = a \cdot n - n \cdot a$ for all $a \in A$, hence, the element $m = b_0^3 - n \in A$, as required.

(ii) ⇒ (i) Suppose that $m \in A$ with $\varphi(m) = 1$ and $ma = \varphi(u^3 a)m$ for all $a \in A$. Let X be an A-bimodule with actions $a \cdot x = \varphi(u^3 a)x, x \cdot a = xa$ for all $a \in A$ and $x \in X$. Put $x_0 = D(m)$. Then

$$x_0 \cdot a = D(ma) - m \cdot D(a) = D(ma) - \varphi(u^3 m)D(a) = \varphi(u^3 m)D(m) - \varphi(u^2)D(a).$$

Hence $D(a) = a \cdot x - x \cdot a$, where $x = \frac{1}{\varphi(u^2)}x_0$. Therefore, A is 3-φ-contractible. □

Now, our remark following Theorem 2.1 leads to the next definition.

Definition 3.2. *A Banach algebra A is called* 3-homomorphism contractible, *if it is 3-φ-contractible for every $\varphi \in \sigma^{(3)}(A) \cup \{0\}$.*

Proposition 3.1. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A)$ such that $\varphi(u) = 1$ for some $u \in A$. If $I(\varphi)$ has a left identity, then A is 3-φ-contractible (at u).*

Proof. Choose $v_0 \in A$ such that $\varphi(v_0) = 1$. Put $a_0 = v_0^4 - \varphi(u^3 v_0)v_0^3$. Then

$$\begin{aligned}\varphi(a_0) &= \varphi(v_0^4) - \varphi(u^3 v_0)\varphi(v_0^3) \\ &= \varphi(v_0^4) - \varphi(u^2)\varphi(v_0)\varphi(v_0^3) \\ &= \varphi(v_0^4) - \varphi(u^2 v_0^4) \\ &= \varphi(v_0^4) - \varphi(u)\varphi(u)\varphi(v_0^4) = 0.\end{aligned}$$

Thus $a_0 \in I(\varphi)$=ker φ. Let $m = v_0^3 - b_0 v_0^3$, where b_0 is a left identity for $I(\varphi)$. Obviously, $\varphi(m) = 1$. Now, if $a \in$ ker φ, then

$$ma = (v_0^3 - b_0 v_0^3)a = v_0^3 a - b_0 v_0^3 a = 0 = \varphi(u^3 a)m.$$

Moreover,

$$\begin{aligned}mv_0 - \varphi(u^3 v_0)m &= (v_0^3 - b_0 v_0^3)v_0 - \varphi(u^3 v_0)(v_0^3 - b_0 v_0^3) \\ &= v_0^4 - b_0 v_0^4 - \varphi(u^3 v_0)v_0^3 + \varphi(u^3 v_0)b_0 v_0^3 \\ &= v_0^4 - \varphi(u^3 v_0)v_0^3 - b_0(v_0^4 - \varphi(u^3 v_0)v_0^3) \\ &= a_0 - b_0 a_0 = 0.\end{aligned}$$

It follows from the equality $A = \mathbb{C}v_0 \oplus I(\varphi)$ and from Theorem 3.1 that A is 3-φ-contractible. □

Proposition 3.2. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A)$ such that $\varphi(u) = 1$ for some $u \in A$. If A 3-φ-contractible and has a left identity, then $I(\varphi)$ has a left identity.*

Proof. Since $A = \mathbb{C}v_0 \oplus I(\varphi)$ for some $v_0 \in A$, it follows from Theorem 3.1 that there exist a $m_1 = v_0 + a_1$ such that $m_1 a = \varphi(u^3 a)m_1$ for all $a \in A$, where $a_1 \in I(\varphi)$. Set $m_2 = v_0 + a_2$ be a left identity for A, where $a_2 \in I(\varphi)$. Put $e = a_2 - a_1$. Since $m_1 a = \varphi(u^2)\varphi(u)\varphi(a)m_1 = 0$ and $m_2 a = a$ for all $a \in I(\varphi)$, we have

$$ea = a_2 a - a_1 a = (m_2 - v_0)a - (m_1 - v_0)a = a.$$

So e is a left identity for $I(\varphi)$. □

Obviously, a Banach algebra A is 3-0-contractible if and only if it has a left identity. Thus, Propositions 3.1 and 3.2 display the following result and we omit its proof.

Corollary 3.1. *Let A be a Banach algebra and $\varphi \in \sigma^{(3)}(A)$. Then A is* 3-homomorphism contractible *if and only if $I(\varphi)$ has a left identity.*

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