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COMMON FIXED POINTS FOR FOUR MULTIVALUED NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

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In this paper, we introduce and study an iteration process for four multi-valued mappings in Kohlenbach hyperbolic spaces and establish a necessary and sufficient condition for strong convergence and Δ-convergence to common fixed points of the new iteration scheme. Also, we give some applications of our results. The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

Keywords: Multi-valued nonexpansive mapping, three-step iteration scheme, common fixed point, hyperbolic space, strong convergence, Δ-convergence.

MSC2010: 47H09, 47H10.

1. **Introduction and Preliminaries**

Let $\mathcal{K}$ be a nonempty bounded closed convex subset of a Banach space $\mathcal{X}$. A mapping $\mathcal{T}\colon \mathcal{K} \to \mathcal{K}$ is said to be nonexpansive if

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \|x - y\|$$

for all $x, y \in \mathcal{K}$.

It has been shown that if $\mathcal{X}$ is uniformly convex then every nonexpansive mapping $\mathcal{T}\colon \mathcal{K} \to \mathcal{K}$ has a fixed point (see Browder [2], cf. also Kirk [12]).

In 1953, Mann [19] introduced an iteration process for single valued nonexpansive mappings in Banach space as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}x_n, \ n \geq 1, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

In 1974, Ishikawa [9] introduced a new iteration process for single valued nonexpansive mappings in Banach space as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}x_n, \ n \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. This iteration scheme reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

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In 2000, Noor [23] introduced a three-step iteration process for single valued nonexpansive mappings in Banach space as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T} y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T} z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T} x_n, \; n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

In 2012, Saluja [27] studied the following iteration scheme for four nonexpansive mappings in uniformly convex Banach spaces. The scheme is as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = \alpha_n \mathcal{R} x_n + \beta_n \mathcal{S} y_n + \gamma_n u_n, \\ y_n = \alpha'_n \mathcal{R} x_n + \beta'_n \mathcal{T} z_n + \gamma'_n v_n, \\ z_n = \alpha''_n \mathcal{R} x_n + \beta''_n \mathcal{U} x_n + \gamma''_n w_n, \; n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$, $\{\gamma''_n\}$ are sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in $\mathcal{K}$.

If we put $\gamma_n = \gamma'_n = \gamma''_n = 0$ for all $n \geq 1$, then the iteration scheme (4) reduces to the following scheme

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = \alpha_n \mathcal{R} x_n + (1-\alpha_n) \mathcal{S} y_n, \\ y_n = \alpha'_n \mathcal{R} x_n + (1-\alpha'_n) \mathcal{T} z_n, \\ z_n = \alpha''_n \mathcal{R} x_n + (1-\alpha''_n) \mathcal{U} x_n, \; n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$ are sequences in $[0,1]$.

Let $\mathcal{K}$ be a subset of a metric space $\mathcal{X}$. A subset $\mathcal{K}$ is called proximal if for each $x \in \mathcal{X}$, there exists an element $k \in \mathcal{K}$ such that $d(x,k) = \inf\{\|x-y\| : y \in \mathcal{K}\} = d(x,\mathcal{K})$. It is well known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are Proximal.

We shall denote $CB(\mathcal{K})$, $C(\mathcal{K})$ and $P(\mathcal{K})$ by the families of all nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximal subsets of $\mathcal{K}$, respectively. Let H denote the Hausdorff metric induced by the metric d of $\mathcal{X}$, that is,

$$H(A,B) = \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}$$

for every $A, B \in CB(\mathcal{X})$, where $d(x,B) = \inf\{\|x-y\| : y \in B\}$.

A multivalued mapping $\mathcal{T} \colon \mathcal{K} \to CB(\mathcal{K})$ is said to be a *contraction* if there exists a constant $t \in [0,1)$ such that for any $x, y \in \mathcal{K}$,

$$H(\mathcal{T}x, \mathcal{T}y) \leq t\, d(x,y),$$

and $\mathcal{T}$ is said to be *nonexpansive* if

$$H(\mathcal{T}x, \mathcal{T}y) \leq d(x,y),$$

for all $x, y \in \mathcal{K}$. A point $x \in \mathcal{K}$ is called a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}x$. Denote the set of all fixed points of $\mathcal{T}$ by $F(\mathcal{T})$ and $P_{\mathcal{T}}(x) = \{y \in \mathcal{T}x : d(x,y) = d(x,\mathcal{T}x)\}$.

We consider the following notion of a hyperbolic space introduced by Kohlenbach [15].

Definition 1.1. *A metric space* $(\mathcal{X}, d)$ *is a hyperbolic space if there exists a map* $\mathcal{W}\colon \mathcal{X}^2 \times [0,1] \to \mathcal{X}$ *satisfying*

(i) $d(u, \mathcal{W}(x, y, \alpha)) \leq \alpha\, d(u, x) + (1-\alpha) d(u, y)$,
(ii) $d(\mathcal{W}(x, y, \alpha), \mathcal{W}(x, y, \beta)) \leq |\alpha - \beta| d(x, y)$,
(iii) $\mathcal{W}(x, y, \alpha) = \mathcal{W}(x, y, (1-\alpha))$,
(iv) $d(\mathcal{W}(x, z, \alpha), \mathcal{W}(y, w, \alpha)) \leq \alpha\, d(x, y) + (1-\alpha) d(z, w)$

for all $x, y, z, w \in \mathcal{X}$ and $\alpha, \beta \in [0,1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [15] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov [8]. An important example of a hyperbolic space is the open unit ball B_H in a real Hilbert space H is as follows.

Let B_H be the open unit ball in H. Then

$$k_{B_H}(x, y) = arg\,tanh(1 - \sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{|1 - \langle x, y \rangle|^2}$$

for all $x,\ y \in B_H$, defines a metric on B_H (also known as Kobayashi distance).

In the sequel, we shall use the term hyperbolic space instead of Kohlenbach hyperbolic space for convenience.

§ A metric space $(\mathcal{X}, d)$ is called a convex metric space introduced by Takahashi [33] if it satisfies only condition (i).

§ A metric space $(\mathcal{X}, d)$ satisfies (i)-(iii), then we obtain the notion of space of hyperbolic type in the sense of Goebel and Kirk [6].

Definition 1.2. *A subset* $\mathcal{K}$ *of a hyperbolic space* $\mathcal{X}$ *is convex if* $\mathcal{W}(x, y, \alpha) \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$ *and* $\alpha \in [0,1]$.

Definition 1.3. *A hyperbolic space* $(\mathcal{X}, d, \mathcal{W})$ *is uniformly convex* [30] *if for any* $u, x, y \in \mathcal{X}$, $r > 0$ *and* $\varepsilon \in (0, 2]$, *there exists a* $\rho \in (0,1]$ *such that* $d(\mathcal{W}(x, y, \frac{1}{2}), u) \leq (1-\rho) r$ *whenever* $d(x, u) \leq r$, $d(y, u) \leq r$ *and* $d(x, y) \geq \varepsilon r$.

A map $\eta\colon (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\rho = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is known as the modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Different notion of hyperbolic space can be found in the literature (see for example [6, 13, 15, 18]). The hyperbolic space introduced by Kohlenbach [15] is slightly restrictive than the space of hyperbolic type [6] but general than hyperbolic space of [25].

The study of fixed points for multivalued nonexpansive mappings using Hausdorff metric was initiated by Markin [20] (see, also [21]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [7] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [17]. The theory of multivalued nonexpansive mappings are harder than the corresponding theory of single-valued nonexpansive mappings. Different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings.

Sastry and Babu [28] in 2005, considered Mann and Ishikawa type iterates for multivalued mappings with a fixed point. They also gave an example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0. They considered the following:

Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{X}$, $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ is a multivalued mapping with $p \in \mathcal{T}p$.

(S_1) The sequence of *Mann iterates* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n s_n, \ n \geq 1, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $s_n \in \mathcal{T}x_n$ such that $\|s_n - p\| = d(p, \mathcal{T}x_n)$.

(S_2) The sequence of *Ishikawa iterates* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \\ y_n = (1-\beta_n)x_n + \beta_n s_n, \ n \geq 1, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$, $\|s_n - r_n\| = d(\mathcal{T}x_n, \mathcal{T}y_n)$ and $\|r_n - p\| = d(p, \mathcal{T}y_n)$ for $s_n \in \mathcal{T}x_n$ and $r_n \in \mathcal{T}y_n$. They established some strong and weak convergence results of the above iterates for multivalued nonexpansive mappings $\mathcal{T}$ under some appropriate conditions.

The following is a useful lemma due to Nadler [21].

Lemma 1.1. *Let $A, B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \eta$.*

Panyanak [24] in 2007, proved some results using Ishikawa type iteration process without the condition $T(p) = \{p\}$ on the mapping $\mathcal{T}$. Later in 2008, Song and Wang [31] proved strong convergence theorems of Mann and Ishikawa iterates for multivalued nonexpansive mappings under some appropriate control conditions. Furthermore, they also gave an affirmative answer to Panyanak's open question in [24].

Recently, Shahzad and Zegeye [29] pointed out that the assumption $T(p) = \{p\}$ for any $p \in F(\mathcal{T})$ is quite strong. In order to get rid of the condition $T(p) = \{p\}$ for any $p \in F(\mathcal{T})$, they used $P_{\mathcal{T}}(x) := \{y \in \mathcal{T}x : \|x - y\| = d(x, \mathcal{T}x))\}$ for a multivalued map $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ and proved some strong convergence results using Mann and Ishikawa type iterative process. Song and Cho [32] improved the results of [29].

In this paper, we first introduce four multivalued mappings version of the iterative process (5) in hyperbolic spaces and use $P_{\mathcal{T}}(x) := \{y \in \mathcal{T}x : d(x,y) = d(x, \mathcal{T}x))\}$ instead of a stronger condition $T(p) = \{p\}$ for any $p \in F(\mathcal{T})$ to approximate common fixed points of four multivalued nonexpansive mappings.

Let $\mathcal{K}$ be a nonempty convex subset of a hyperbolic space $\mathcal{X}$. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}\colon \mathcal{K} \to P(\mathcal{K})$ be four multivalued mappings and $P_{\mathcal{T}}(x) := \{y \in \mathcal{T}x : d(x,y) = d(x, \mathcal{T}x))\}$. Choose $x_0 \in \mathcal{K}$

and define $\{x_n\}$ as

$$\begin{cases} z_n = \mathcal{W}(u_n, t_n, \alpha''_n), \\ y_n = \mathcal{W}(u_n, w_n, \alpha'_n), \\ x_{n+1} = \mathcal{W}(u_n, v_n, \alpha_n), \ n \geq 1, \end{cases} \tag{8}$$

where $u_n \in P_{\mathcal{R}}(x_n)$, $v_n \in P_{\mathcal{S}}(y_n)$, $w_n \in P_{\mathcal{T}}(z_n)$, $t_n \in P_{\mathcal{U}}(x_n)$ and $\{\alpha_n\}, \{\alpha'_n\}$, $\{\alpha''_n\} \in [0, 1]$.

It follows from the definition of $P_{\mathcal{T}}$ that $d(x, \mathcal{T}x) \leq d(x, P_{\mathcal{T}}(x))$ for any $x \in \mathcal{K}$.

The concept of Δ-convergence in a general metric space was introduced by Lim [18]. In 2008, Kirk and Panyanak [14] used the notion of Δ-convergence introduced by Lim [18] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [4] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the notion of Δ-convergence has been widely studied and a number of articles have appeared e.g., [1, 3, 4, 10, 22, 26].

Now, we recall some definitions.

Let K be a nonempty subset of metric space $\mathcal{X}$. Let $\{x_n\}$ be a bounded sequence in a metric space $\mathcal{X}$. For $x \in \mathcal{X}$, define a continuous functional $r(., \{x_n\})\colon \mathcal{X} \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. Then

(a) $r_{\mathcal{K}}(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{X}\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $\mathcal{K} \subset \mathcal{X}$.

(b) for any $y \in \mathcal{K}$, the set $A_{\mathcal{K}}(\{x_n\}) = \big\{x \in \mathcal{X} : r(x, \{x_n\}) \leq r(y, \{x_n\})\big\}$ is called the asymptotic center of $\{x_n\}$ with respect to $\mathcal{K} \subset \mathcal{X}$.

If the asymptotic radius of the asymptotic center are taken with respect to $\mathcal{X}$, then these are simply denoted by $r(\{x_n\})$ and $A(\{x_n\})$, respectively. In general, $A(\{x_n\})$ may be empty or may even contain infinitely many points. It is well known that a complete uniformly convex hyperbolic space with monotone modulus of convexity enjoys the property that bounded sequences have unique asymptotic center with respect to closed convex subsets ([16]).

Definition 1.4. *A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in \mathcal{X}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$* [14]. *In this case, we write Δ-$\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.*

In the sequel we need the following key results to be used in our main results.

Lemma 1.2. *(*[11]*) Let $(\mathcal{X}, d, \mathcal{W})$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathcal{X}$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{X}$ such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(\mathcal{W}(x_n, y_n, \alpha_n), x)$ $= r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.3. *(*[11]*) Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $\mathcal{X}$ and $\{x_n\}$ a bounded sequence in $\mathcal{K}$ such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in $\mathcal{K}$ such that $\lim_{m\to\infty} r(y_m, \{x_n\})$ $= r(y, \{x_n\})$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.4. *([34]) Let $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$. Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 1.5. *(See [5]) Let $\mathcal{K}$ be a nonempty subset of a metric space $\mathcal{X}$. Let $\mathcal{T} \colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping and $P_{\mathcal{T}}(x) = \{y \in \mathcal{T}x : d(x, y) = d(x, \mathcal{T}x)\}$. Then the following are equivalent:*

(1) $x \in F(\mathcal{T})$, that is, $x \in \mathcal{T}x$;

(2) $P_{\mathcal{T}}(x) = \{x\}$, that is, $x = y$ for each $y \in P_{\mathcal{T}}(x)$;

(3) $x \in F(P_{\mathcal{T}})$, that is, $x \in P_{\mathcal{T}}(x)$.

Moreover, $F(\mathcal{T}) = F(P_{\mathcal{T}})$.

2. Main Results

In this section we prove some strong and a Δ-convergence theorems using iteration scheme (8). Assume that $\mathcal{F} = F(\mathcal{R}) \cap F(\mathcal{S}) \cap F(\mathcal{T}) \cap F(\mathcal{U})$ denotes the set of all common fixed points of the multivalued mappings $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$. First, we need the following lemmas to prove our main results.

Lemma 2.1. *Let $\mathcal{X}$ be a hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U} \colon \mathcal{K} \to P(\mathcal{K})$ be four multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}$ are real sequences in $[0, 1]$. Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$. Then $p \in P_{\mathcal{R}}(p) = \{p\}$, $p \in P_{\mathcal{S}}(p) = \{p\}$, $p \in P_{\mathcal{T}}(p) = \{p\}$ and $p \in P_{\mathcal{U}}(p) = \{p\}$ by Lemma 1.5. Using (8), we have

$$\begin{aligned} d(z_n, p) &= d\Big(\mathcal{W}(u_n, t_n, \alpha''_n), p\Big) \\ &\leq (1 - \alpha''_n)d(u_n, p) + \alpha''_n d(t_n, p) \\ &\leq (1 - \alpha''_n)d(u_n, P_{\mathcal{R}}(p)) + \alpha''_n d(t_n, P_{\mathcal{U}}(p)) \\ &\leq (1 - \alpha''_n)H(P_{\mathcal{R}}(x_n), P_{\mathcal{R}}(p)) + \alpha''_n H(P_{\mathcal{U}}(x_n), P_{\mathcal{U}}(p)) \\ &\leq (1 - \alpha''_n)d(x_n, p) + \alpha''_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{9}$$

Now using (8) and (9), we have

$$\begin{aligned} d(y_n, p) &= d\Big(\mathcal{W}(u_n, w_n, \alpha'_n), p\Big) \\ &\leq (1 - \alpha'_n)d(u_n, p) + \alpha'_n d(w_n, p) \\ &\leq (1 - \alpha'_n)d(u_n, P_{\mathcal{R}}(p)) + \alpha'_n d(t_n, P_{\mathcal{T}}(p)) \\ &\leq (1 - \alpha'_n)H(P_{\mathcal{R}}(x_n), P_{\mathcal{R}}(p)) + \alpha'_n H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq (1 - \alpha'_n)d(x_n, p) + \alpha'_n d(z_n, p) \\ &\leq (1 - \alpha'_n)d(x_n, p) + \alpha'_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{10}$$

Again using (8) and (10), we have

$$\begin{aligned} d(x_{n+1},p) &= d\Big(\mathcal{W}(u_n,v_n,\alpha_n),p\Big) \\ &\leq (1-\alpha_n)d(u_n,p)+\alpha_n d(v_n,p) \\ &\leq (1-\alpha_n)d(u_n,P_{\mathcal{R}}(p))+\alpha_n d(v_n,P_{\mathcal{S}}(p)) \\ &\leq (1-\alpha_n)H(P_{\mathcal{R}}(x_n),P_{\mathcal{R}}(p))+\alpha_n H(P_{\mathcal{S}}(y_n),P_{\mathcal{S}}(p)) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned}$$

That is,

$$d(x_{n+1},p) \leq d(x_n,p). \tag{11}$$

It follows from Lemma 1.4 that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in \mathcal{F}$. This completes the proof. □

Lemma 2.2. *Let $\mathcal{X}$ be a hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{R},\mathcal{S},\mathcal{T},\mathcal{U}\colon \mathcal{K}\to P(\mathcal{K})$ be four multivalued mappings and $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (8), where $\{\alpha_n\},\{\alpha'_n\},\{\alpha''_n\}$ are real sequences in $[0,1]$ satisfying $0<a_1\leq \alpha_n,\alpha'_n,\alpha''_n\leq a_2<1$ for some $a_1,a_2\in(0,1)$. If $\mathcal{F}\neq\emptyset$ and*

$$d(x,\mathcal{S}y)\leq d(\mathcal{R}x,\mathcal{S}y),\ \ \forall\, x,y\in\mathcal{K}. \tag{12}$$

Then $\lim_{n\to\infty} d(x_n,P_{\mathcal{R}}(x_n))=0=\lim_{n\to\infty} d(x_n,P_{\mathcal{S}}(y_n))=\lim_{n\to\infty} d(x_n,P_{\mathcal{T}}(z_n))$ $=\lim_{n\to\infty} d(x_n,P_{\mathcal{U}}(x_n))$.

Proof. By Lemma 2.1, $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in\mathcal{F}$. Assume that $\lim_{n\to\infty} d(x_n,p)=c$ for some $c\geq 0$. If $c=0$, then the result is trivial. Suppose $c>0$.

Now $\lim_{n\to\infty} d(x_{n+1},p)=c$ can be written as

$$\lim_{n\to\infty} d\Big(\mathcal{W}(u_n,v_n,\alpha_n),p\Big)=c. \tag{13}$$

Now taking lim sup on both side of (9) and (10), we get

$$\limsup_{n\to\infty} d(z_n,p)\leq c, \tag{14}$$

and

$$\limsup_{n\to\infty} d(y_n,p)\leq c. \tag{15}$$

Since $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$ are nonexpansive, we have

$$\begin{aligned} d(u_n,p) &= d(u_n,P_{\mathcal{R}}(p)) \\ &\leq H(P_{\mathcal{R}}(x_n),P_{\mathcal{R}}(p)) \\ &\leq d(x_n,p). \end{aligned}$$

Hence

$$\limsup_{n\to\infty} d(u_n,p)\leq c. \tag{16}$$

Next,

$$\begin{aligned} d(v_n,p) &= d(v_n,P_{\mathcal{S}}(p)) \\ &\leq H(P_{\mathcal{S}}(y_n),P_{\mathcal{S}}(p)) \\ &\leq d(y_n,p)\leq d(x_n,p), \end{aligned}$$

and so

$$\limsup_{n\to\infty} d(v_n, p) \leq c. \tag{17}$$

Again, note that

$$\begin{aligned} d(w_n, p) &= d(w_n, P_{\mathcal{J}}(p)) \\ &\leq H(P_{\mathcal{J}}(z_n), P_{\mathcal{J}}(p)) \\ &\leq d(z_n, p) \leq d(x_n, p), \end{aligned}$$

and so

$$\limsup_{n\to\infty} d(w_n, p) \leq c. \tag{18}$$

Further, note that

$$\begin{aligned} d(t_n, p) &= d(t_n, P_{\mathcal{U}}(p)) \\ &\leq H(P_{\mathcal{U}}(x_n), P_{\mathcal{U}}(p)) \\ &\leq d(x_n, p), \end{aligned}$$

and so

$$\limsup_{n\to\infty} d(t_n, p) \leq c. \tag{19}$$

From (13), (16), (17) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(u_n, v_n) = 0. \tag{20}$$

Using (12) and (20), we obtain

$$\begin{aligned} d(u_n, x_n) &\leq d(u_n, v_n) + d(v_n, x_n) \\ &\leq d(u_n, v_n) + d(u_n, v_n) = 2d(u_n, v_n) \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{21}$$

and hence

$$\begin{aligned} d(v_n, x_n) &\leq d(v_n, u_n) + d(u_n, x_n) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{22}$$

Again, we observe that for each $n \geq 1$

$$\begin{aligned} d(x_n, p) &\leq d(x_n, v_n) + d(v_n, p) \\ &\leq d(x_n, v_n) + d(v_n, P_{\mathcal{S}}(p)) \\ &\leq d(x_n, v_n) + H(P_{\mathcal{S}}(y_n), P_{\mathcal{S}}(p)) \\ &\leq d(x_n, v_n) + d(y_n, p). \end{aligned}$$

Using (22), we obtain

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{23}$$

Combining (23) together with (15) gives

$$\lim_{n\to\infty} d(y_n, p) = c.$$

That is,

$$\lim_{n\to\infty} d\Big(\mathcal{W}(u_n, w_n, \alpha'_n), p\Big) = c. \tag{24}$$

From (16), (18), (24) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(u_n, w_n) = 0, \tag{25}$$

and hence

$$d(w_n, x_n) \leq d(w_n, u_n) + d(u_n, x_n) \to 0 \text{ as } n \to \infty. \tag{26}$$

Again note that

$$\begin{aligned} d(x_n, p) &\leq d(x_n, w_n) + d(w_n, p) \\ &\leq d(x_n, w_n) + d(w_n, P_{\mathcal{T}}(p)) \\ &\leq d(x_n, w_n) + H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq d(x_n, w_n) + d(z_n, p). \end{aligned}$$

Using (26), we obtain

$$c \leq \liminf_{n\to\infty} d(z_n, p). \tag{27}$$

Combining (27) together with (14) gives

$$\lim_{n\to\infty} d(z_n, p) = c.$$

That is,

$$\lim_{n\to\infty} d\Big(\mathcal{W}(u_n, t_n, \alpha''_n), p\Big) = c. \tag{28}$$

From (16), (19), (28) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(u_n, t_n) = 0. \tag{29}$$

Now using (21) and (29), we obtain

$$d(x_n, t_n) \leq d(x_n, u_n) + d(u_n, t_n) \to 0 \text{ as } n \to \infty. \tag{30}$$

Since $d(x, P_{\mathcal{R}}(x)) = \inf_{r\in P_{\mathcal{R}}(x)} d(x, r)$, therefore $d(x_n, P_{\mathcal{R}}(x_n)) \leq d(x_n, u_n) \to 0$ as $n \to \infty$. Similarly $d(x_n, P_{\mathcal{S}}(y_n)) \leq d(x_n, v_n) \to 0$ as $n \to \infty$, $d(x_n, P_{\mathcal{T}}(z_n)) \leq d(x_n, w_n) \to 0$ as $n \to \infty$, and $d(x_n, P_{\mathcal{U}}(x_n)) \leq d(x_n, t_n) \to 0$ as $n \to \infty$. This completes the proof. □

We now prove Δ-convergence theorem of the iteration process (8).

Theorem 2.1. *Let $\mathcal{X}$ be a uniformly convex hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ with monotone modulus of uniform convexity η and $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}$, $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$, $P_{\mathcal{U}}$, $\{x_n\}$ and condition (12)be as in Lemma 2.2. Then $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ (or $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$).*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded, therefore $\{x_n\}$ has a unique asymptotic center. Thus $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then $\lim_{n\to\infty} d(u_n, P_{\mathcal{R}}(u_n)) = 0 = \lim_{n\to\infty} d(u_n, P_{\mathcal{S}}(u_n)) = \lim_{n\to\infty} d(u_n, P_{\mathcal{T}}(u_n)) = \lim_{n\to\infty} d(u_n, P_{\mathcal{U}}(u_n))$ by Lemma 2.2. We now prove that u is a common fixed point of $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$. For this, take $\{v_m\}$ in $P_{\mathcal{R}}(u)$. Then

$$\begin{aligned} r(v_m, \{u_n\}) &= \limsup_{n\to\infty} d(v_m, u_n) \leq \limsup_{n\to\infty} [d(v_m, P_{\mathcal{R}}(u_n)) + d(P_{\mathcal{R}}(u_n), u_n)] \\ &\leq \limsup_{n\to\infty} H(P_{\mathcal{R}}(u), P_{\mathcal{R}}(u_n)) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}). \end{aligned}$$

This yields $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. Lemma 1.3 gives $\lim_{n\to\infty} v_m = u$. Note that $\mathcal{R}u \in P(\mathcal{K})$ being proximal is closed, hence $P_{\mathcal{R}}(u)$ is closed. Moreover, $P_{\mathcal{R}}(u)$ is bounded. Consequently $\lim_{n\to\infty} v_m = u \in P_{\mathcal{R}}(u)$. Hence $u \in F(P_{\mathcal{R}})$. Similarly, $u \in F(P_{\mathcal{S}})$,

$u \in F(P_{\mathcal{T}})$ and $u \in F(P_{\mathcal{U}})$. Hence $u \in \mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1, therefore by the uniqueness of asymptotic center, we have

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) = \limsup_{n\to\infty} d(u_n, u),\end{aligned}$$

a contradiction. Hence $x = u$. Thus $A(\{u_n\}) = \{u\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. This shows that $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ (or $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$). This completes the proof. □

The next result is a necessary and sufficient condition for the strong convergence of the iteration scheme (8).

Theorem 2.2. *Let $\mathcal{X}$ be a complete hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}$, $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$, $P_{\mathcal{U}}$ and $\{x_n\}$ be as in Lemma 2.1. Then the sequence $\{x_n\}$ converges strongly to $p \in \mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. If $\{x_n\}$ converges to $p \in \mathcal{F}$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, \mathcal{F}) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Now to prove that the condition is also sufficient, assume that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Lemma 2.1 $d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F})$, and so $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists by Lemma 1.4. By hypothesis $\liminf_{n\to\infty} d(x_n, \mathcal{F})$
$= 0$, thus $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ by Lemma 1.4.

We now show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$. Let $m, n \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of all positive integers, and assume that $m > n$. Then it follows from equation (11) of Lemma 2.1 that $d(x_m, p) \leq d(x_n, p)$ for all $p \in \mathcal{F}$. Thus, we have $d(x_m, x_n) \leq d(x_m, p) + d(x_n, p) \leq 2d(x_n, p)$. Taking inf on the set $\mathcal{F}$, we have $d(x_m, x_n) \leq d(x_n, \mathcal{F})$. On letting $m \to \infty$, $n \to \infty$, the inequality $d(x_m, x_n) \leq d(x_n, \mathcal{F})$ shows that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$ and hence converges, say to $p_* \in \mathcal{K}$. Now it remains to show that $p_* \in \mathcal{F}$. Indeed, by $d(x_n, F(P_{\mathcal{R}})) = \inf_{z\in F(P_{\mathcal{R}})} d(x_n, z)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(P_{\mathcal{R}})$ such that $d(x_n, p_n^{(\varepsilon)}) < d(x_n, F(P_{\mathcal{R}})) + \frac{\varepsilon}{3}$. This implies $\lim_{n\to\infty} d(x_n, p_n^{(\varepsilon)}) \leq \frac{\varepsilon}{3}$. From $d(p_n^{(\varepsilon)}, p_*) \leq d(x_n, p_n^{(\varepsilon)}) + d(x_n, p_*)$, it follows that $\limsup_{n\to\infty} d(p_n^{(\varepsilon)}, p_*) \leq \frac{\varepsilon}{3}$. Finally, we have

$$\begin{aligned}d(P_{\mathcal{R}}(p_*), p_*) &\leq d(p_*, p_n^{(\varepsilon)}) + d(p_n^{(\varepsilon)}, P_{\mathcal{R}}(p_*)) \\ &\leq d(p_*, p_n^{(\varepsilon)}) + H(P_{\mathcal{R}}(p_n^{(\varepsilon)}), P_{\mathcal{R}}(p_*)) \leq 2\, d(p_*, p_n^{(\varepsilon)})\end{aligned}$$

yields that $d(P_{\mathcal{R}}(p_*), p_*) < \varepsilon$. Since ε is arbitrary, therefore $d(P_{\mathcal{R}}(p_*), p_*) = 0$. Similarly, we can show that $d(P_{\mathcal{S}}(p_*), p_*) = 0$, $d(P_{\mathcal{T}}(p_*), p_*) = 0$ and $d(P_{\mathcal{U}}(p_*), p_*) = 0$. Since $\mathcal{F}$ is closed, $p_* \in \mathcal{F}$. This shows that the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

For our next results, we need the following definitions.

Definition 2.1. *A mapping $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ is semi $-$ compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, \mathcal{T}x_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

We would also like to give here the definition of the so-called condition (A^*).

Definition 2.2. *Let φ be a nondecreasing self-map on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(r) > 0$ for all $r \in (0, \infty)$ and let $d(x, \mathcal{F}) = \inf\{d(x, y) : y \in \mathcal{F}\}$. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}\colon \mathcal{K} \to P(\mathcal{K})$ be four multivalued maps with $\mathcal{F} \neq \emptyset$. Then the four maps are said to satisfy condition (A^*) if*

$$d(x, \mathcal{R}x) \geq \varphi(d(x, \mathcal{F})) \text{ or } d(x, \mathcal{S}x) \geq \varphi(d(x, \mathcal{F})) \text{ or } d(x, \mathcal{T}x) \geq \varphi(d(x, \mathcal{F}))$$

$$\text{or } d(x, \mathcal{U}x) \geq \varphi(d(x, \mathcal{F})) \text{ for all } x \in \mathcal{K}.$$

Applying Lemma 2.2 and Theorem 2.2, we can easily obtain the following results.

Theorem 2.3. *Let $\mathcal{X}$ be a complete hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ with monotone modulus of uniform convexity η and $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}$, $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$, $P_{\mathcal{U}}$, $\{x_n\}$ and condition (12)be as in Lemma 2.2. Suppose that the mappings $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$ satisfies the condition (A^*), then the sequence $\{x_n\}$ defined in (8) converges strongly to $p \in \mathcal{F}$.*

Theorem 2.4. *Let $\mathcal{X}$ be a complete hyperbolic space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ with monotone modulus of uniform convexity η and $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}$, $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$, $P_{\mathcal{U}}$, $\{x_n\}$ and condition (12)be as in Lemma 2.2. Suppose that one of the mappings in $P_{\mathcal{R}}$, $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{U}}$ is semi-compact, then the sequence $\{x_n\}$ defined in (8) converges strongly to $p \in \mathcal{F}$.*

Example 2.1. *Let $\mathcal{K} = [0,1]$ be equipped with the Euclidean metric. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U} \colon \mathcal{K} \to CB(\mathcal{K})$ (family of closed and bounded subset of $\mathcal{K}$) be defined by $\mathcal{R}(x) = [0, \frac{x}{2}]$, $\mathcal{S}(x) = [0, \frac{x}{4}]$, $\mathcal{T}(x) = [0, \frac{x}{5}]$ and $\mathcal{U}(x) = [0, \frac{x}{6}]$. It is easy to see that for any $x, y \in \mathcal{K}$*

$$H(\mathcal{R}(x), \mathcal{R}(y)) = \max\left\{\left|\frac{x}{2} - \frac{y}{2}\right|, 0\right\} = \left|\frac{x}{2} - \frac{y}{2}\right| = \left|\frac{x-y}{2}\right| \leq |x-y|.$$

In a similarly way, we obtain

$$H(\mathcal{S}(x), \mathcal{S}(y)) = \max\left\{\left|\frac{x}{4} - \frac{y}{4}\right|, 0\right\} = \left|\frac{x}{4} - \frac{y}{4}\right| = \left|\frac{x-y}{4}\right| \leq |x-y|,$$

$$H(\mathcal{T}(x), \mathcal{T}(y)) = \max\left\{\left|\frac{x}{5} - \frac{y}{5}\right|, 0\right\} = \left|\frac{x}{5} - \frac{y}{5}\right| = \left|\frac{x-y}{5}\right| \leq |x-y|,$$

and

$$H(\mathcal{U}(x), \mathcal{U}(y)) = \max\left\{\left|\frac{x}{6} - \frac{y}{6}\right|, 0\right\} = \left|\frac{x}{6} - \frac{y}{6}\right| = \left|\frac{x-y}{6}\right| \leq |x-y|,$$

showing that $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ are multivalued nonexpansive mappings. Clearly, $F(\mathcal{R}) \cap F(\mathcal{S}) \cap F(\mathcal{T}) \cap F(\mathcal{U}) = \{0\}$. Hence, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ have a unique common fixed point.

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