$(\mathscr{F}_1, \mathscr{F}_2)$-CHAOS AND SENSITIVITY FOR TIME-VARYING DISCRETE SYSTEMS

Xinxing Wu[1], Yang Luo[1], Lidong Wang[2], Jianhua Liang[3]

*We prove that the $(\mathscr{F}_1, \mathscr{F}_2)$-chaoticity and sensitivity of two uniformly topological equiconjugate time-varying discrete systems are equivalent, improving the main result in [Annales Polonici Mathematici, **107** (2013), 49–57]. Moreover, some examples are given to show that Li-Yorke chaos, distributional chaos, and distributional chaos in a sequence on general metric spaces are not preserved under topological conjugation.*

Keywords: Time-varying discrete system, $(\mathscr{F}_1, \mathscr{F}_2)$-chaos, sensitivity, topological conjugation.
MSC2010: 54H20, 37B99, 54B20

1. Introduction

Li and Yorke firstly gave the concept of 'chaos' in their famous paper [7] in 1975. Meanwhile, it was the first description the conception 'chaos' with strict mathematical language. A *dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and $f : X \to X$ is a continuous map. A subset $D \subset X$ is called a *Li-Yorke scrambled set* of f if any different points $x, y \in D$ satisfy

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \quad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

(X, f) is *chaotic in the sense of Li-Yorke* (or *Li-Yorke chaotic*) if there exists an uncountable Li-Yorke scrambled set. Since then, the research of chaos has greatly influenced dynamical systems. Various definitions of chaos had been given according to property of iterative mapping, such as Devaney chaos [4], distributional chaos [10], Li-Yorke sensitivity [2], distributional chaos in a sequence [13], etc.

Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. A time-varying discrete system (TVDS) can be written in the following form

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}^+, \tag{1}$$

[1] X. Wu, School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, P. R. China, e-mail: wuxinxing5201314@163.com

[1] Y. Luo, School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, P. R. China, e-mail: 1321488002@qq.com

[2] L. Wang (Corresponding author), Zhuhai College of Jilin University, Zhuhai 519041, P. R. China, e-mail: wld0707@126.com

[3] J. Liang, Mathematics School and Institute, Jilin University, Changchun, Jilin 130012, P. R. China, e-mail: liangjianhua995@163.com

where $f_n : D_n \longrightarrow D_{n+1}$ is a map and D_n is a subset of a metric space (X, d). Consider another TVDS

$$y_{n+1} = g_n(y_n), \quad n \in \mathbb{Z}^+, \tag{2}$$

where $g_n : E_n \longrightarrow E_{n+1}$ is a map and E_n is a subset of a metric space (Y, ϱ).

Definition 1.1. *System (1) is said to be topologically $\{h_n\}_{n=0}^{\infty}$ conjugate to system (2) if for each $n \geq 0$, there exists a homeomorphism $h_n : D_n \to E_n$ such that $h_{n+1} \circ f_n = g_n \circ h_n, n \geq 0$. The sequence $\{h_n\}_{n=0}^{\infty}$ is said to be uniformly equicontinuous in $\{D\}_{n=0}^{\infty}$ if for any $\varepsilon > 0$, there exists a positive constant δ such that $\rho(h_n(x), h_n(y)) < \varepsilon$ for all $n \geq 0$ and $x, y \in D_n$ with $d(x, y) < \delta$. System (1) is said to be uniformly topologically $\{h_n\}_{n=0}^{\infty}$ conjugate (resp. equiconjugate) to system (2) if $\{h_n\}_{n=0}^{\infty}$ and $\{h_n^{-1}\}_{n=0}^{\infty}$ are uniformly continuous (resp. equicontinuous) in $\{D_n\}_{n=0}^{\infty}$ and $\{E_n\}_{n=0}^{\infty}$.*

$$\begin{array}{ccccccccccc}
D_0 & \xrightarrow{f_0} & D_1 & \xrightarrow{f_1} & D_2 & \xrightarrow{f_2} & D_3 \ldots\ldots D_{n-1} & \xrightarrow{f_{n-1}} & D_n & \xrightarrow{f_n} & D_{n+1} \ldots\ldots \\
\downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & \downarrow & & \downarrow h_n & & \downarrow h_{n+1} \\
E_0 & \xrightarrow{g_0} & E_1 & \xrightarrow{g_1} & E_2 & \xrightarrow{g_2} & E_3 \ldots\ldots E_{n-1} & \xrightarrow{g_{n-1}} & E_n & \xrightarrow{g_n} & E_{n+1} \ldots\ldots
\end{array}$$

For any $n \geq 2$, denote $f_0^{(n)} = f_{n-1} \circ \cdots f_1 \circ f_0$ and $f_0^{(1)} = f_0$, $f_0^{(0)} = \mathrm{id}_{D_0}$. Clearly, $x_n = f_0^{(n)}(x_0)$. These TVDS have been considered by several mathematicians ([3, 11, 14, 19, 29]). For example, Wu and Zhu [29] proved that some chaotic properties are preserved under iterations for TVDS which is uniformly converges. Wang et al. [14] studied distributional chaos for TVDS and proved that two uniormly topological equiconjugate time-varying discrete systems have simultaneously the distributional chaos in a sequence and the weakly mixing property. Recently, Shao et al. [11] obtained more results on distributional chaos for TVDS.

Recently, the notion of $(\mathscr{F}_1, \mathscr{F}_2)$-chaos via a Furstenberg family couple $\mathscr{F}_1$ and $\mathscr{F}_2$ was introduced by Tan and Xiong [12]. In this paper, we prove that the $(\mathscr{F}_1, \mathscr{F}_2)$-chaoticity and sensitivity of two uniformly topological equi-conjugate time-varying systems are equivalent, improving the main result in [14]. Finally, we give some examples to show that Li-Yorke chaos, distributional chaos, and distributional chaos in a sequence are not preserved under topological conjugation.

2. **Basic definitions**

We recall some basic concepts related to the Furstenberg families (see [1] for more details). Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}^+$. A collection $\mathscr{F} \subset \mathcal{P}$ is called a *Furstenberg family* if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$. A family $\mathscr{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e., neither empty nor the whole $\mathcal{P}$. It is easy to see that $\mathscr{F}$ is proper if and only if $\mathbb{Z}^+ \in \mathscr{F}$ and $\emptyset \notin \mathscr{F}$. Let $\mathscr{F}_{inf}$ be the collection of all infinite subsets of $\mathbb{Z}^+$. All the families considered below are assumed to be proper.

For $A \subset \mathbb{Z}^+$, define $\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap [0, n-1]|$, and $\underline{d}(A) = \liminf_{n\to\infty} \frac{1}{n} |A \cap [0, n-1]|$. Then, $\overline{d}(A)$ and $\underline{d}(A)$ are *the upper density* and *the lower density* of A, respectively. Fix any $\alpha \in [0, 1]$ and denote by $\widehat{\mathscr{M}^{\alpha}}$ (resp. $\widehat{\mathscr{M}_{\alpha}}$) the family consisting of sets $A \subset \mathbb{Z}^+$ with $\overline{d}(A) \geq \alpha$ (resp. $\underline{d}(A) \geq \alpha$).

Definition 2.1. [12] *Let* (X, T) $\mathscr{F}_1$, $\mathscr{F}_2$ *be Furstenberg families.* $D \subset D_0$.

(1) D *is a* $(\mathscr{F}_1, \mathscr{F}_2)$*-scrambled subset of system (1) if for any two different points* $u, v \in D$, *there exists* $\delta > 0$ *such that*

(a) $\{n \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) < \varepsilon\} \in \mathscr{F}_1$ *for all* $\varepsilon > 0$;

(b) $\{n \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) > \delta\} \in \mathscr{F}_2$.

(2) System (1) is $(\mathscr{F}_1, \mathscr{F}_2)$*-chaotic if it admits an uncountable* $(\mathscr{F}_1, \mathscr{F}_2)$*-scrambled subset.*

It follows directly from Definition 2.1 that system (1) is Li-Yorke chaotic (resp., distributionally chaotic) if and only if it is $(\mathscr{F}_{inf}, \mathscr{F}_{inf})$-chaotic (resp., $(\widehat{\mathscr{M}^1}, \widehat{\mathscr{M}^1})$-chaotic).

For $U \subset X$ and $\varepsilon > 0$, let $N(U, \varepsilon) = \{n \in \mathbb{Z}^+ : \operatorname{diam}(f_0^{(n)}(U)) > \varepsilon\}$. It is easy to see that system (1) is sensitive if and only if there exists $\varepsilon > 0$ such that, for any nonempty open subset $U \subset X$, $N(U, \varepsilon) \neq \emptyset$. For a dynamical system, Moothathu [8] initiated a preliminary study of stronger forms of sensitivity formulated in terms of some subsets of $\mathbb{Z}^+$, namely the syndetical sensitivity and cofinite sensitivity. Recently, Li [6] introduced the concept of ergodic sensitivity. Let $\mathscr{F}$ be a Furstenberg family. System (1) is said to be $\mathscr{F}$*-sensitive* if there exists $\varepsilon > 0$ such that for any nonempty open subset $U \subset X$, $N(U, \varepsilon) \in \mathscr{F}$. More results on sensitivity can be found in [5, 9, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30].

Akin and Kolyada [2] introduced the concept of Li-Yorke sensitivity which links the Li-Yorke chaos with the notion of sensitivity and proved that every weakly mixing dynamical system is Li-Yorke sensitive. According to Akin and Kolyada [2], system (1) is *Li-Yorke sensitive* if there exists some $\delta > 0$ such that any neighbourhood of any $x \in X$ contains a point y satisfying $\liminf_{n\to\infty} d(f_0^{(n)}(x), f_0^{(n)}(y)) = 0$ and $\limsup_{n\to\infty} d(f_0^{(n)}(x), f_0^{(n)}(y)) > \delta$.

3. Chaoticity of uniformly topological conjugate systems

This section proves that both $(\mathscr{F}_1, \mathscr{F}_2)$-chaoticity and sensitivity are preserved under uniform topological equiconjugation for TVDS.

Theorem 3.1. *Let* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ *be two proper Furstenberg families. If systems (1) and (2) are uniformly topologically equiconjugate, then system (1) is* $(\mathscr{F}_1, \mathscr{F}_2)$*-chaotic if and only if system (2) is* $(\mathscr{F}_1, \mathscr{F}_2)$*-chaotic.*

Proof. It suffices to check the necessity, because the sufficiency can be verified similarly. Take an uncountable $(\mathscr{F}_1, \mathscr{F}_2)$-scrambled subset $D \subset D_0$ of system (1) and choose $E = h_0(D)$. We shall show that E is a $(\mathscr{F}_1, \mathscr{F}_2)$-scrambled subset $D \subset D_0$ of system (2).

Given any two distinct points $x, y \in E$, there exist $u, v \in D$ such that $h_0(u) = x$ and $h_0(v) = y$. The $(\mathscr{F}_1, \mathscr{F}_2)$-chaoticity of system (1) implies that there exists $\delta > 0$ such that

(a) $\{i \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) < \varepsilon\} \in \mathscr{F}_1$ for all $\varepsilon > 0$;

(b) $\{i \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) > \delta\} \in \mathscr{F}_2$.

For any $\varepsilon > 0$, noting that $\{h_n\}_{n=0}^{\infty}$ is uniformly equicontinuous, it follows that there exists $0 < \varepsilon_1 < \varepsilon$ such that $\varrho(h_n(x_1), h_n(x_2)) < \varepsilon$ holds for all $n \in \mathbb{Z}^+$ and $x_1, x_2 \in D_n$ with

$d(x_1, x_2) < \varepsilon_1$. This implies that

$$\begin{aligned} & \{i \in \mathbb{Z}^+ : \varrho(g_0^{(i)}(x), g_0^{(i)}(y)) < \varepsilon\} \\ = & \{i \in \mathbb{Z}^+ : \varrho(h_i(f_0^{(i)}(u)), h_i(f_0^{(i)}(v))) < \varepsilon\} \\ \supset & \{i \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) < \varepsilon_1\} \in \mathscr{F}_1. \end{aligned}$$

So, $\{i \in \mathbb{Z}^+ : \varrho(g_0^{(i)}(x), g_0^{(i)}(y)) < \varepsilon\} \in \mathscr{F}_1$, as $\mathscr{F}_1$ is hereditary upwards.

Similarly, since $\{h_n^{-1}\}_{n=0}^{\infty}$ is uniformly equicontinuous, then there there exists $0 < \varepsilon_2 < \delta$ such that $d(h_n^{-1}(x_1), h_n^{-1}(x_2)) \leq \delta$ holds for all $n \in \mathbb{Z}^+$ and $x_1, x_2 \in E_n$ with $\varrho(x_1, x_2) < \varepsilon_2$. This, together with (b), implies that

$$\begin{aligned} & \{i \in \mathbb{Z}^+ : d(f_0^{(i)}(u), f_0^{(i)}(v)) > \delta\} \\ = & \{i \in \mathbb{Z}^+ : d(h_i^{-1}(g_0^{(i)}(x)), h_i^{-1}(g_0^{(i)}(y))) > \delta\} \\ \subset & \{i \in \mathbb{Z}^+ : \varrho(g_0^{(i)}(x), g_0^{(i)}(y)) > \varepsilon_2\} \in \mathscr{F}_2. \end{aligned}$$

□

Corollary 3.1. *If systems (1) and (2) are uniformly topologically equiconjugate, then system (1) is Li-Yorke chaotic (resp., distributionally chaotic, distributionally chaotic in a sequence) if and only if system (2) is (resp., distributionally chaotic, distributionally chaotic in a sequence) .*

According to the proof of Theorem 3.1, we can obtained the following.

Theorem 3.2. *Let $\mathscr{F}$ be a Furstenberg families. If systems (1) and (2) are uniformly topologically equiconjugate, then system (1) is $\mathscr{F}$-sensitive (resp., sensitive, multi-sensitive, Li-Yorke sensitive) if and only if system (2) is $\mathscr{F}$-sensitive (resp., sensitive, multi-sensitive, Li-Yorke sensitive).*

4. **Examples**

In [14, Theorem 2.12], Wang et al. proved that distributional chaos in a sequence is preserved under uniform topological equiconjugation. Here, we use some examples to show that Li-Yorke chaos, distributional chaos, and distributional chaos in a sequence are not preserved under topological conjugation. Firstly, we define α-map, β-map, and γ-map as follows:

α-map. Let $X = [0, +\infty)$. Define $\varrho_1 : X \times X \longrightarrow [0, 1]$ by

$$\varrho_1(x, y) = \begin{cases} 0, & x = y, \\ \frac{1}{2^{[x]}}, & [x] = [y] \equiv 1 \pmod 2 \text{ and } x \neq y, \\ 1, & \text{otherwise.} \end{cases}$$

Define $\alpha : (X, \varrho_1) \longrightarrow (X, \varrho_1)$ by

$$\alpha(x) = \begin{cases} 0, & x = 0, \\ \frac{n(n-1)}{2}, & x = \frac{n(n-1)}{2} + (n-1) \text{ for some } n \geq 2, \\ x + 1, & \text{otherwise.} \end{cases}$$

β-map. Define $\varrho_2 : X \times X \longrightarrow [0,1]$ by

$$\varrho_2(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

and choose $\beta : (X, \varrho_2) \to (X, \varrho_2)$ such that $\beta(x) = \alpha(x)$ for any $x \in X$.

γ-map. Define $\varrho_3 : X \times X \longrightarrow [0,1]$ by

$$\varrho_3(x,y) = \begin{cases} 0, & x = y, \\ \frac{1}{2^{2k}}, & x \neq y \in [0, b_1) \text{ or } x \neq y \in \left[\sum_{j=1}^{2k} b_j, \sum_{j=1}^{2k+1} b_j\right), k \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$

where $b_1 = 2$, $b_i = 2^{b_1 + \cdots + b_{i-1}}$ $(i \geq 2)$ and take $\gamma : (X, \varrho_3) \longrightarrow (X, \varrho_3)$ as $\gamma(x) = \alpha(x)$ for any $x \in X$.

Clearly, each ϱ_i is a discrete metric on X. So, α-map, β-map, and γ-map are all continuous. From the definitions of α-map, β-map, and γ-map, it can be verified that the following result holds.

Proposition 4.1. *Any pair of α-map, β-map, and γ-map are topologically conjugate.*

Proposition 4.2. *α-map is distributioally chaotic in a sequence.*

Proof. Take $D = (0,1) \subset X$ and let $b_1 = 2$, $b_i = 2^{b_1 + \cdots + b_{i-1}}$ $(i \geq 2)$,

$$p_k = \begin{cases} 2^k + 1, & k \leq b_1 \text{ or } \sum_{j=1}^{2k} b_j < k \leq \sum_{j=1}^{2k+1} b_j \ (k \in \mathbb{N}), \\ 2^k, & \sum_{j=1}^{2k-1} b_j < k \leq \sum_{j=1}^{2k} b_j \ (k \in \mathbb{N}). \end{cases}$$

Clearly, $\{p_k\}_{k \in \mathbb{N}}$ is an increasing sequence of positive integers.

Given any two distinct points $x, y \in D$, we claim that (x,y) is a distributionally chaotic pair of α-map along $\{p_k\}_{k \in \mathbb{N}}$.

For any $t > 0$, choose a sufficient large integer $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < t$. It can be verified that for any $k \geq k_0$ and any $\sum_{j=1}^{2k} b_j < i \leq \sum_{j=1}^{2k+1} b_j$, $\varrho_1(\alpha^{p_i}(x), \alpha^{p_i}(y)) = \frac{1}{2^{(2^i+1)\cdot(2\cdot n_0 - 1)}} < \frac{1}{2^{k_0}} < t$, implying that

$$\begin{aligned} & F^*_{x,y}(t, \{p_k\}_{k \in \mathbb{N}}, \alpha) \\ = \ & \limsup_{n \to \infty} \frac{1}{n} |\{1 \leq k \leq n : \varrho_1(\alpha^{p_k}(x), \alpha^{p_k}(y)) < t\}| \\ \geq \ & \limsup_{i \to \infty} \frac{1}{\sum_{h=1}^{2i+1} b_h} \left| \left\{ 1 \leq k \leq \sum_{h=1}^{2i+1} b_h : \varrho_1(\alpha^{p_k}(x), \alpha^{p_k}(y)) < t \right\} \right| \\ \geq \ & \lim_{i \to \infty} \frac{b_{2i+1}}{l_i} = \lim_{i \to \infty} \frac{2^{b_1 + \cdots + b_{2i}}}{b_1 + \cdots + b_{2i} + 2^{b_1 + \cdots + b_{2i}}} = 1. \end{aligned}$$

Observe that for any $k \in \mathbb{N}$ and any $\sum_{j=1}^{2k-1} b_j \le i < \sum_{j=1}^{2k} b_j$, $\varrho_1(\alpha^{p_i}(x), \alpha^{p_i}(y)) = 1$. Then,

$$\begin{aligned} & F_{x,y}(1/2, \{p_k\}_{k\in\mathbb{N}}, \alpha) \\ = \; & \liminf_{n\to\infty} \frac{1}{n} |\, \{1 \le k \le n : \varrho_1(\alpha^{p_k}(x), \alpha^{p_k}(y)) < 1/2\} \,| \\ \le \; & \liminf_{i\to\infty} \frac{1}{\sum_{h=1}^{2i} b_h} \left| \left\{ 1 \le k \le \sum_{h=1}^{2i} b_h : \varrho_1(\alpha^{p_k}(x), \alpha^{p_k}(y)) < 1/2 \right\} \right| \\ \le \; & \lim_{i\to\infty} \frac{b_1 + \cdots + b_{2i-1}}{j_i} = \lim_{i\to\infty} \frac{b_1 + \cdots + b_{2i-1}}{b_1 + \cdots + b_{2i-1} + 2^{b_1 + \cdots + b_{2i-1}}} = 0. \end{aligned}$$

Therefore, α-map is distributionally chaotic along $\{p_k\}_{k\in\mathbb{N}}$. □

Proposition 4.3. *β-map is not Li-Yorke chaotic.*

Proof. The results follows directly from the definition of ϱ_2. □

Proposition 4.4. *γ-map is distributionally chaotic.*

Proof. Let $D = (0, 1)$ and take any two distinct points $x, y \in D$. For any $t > 0$, choose a sufficient large integer $h_0 \in \mathbb{N}$ such that $\frac{1}{2^{2h_0}} < t$. It is not difficult to check that for any $h \ge h_0$ and any $\sum_{j=1}^{2h} b_j \le m < (\sum_{j=1}^{2h+1} b_j) - 1$, $\varrho_3(\gamma^m(x), \gamma^m(y)) = \frac{1}{2^{2h}} \le \frac{1}{2^{2h_0}} < t$. This implies that

$$\begin{aligned} & F^*_{x,y}(t, \{k\}_{k\in\mathbb{N}}, \gamma) \\ = \; & \limsup_{n\to\infty} \frac{1}{n} |\{1 \le k \le n : \varrho_3(\gamma^k(x), \gamma^k(y)) < t\}| \\ \ge \; & \limsup_{j\to\infty} \frac{1}{\sum_{h=1}^{2j+1} b_h} \left| \left\{ 1 \le k \le \sum_{h=1}^{2j+1} b_h : \varrho_3(\gamma^k(x), \gamma^k(y)) < t \right\} \right| \\ \ge \; & \lim_{j\to\infty} \frac{b_{2j+1} - 1}{l_j} = \lim_{j\to\infty} \frac{2^{b_1 + \cdots + b_{2j}} - 1}{b_1 + \cdots + b_{2j} + 2^{b_1 + \cdots + b_{2j}}} = 1. \end{aligned}$$

Since for any $k \in \mathbb{N}$ and any $\sum_{j=1}^{2k-1} b_j \le i < \sum_{j=1}^{2k} b_j$, $\varrho_3(\gamma^i(x), \gamma^i(y)) = 1$, it follows that

$$\begin{aligned} & F_{x,y}(1/2, \{k\}_{k\in\mathbb{N}}, \gamma) \\ = \; & \liminf_{n\to\infty} \frac{1}{n} |\{1 \le k \le n : \varrho_3(\gamma^k(x), \gamma^k(y)) < 1/2\}| \\ \le \; & \liminf_{i\to\infty} \frac{1}{\sum_{h=1}^{2i} b_h} \left| \left\{ 1 \le k \le \sum_{h=1}^{2i} b_h : \varrho_3(\gamma^k(x), \gamma^k(y)) < 1/2 \right\} \right| \\ \le \; & \lim_{i\to\infty} \frac{b_1 + \cdots + b_{2i-1} + 1}{j_i} = \lim_{i\to\infty} \frac{b_1 + \cdots + b_{2i-1} + 1}{b_1 + \cdots + b_{2i-1} + 2^{b_1 + \cdots + b_{2i-1}}} = 0. \end{aligned}$$

Hence, γ-map is distributionally chaotic. □

Summing up Proposition 4.1–Proposition 4.4, it easy to see that Li-Yorke chaos, distributional chaos, and distributional chaos in a sequence are not preserved under topological conjugation.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11601449, 11271061), the National Nature Science Foundation of China (Key Program) (No. 51534006), the Hong Kong Scholars Program, Science and Technology Innovation Team of Education Department of Sichuan for Dynamical System and its Applications (No. 18TD0013), Youth Science and Technology Innovation Team of Southwest Petroleum University for Nonlinear Systems (No. 2017CXTD02), and Independent Research Foundation of the Central Universities (No. DC201502050201).

REFERENCES

[1] *E. Akin*, Recurrence in Topological Dynamics: Furstenberg and Ellis Actions, Plenum Press, New York; 1997.

[2] *E. Akin, S. Kolyada*, Li-Yorke sensitivity, Nonlinearity, **16** (2003), No. 4, 1421-1433.

[3] *J.S. Cánovas*, Li-Yorke chaos in a class of nonautonomous discrete systems, J. Difference Equ. Appl., **17** (2011), No. 4, 479-486.

[4] *R.L. Devaney*, An Introduction to Chaotic Dynamical Systems, Redwood City: Addison-Wesley, 1989.

[5] *J. Li, P. Oprocha, X. Wu*, Furstenberg families, sensitivity and the space of probability measures, Nonlinearity, **30** (2017), No. 3, 987-1005.

[6] *R. Li*, The large deviations theorem and ergodic sensitivity, Commun. Nonlinear Sci. Numer. Simulat., **18** (2013), No. 4, 819-825.

[7] *T.Y. Li, J.A. Yorke*, Period three implies chaos, Amer. Math. Monthly, **82** (1975), No. 10, 985-992.

[8] *T.K.S. Moothathu*, Stronger forms of sensitivity for dynamical systems, Nonlinearity, **20** (2007), No. 9, 2115-2126.

[9] *P. Oprocha, X. Wu*, On average tracing of periodic average pseudo orbits, Discrete Contin. Dyn. Syst., **37** (2017), No. 9, 4943-4957.

[10] *B. Schweitzer, J. Smital*, Measures of chaos and spectral decomposition of dynamical systems of the interval, Trans. Amer. Math. Soc., **344** (1994), No. 2, 737-754.

[11] *H. Shao, Y. Shi, H. Zhu*, On distributional chaos in non-autonomous discrete systems, Chaos Solitons Fractals, **107** (2018), 234-243.

[12] *F. Tan, J. Xiong*, Chaos via Furstenberg family couple, Topology Appl., **156** (2009), No. 3, 525-532.

[13] *L. Wang, G. Huang, S. Huan*, Distributional chaos in a sequence, Nonlinear Anal., **67** (2007), No. 7, 2131-2136.

[14] *L. Wang, Y. Li, Y. Gao, H. Liu*, Distributional chaos of time-varying discrete dynamical systems, Ann. Polon. Math., **107** (2013), No. 1, 49-57.

[15] *X. Wu*, Chaos of transformations induced on the space of probability measures, Int. J. Bifurcation and Chaos, **26** (2016), No. 13, 1650227 (12 pages).

[16] *X. Wu*, A remark on topological sequence entropy, Int. J. Bifurcation and Chaos, **27** (2017), No. 7, 1750107 (7 pages).

[17] *X. Wu, X. Ding, T. Lu, J. Wang*, Topological dynamics of Zadeh's extension on upper semi-continuous fuzzy sets, Int. J. Bifurcation and Chaos, **27** (2017), No. 10, 1750165 (13 pages).

[18] *X. Wu, Y. Luo*, Invariance of distributional chaos for backward shifts, Oper. Matrices (in press).

[19] *X. Wu, Y. Luo, X. Ma, T. Lu*, Rigidity and sensitivity on uniform spaces, Topology Appl., **252** (2019), 145-157.

[20] *X. Wu, X. Ma, Z. Zhu, T. Lu*, Topological ergodic shadowing and chaos on uniform spaces, Int. J. Bifurcation and Chaos, **28** (2018), No. 3, 1850043 (9 pages).

[21] *X. Wu, P. Oprocha, G. Chen*, On various definitions of shadowing with average error in tracing, Nonlinearity, **29** (2016), No. 7, 1942-1972.

[22] *X. Wu, J. Wang, G. Chen*, $\mathscr{F}$-sensitivity and multi-sensitivity of hyperspatial dynamical systems, J. Math. Anal. Appl., **429** (2015), No. 1, 16-26.

[23] *X. Wu, L. Wang, G. Chen*, Weighted backward shift operators with invariant distributionally scrambled subsets, Ann. Funct. Anal., **8** (2017), No. 2, 199-210.

[24] *X. Wu, L. Wang, J. Liang*, The chain properties and average shadowing property of iterated function systems, Qual. Theory Dyn. Syst., **17** (2018), No. 1, 219-227.

[25] *X. Wu, L. Wang, J. Liang*, The chain properties and Li-Yorke sensitivity of Zadeh's extension on the space of upper semi-continuous fuzzy sets, Iran. J. Fuzzy Syst., **15** (2018), No. 6, 83-95.

[26] *X. Wu, X. Wang*, On the iteration properties of large deviations theorem, Int. J. Bifurcation and Chaos, **26** (2016), No. 3, 1650054 (6 pages).

[27] *X. Wu, X. Wang, G. Chen*, On the large deviations theorem of weaker types, Int. J. Bifurcation and Chaos, **27** (2017), No. 8, 1750127 (12 pages).

[28] *X. Wu, X. Zhang, X. Ma*, Various shadowing in linear dynamical systems, Int. J. Bifurcation and Chaos, **29** (2019), No. 3, 1950042 (10 pages).

[29] *X. Wu, P. Zhu*, Chaos in a class of non-autonomous discrete systems, Appl. Math. Lett., **26** (2013), No. 4, 431-436.

[30] *X. Ye, R. Zhang*, On sensitive sets in topological dynamics, Nonlinearity, **21** (2008), No. 7, 1601-1620.