# MULTIPLICITY RESULTS FOR A CLASS OF NAVIER DOUBLY EIGENVALUE BOUNDARY VALUE SYSTEMS DRIVEN BY A $(p_1, \ldots, p_n)$-BIHARMONIC OPERATOR

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*Existence results of three weak solutions for a Navier doubly eigenvalue boundary value system involving the $(p_1, \ldots, p_n)$-biharmonic operator, under suitable assumptions, are established. The approach is fully based on Ricceri's Variational Principle.*

**Keywords:** $(p_1, \ldots, p_n)$-biharmonic, Navier condition, Multiple solutions, Ricceri's Variational Principle.

**MSC2010:** 35D30, 35B38, 35J50.

## 1. Introduction and Preliminaries

In this paper we are interested in ensuring the existence of at least three weak solutions for the following Navier doubly eigenvalue boundary value system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(x, u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $1 \le i \le n$, where $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class $C^1$, $\lambda$ and $\mu$ are positive parameters and $p_i > \max\{1, N/2\}$ for $1 \le i \le n$. Here, $F, G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions with respect to $x \in \Omega$ for every $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and are $C^1$ with respect to $(t_1, \ldots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$, and $F_{u_i}$ and $G_{u_i}$ denotes the partial derivative of $F$ and $G$ with respect to $u_i$, respectively.
Moreover, $F$ and $G$ satisfy the following additional assumptions:

($F_1$) for every $M > 0$ and every $1 \le i \le n$,

$$\sup_{|(t_1,\ldots,t_n)|\le M} |F_{u_i}(x, t_1, \ldots, t_n)| \in L^1(\Omega).$$

($F_2$) $F(x, 0, \ldots, 0) = 0$ for a.e. $x \in \Omega$.

(G) for every $M > 0$ and every $1 \le i \le n$,

$$\sup_{|(t_1,\ldots,t_n)|\le M} |G_{u_i}(x, t_1, \ldots, t_n)| \in L^1(\Omega).$$

Here and in what follows, we let $X$ be the Cartesian product of the $n$ Sobolev spaces $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$ for $1 \le i \le n$, i.e.,

$$X := \left(W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)\right) \times \cdots \times \left(W^{2,p_n}(\Omega) \cap W_0^{1,p_n}(\Omega)\right)$$

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equipped with the norm

$$\|u\| := \sum_{i=1}^{n} \|u_i\|_{p_i}, \quad u = (u_1, u_2, \dots, u_n),$$

where for $1 \le i \le n$,

$$\|u_i\|_{p_i} := \left[ \int_\Omega |\Delta u_i(x)|^{p_i} dx \right]^{\frac{1}{p_i}}.$$

Let us recall that for any positive integer $k$ and any $1 \le i \le n$, $W_0^{1,p_i}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ if $p_i > N/k$, and that for $1 \le i \le n$, $W^{2,p_i}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ if $p_i > \max\{1, N/2\}$ (see [22, page 1026]). So, if $p_i > \max\{1, N/2\}$ for $1 \le i \le n$, the embedding $X \hookrightarrow (C^0(\overline{\Omega}))^n$ is compact.

Let

$$c := \max \left\{ \sup_{u_i \in W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} : \text{ for } 1 \le i \le n \right\}. \tag{2}$$

In the case $p_i > \max\{1, N/2\}$ for $1 \le i \le n$, since the embedding $X \hookrightarrow (C^0(\overline{\Omega}))^n$ is compact, one has $c < +\infty$.

As usual, a weak solution of system (1) is any $u = (u_1, u_2, \dots, u_n) \in X$ such that

$$\int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i - 2} \Delta u_i(x) \Delta v_i(x) dx - \lambda \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$
$$- \mu \int_\Omega \sum_{i=1}^{n} G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx = 0$$

for every $v = (v_1, v_2, \dots, v_n) \in X$ (see [17, 21]).

Moreover, let

$$D := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega).$$

Simple calculations show that there is $x^0 \in \Omega$ such that $B(x^0, D) \subseteq \Omega$, where $B(x, r)$ stands for the open ball in $\mathbb{R}^N$ of radius $r$ centered at $x$.

Put

$$\sigma_i := \frac{144(N+2)^2}{D^2} \left( \frac{c D^N \pi^{N/2} (2^N - 1)}{2^N \Gamma(1 + N/2)} \right)^{1/p_i}, \tag{3}$$

$$\kappa_i := \begin{cases} \frac{4N}{D^2} \left( \frac{c D^N \pi^{N/2} (3^N - 2^N)}{2^{2N} \Gamma(1+N/2)} \right)^{1/p_i}, & N < 4, \\ \\ \frac{16}{D^2} \left( \frac{c D^N \pi^{N/2} (3^N - 2^N)}{2^{2N} \Gamma(1+N/2)} \right)^{1/p_i}, & N \ge 4, \end{cases} \tag{4}$$

for $1 \le i \le n$, where $\Gamma$ denotes the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} dz$$

for all $t > 0$.

There seems to be increasing interest in studying fourth-order boundary value problems, because the static form change of beam or the sport of rigid body can be described by a fourth-order equation, and specially a model to study travelling waves in suspension bridges can be furnished by the fourth-order equation of nonlinearity, so it is important to Physics (see [14]). More general nonlinear fourth-order elliptic boundary value problems have been studied in recent years. Several results are known concerning the existence of multiple solutions for fourth-order boundary value problems, and we refer the reader to [2, 3, 4, 5, 6, 8, 11, 12, 15, 16] and references therein.

For example in [12], based on a recent three critical points theorem, the authors proved the existence of at least three weak solutions for the following $(p_1,\ldots,p_n)$-biharmonic system with Navier boundary condition

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(x,u_1,\ldots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

for $1 \le i \le n$, where $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class $C^1$, $\lambda$ is a positive parameter, $p_i > \max\{1, N/2\}$ for $1 \le i \le n$, $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for every $(t_1,\ldots,t_n) \in \mathbb{R}^n$ and is $C^1$ with respect to $(t_1,\ldots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$, satisfying the condition

$$\sup_{|(t_1,\ldots,t_n)|\le M} |F_{u_i}(x,t_1,\ldots,t_n)| \in L^1(\Omega)$$

for every $M > 0$ and every $1 \le i \le n$, and $F(x,0,\ldots,0) = 0$ for a.e. $x \in \Omega$.

In [15], Li and Tang considered the following $p$-biharmonic equation with Navier boundary condition

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda, \mu \in [0,+\infty[$, $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class $C^1$, $p > \max\{1, N/2\}$, $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Using the modified three critical points theorem of Ricceri [18], they established the existence of an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number $\rho$ such that, for each $\lambda \in \Lambda$, problem (6) admits at least three weak solutions whose norms in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ are less than $\rho$. Also in [16], the authors unified and generalized Li and Tang's problem and established the existence of at least three solutions to a Navier boundary problem involving the $(p,q)$-biharmonic systems.

The goal of this work is to establish some new criteria for system (1) to have at least three weak solutions in $X$, by means of a very recent abstract critical point result of Ricceri [19]. We first recall the following three critical points theorem that follows from a combination of [7, Theorem 3.6] and [19, Theorem 1]. We also refer the reader to the recent papers [1] and [10] where an analogous variational approach has been developed on studying elliptic problems.

**Lemma 1.1.** *Let $X$ be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, bounded on bounded subsets of $X$; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

*Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that*

($a_1$) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

($a_2$) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

*Then, for each compact interval $[a,b] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every $C^1$ functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation*

$$\Phi'(x) - \lambda\Psi'(x) - \mu J'(x) = 0$$

*has at least three solutions in $X$ whose norms are less than $\rho$.*

For other basic notations and definitions, we refer the reader to [9, 13, 22].

## 2. **Main results**

In the present section we discuss the existence of multiple solutions for system (1). For any $\gamma > 0$, we denote by $K(\gamma)$ the set

$$\left\{(t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le \gamma\right\}.$$

This set will be used in some of our hypotheses with appropriate choices of $\gamma$.

We formulate our main result as follows.

**Theorem 2.1.** *Assume that there exist two positive constants $\theta$ and $\delta$ with $\sum_{i=1}^n \frac{(\delta\kappa_i)^{p_i}}{p_i} > \frac{\theta}{\prod_{i=1}^n p_i}$ such that*

(b$_1$) *$F(x,t_1,\dots,t_n) \ge 0$ for a.e. $x \in \Omega\backslash B(x^0, D/2)$ and all $t_i \in [0,\delta]$ for $1 \le i \le n$;*

(b$_2$)

$$\frac{\theta}{\prod_{i=1}^n p_i}\int_{B(x^0,D/2)} F(x,\delta,\dots,\delta)dx \\ -m(\Omega)\sum_{i=1}^{n}\frac{(\delta\sigma_i)^{p_i}}{p_i} \sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n) > 0,$$

*where $m(\Omega)$ is the Lebesgue measure of the set $\Omega$;*

(b$_3$)

$$\limsup_{(|t_1|,\dots,|t_n|)\to(+\infty,\dots,+\infty)} \frac{F(x,t_1,\dots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} < \frac{\left(\prod_{i=1}^n p_i\right) \sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)}{\theta}$$

*uniformly with respect to $x \in \Omega$.*

*Then, setting*

$$\Lambda := \left]\frac{\sum_{i=1}^n \frac{(\delta\sigma_i)^{p_i}}{p_i}}{c\int_{B(x^0,D/2)} F(x,\delta,\dots,\delta)dx}, \frac{\theta}{(c\prod_{i=1}^n p_i)m(\Omega) \sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})} F(x,t_1,\dots,t_n)}\right[,$$

*for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, system (1) admits at least three weak solutions in $X$ whose norms are less than $\rho$.*

*Proof.* Our aim is to apply Lemma 1.1 to our problem. To this end, for each $u = (u_1,\dots,u_n) \in X$, we let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$$

and

$$\Psi(u) := \int_\Omega F(x, u_1(x), \dots, u_n(x))dx.$$

Clearly, $\Phi$ is bounded on each bounded subset of $X$ and it is known that $\Phi$ and $\Psi$ are well defined and continuously Gâteaux differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\Phi'(u)(v) = \int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x)dx$$

$\Big($since $\nabla(\frac{1}{p}|\Delta u|^p) = \varphi(\Delta u)$, where $\varphi(\Delta u) := \begin{cases} |\Delta u|^{p-2}\Delta u, & \nabla u \neq 0, \\ 0, & \nabla u = 0. \end{cases}\Big)$ and

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx$$

for every $v = (v_1, \dots, v_n) \in X$, as well as $\Phi$ is sequentially weakly lower semicontinuous (see Proposition 25.20 of [22]). Also, $\Phi' : X \to X^*$ is a uniformly monotone operator in $X$ (for more details, see (2.2) of [20]), and since $\Phi'$ is coercive and hemicontinuous in $X$, by applying Minty-Browder theorem (Theorem 26.A of [22]), $\Phi'$ admits a continuous inverse on $X^*$.

We claim that $\Psi' : X \to X^*$ is a compact operator. To this end, it is enough to show that $\Psi'$ is strongly continuous on $X$. For this, for fixed $(u_1, \dots, u_n) \in X$, let $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in $X$ as $m \to +\infty$. Then we have $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on $\Omega$ as $m \to +\infty$ (see [22]). Since $F(x, \cdot, \dots, \cdot)$ is $C^1$ in $\mathbb{R}^n$ for every $x \in \Omega$, the derivatives of $F$ are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, so for $1 \leq i \leq n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \to +\infty$. By the Lebesgue dominated convergence theorem, $\Psi'(u_{1m}, \dots, u_{nm}) \to \Psi'(u_1, \dots, u_n)$ strongly as $m \to +\infty$. Thus we proved that $\Psi'$ is strongly continuous on $X$. Now, let $(u_{1m}, \dots, u_{nm})$ be a bounded sequence in $X$. Since $X$ is reflexive, there exists a subsequence, still denoted by $(u_{1m}, \dots, u_{nm})$, such that $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in $X$ as $m \to +\infty$. Hence, $\Psi'(u_{1m}, \dots, u_{nm}) \to \Psi'(u_1, \dots, u_n)$ strongly as $m \to +\infty$. Thus, $\Psi'$ is compact and the claim is true.

Moreover, we have

$$\Phi(0) = \Psi(0) = 0.$$

Next, put $w(x) = (w_1(x), \dots, w_n(x))$ such that for $1 \leq i \leq n$,

$$w_i(x) := \begin{cases} 0 & x \in \Omega \setminus B(x^0, D), \\ \frac{16\delta\big(3(l^4-D^4)-6D(l^3-D^3)+3D^2(l^2-D^2)\big)}{3D^4} & x \in B(x^0, D) \setminus B(x^0, D/2), \\ \delta & x \in B(x^0, D/2), \end{cases}$$

where $l := \text{dist}(x, x^0) = \sqrt{\sum_{j=1}^{N}(x_j - x_j^0)^2}$. We have

$$\frac{\partial w_i(x)}{\partial x_j} = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D) \cup B(x^0, D/2), \\ \frac{64\delta}{D^4}\big(l^2(x_j - x_j^0) - \frac{3D}{2}l(x_j - x_j^0) + \frac{D^2}{2}(x_j - x_j^0)\big) & x \in B(x^0, D) \setminus B(x^0, D/2), \end{cases}$$

$$\frac{\partial^2 w_i(x)}{\partial x_j^2} = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D) \cup B(x^0, D/2), \\ \frac{64\delta}{D^4}\big(\frac{D^2}{2} + (2l - \frac{3D}{2})(x_j - x_j^0)^2/l - (\frac{3D}{2} - l)l\big) & x \in B(x^0, D) \setminus B(x^0, D/2), \end{cases}$$

$$\sum_{j=1}^{N} \frac{\partial^2 w_i(x)}{\partial x_j^2} = \begin{cases} 0 & x \in \Omega \setminus B(x^0, D) \cup B(x^0, D/2), \\ \frac{64\delta}{D^4}\left((N+2)l^2 - \frac{3D}{2}(N+1)l + \frac{D^2}{2}N\right) & x \in B(x^0, D) \setminus B(x^0, D/2). \end{cases}$$

Clearly $w = (w_1, \ldots, w_n) \in X$ and, in particular, one has for $1 \le i \le n$,

$$\|w_i\|_{p_i}^{p_i} = \frac{(64\delta)^{p_i} 2\pi^{N/2}}{D^{4p_i}\Gamma(N/2)} \int_{D/2}^{D} |(N+2)r^2 - \frac{3D}{2}(N+1)r + \frac{D^2}{2}N|^{p_i} r^{N-1} dr. \tag{7}$$

Here, we obtain from (3), (4) and (7) that for $1 \le i \le n$,

$$\frac{(\delta\kappa_i)^{p_i}}{c} < \|w_i\|_{p_i}^{p_i} < \frac{(\delta\sigma_i)^{p_i}}{c}. \tag{8}$$

Put $r := \frac{\theta}{c\prod_{i=1}^n p_i}$. By the assumption $\sum_{i=1}^n \frac{(\delta\kappa_i)^{p_i}}{p_i} > \frac{\theta}{\prod_{i=1}^n p_i}$, it follows from (8) that $\Phi(w) > r$.

Since $0 \le w_i(x) \le \delta$ for each $x \in \Omega$ for $1 \le i \le n$, condition ($b_1$) ensures that

$$\int_{\Omega \setminus B(x^0,D)} F(x, w_1(x), \ldots, w_n(x))dx + \int_{B(x^0,D)\setminus B(x^0,D/2)} F(x, w_1(x), \ldots, w_n(x))dx \ge 0.$$

Hence

$$\int_{\Omega} F(x, w_1(x), \ldots, w_n(x))dx \ge \int_{B(x^0,D/2)} F(x, \delta, \ldots, \delta)dx.$$

Now, owing to assumption ($b_2$) and (8), we have

$$\begin{aligned} m(\Omega) \sup_{(x,t_1,\ldots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})} & F(x, t_1, \ldots, t_n) \\ &< \frac{\theta}{\left(\sum_{i=1}^n \frac{(\delta\sigma_i)^{p_i}}{p_i}\right)\left(\prod_{i=1}^n p_i\right)} \int_{B(x^0,D/2)} F(x, \delta, \ldots, \delta)dx \\ &< \frac{\theta}{\left(\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}\right)\left(c\prod_{i=1}^n p_i\right)} \int_{B(x^0,D/2)} F(x, \delta, \ldots, \delta)dx \\ &\le \frac{\theta}{c} \frac{\int_\Omega F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^n \left(\prod_{j=1, j\ne i}^n p_j\right)\|w_i\|_{p_i}^{p_i}}. \end{aligned} \tag{9}$$

Taking into account that for each $u_i \in W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$,

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \le c\|u_i\|_{p_i}^{p_i}$$

for $1 \le i \le n$ (see (2)), we have that

$$\sup_{x\in\Omega} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le c \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} = c\Phi(u) \tag{10}$$

for every $u = (u_1, \dots, u_n) \in X$, and taking into account (9) and (10), it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{\Phi(u)\le r}\int_\Omega F(x,u_1(x),\dots,u_n(x))dx\\ &\le m(\Omega)\sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})}F(x,t_1,\dots,t_n)\\ &< \frac{\theta}{c}\frac{\int_\Omega F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^n\left(\prod_{j=1,j\neq i}^n p_j\right)\|w_i\|_{p_i}^{p_i}}\\ &= \frac{\theta}{c\prod_{i=1}^n p_i}\frac{\int_\Omega F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^n\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}\\ &= r\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Therefore, assumption ($a_1$) of Lemma 1.1 is satisfied.

Now, for fixed $\lambda \in \Lambda$, due to ($b_3$), there exist two constants $\gamma, \vartheta \in \mathbb{R}$ with

$$0<\gamma<\frac{\left(\prod_{i=1}^n p_i\right)\sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})}F(x,t_1,\dots,t_n)}{\theta}$$

such that

$$F(x,t_1,\dots,t_n)\le\gamma\left(\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}\right)+\vartheta$$

for all $x \in \Omega$ and for all $(t_1, \dots, t_n) \in \mathbb{R}^n$. Fix $u = (u_1, \dots, u_n) \in X$. Then

$$F(x,u_1(x),\dots,u_n(x))\le\gamma\left(\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}\right)+\vartheta \tag{11}$$

for all $x \in \Omega$. So, for any fixed $\lambda \in \Lambda$, from (10) and (11) we have

$$\begin{aligned}\Phi(u)-\lambda\Psi(u) &= \sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}-\lambda\int_\Omega F(x,u_1(x),\dots,u_n(x))dx\\ &\ge \sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}-\lambda\gamma\left(\int_\Omega\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}dx\right)-\lambda\vartheta\, m(\Omega)\\ &\ge \sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}-\lambda\gamma\left(c\, m(\Omega)\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}\right)-\lambda\vartheta\, m(\Omega)\\ &\ge \left(1-\frac{\gamma\theta}{\left(\prod_{i=1}^n p_i\right)\sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})}F(x,t_1,\dots,t_n)}\right)\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}\\ &\quad -\frac{\vartheta\theta}{\left(c\prod_{i=1}^n p_i\right)\sup_{(x,t_1,\dots,t_n)\in\Omega\times K(\frac{\theta}{\prod_{i=1}^n p_i})}F(x,t_1,\dots,t_n)},\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}\left(\Phi(u)-\lambda\Psi(u)\right)=+\infty,$$

which means that the functional $\Phi - \lambda\Psi$ is coercive. Then, also condition ($a_2$) of Lemma 1.1 holds.

In addition, since $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for every $(t_1, \dots, t_n) \in \mathbb{R}^n$ and is $C^1$ with respect to $(t_1, \dots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$, satisfying condition (G), the functional

$$J(u) = \int_\Omega G(x, u_1(x), \ldots, u_n(x))dx$$

is well defined and continuously Gâteaux differentiable on $X$ with a compact derivative, and

$$J'(u)(v) = \int_\Omega \sum_{i=1}^n G_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)dx$$

for all $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in X$. Thus, all the hypotheses of Lemma 1.1 are satisfied. Also note that the solutions of the equation

$$\Phi'(u) - \lambda\Psi'(u) - \mu J'(u) = 0$$

are exactly the weak solutions of (1). So, the conclusion follows from Lemma 1.1. □

We now point out the following special case of Theorem 2.1 when $F$ does not depend on $x \in \Omega$.

**Theorem 2.2.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-function and assume that there exist two positive constants $\theta$ and $\delta$ with $\sum_{i=1}^n \frac{(\delta\kappa_i)^{p_i}}{p_i} > \frac{\theta}{\prod_{i=1}^n p_i}$ such that*

(b$_4$) *$F(t_1, \ldots, t_n) \geq 0$ for all $t_i \in [0, \delta]$ for $1 \leq i \leq n$;*

(b$_5$)

$$\frac{\theta\pi^{N/2}}{\Gamma(1+N/2)\prod_{i=1}^n p_i}\Big(\frac{D}{2}\Big)^N F(\delta, \ldots, \delta) - m(\Omega)\sum_{i=1}^n \frac{(\delta\sigma_i)^{p_i}}{p_i} \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta}{\prod_{i=1}^n p_i}\right)} F(t_1, \ldots, t_n) > 0;$$

(b$_6$) $\limsup_{(|t_1|,\ldots,|t_n|)\to(+\infty,\ldots,+\infty)} \frac{F(t_1,\ldots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} \leq 0.$

*Then, setting*

$$\Lambda := \left]\frac{\Gamma(1+N/2)\sum_{i=1}^n \frac{(\delta\sigma_i)^{p_i}}{p_i}}{c\pi^{N/2}F(\delta, \ldots, \delta)}\Big(\frac{2}{D}\Big)^N, \frac{\theta}{(c\prod_{i=1}^n p_i)m(\Omega) \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta}{\prod_{i=1}^n p_i}\right)} F(t_1, \ldots, t_n)}\right[,$$

*for each compact interval $[a, b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the system*

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(u_1, \ldots, u_n) + \mu G_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

*for $1 \leq i \leq n$, admits at least three weak solutions in $X$ whose norms are less than $\rho$.*

*Proof.* Set $F(x, t_1, \ldots, t_n) = F(t_1, \ldots, t_n)$ for all $x \in \Omega$ and $t_i \in \mathbb{R}$ for $1 \leq i \leq n$. Since $\int_{B(x^0,D/2)} F(\delta, \ldots, \delta)dx = \frac{\pi^{N/2}}{\Gamma(1+N/2)}(\frac{D}{2})^N F(\delta, \ldots, \delta)$, Theorem 2.1 ensures the conclusion. □

Let $\sigma = \sigma_1$, $\kappa = \kappa_1$ and $p = p_1$. Then we have the following existence result.

**Corollary 2.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be an $L^1$-Carathéodory function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$ and assume that there exist two positive constants $\theta$ and $\delta$ with $(\delta\kappa)^p > \theta$ such that*

(b$_7$) *$F(t) \geq 0$ for all $t \in [0, \delta]$;*

(b$_8$) $\frac{\theta\pi^{N/2}}{\Gamma(1+N/2)}(\frac{D}{2})^N F(\delta) - m(\Omega)(\delta\sigma)^p \sup_{t\in[-\sqrt[p]{\theta},\sqrt[p]{\theta}]} F(t) > 0$;

(b$_9$) $\limsup_{|t|\to+\infty} \frac{F(t)}{|t|^p} \leq 0$.

*Then, setting*

$$\Lambda := \left]\frac{\Gamma(1+N/2)(\delta\sigma)^p}{(pc)\,\pi^{N/2}F(\delta)}\Big(\frac{2}{D}\Big)^N, \frac{\theta}{m(\Omega)(pc)\sup_{t\in[-\sqrt[p]{\theta},\sqrt[p]{\theta}]} F(t)}\right[,$$

*for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem*

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

*admits at least three weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ whose norms are less than $\rho$.*

If $N = 1$, we can get a better result than Theorem 2.2. For simplicity, we fix $\Omega = (0,1)$ and Note that in this situation we have $p_i > 1$ for $1 \leq i \leq n$.

**Theorem 2.3.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-function and assume that there exist two positive constants $\theta$ and $\delta$ with $\sum_{i=1}^n \frac{(32\delta)^{p_i}}{2cp_i} > \frac{\theta}{\prod_{i=1}^n p_i}$ such that Assumptions (b$_4$) and (b$_6$) in Theorem 2.2 holds, and*

(b$_{10}$) $\frac{\theta}{\prod_{i=1}^n p_i} F(\delta,\dots,\delta) - \sum_{i=1}^n \frac{(32\delta)^{p_i}}{cp_i} \sup_{(t_1,\dots,t_n)\in K(\frac{\theta}{\prod_{i=1}^n p_i})} F(t_1,\dots,t_n) > 0$.

*Then, setting*

$$\Lambda := \left]\frac{\sum_{i=1}^n \frac{(32\delta)^{p_i}}{p_i}}{F(\delta,\dots,\delta)}, \frac{\theta}{\left(c\prod_{i=1}^n p_i\right)\sup_{(t_1,\dots,t_n)\in K(\frac{\theta}{\prod_{i=1}^n p_i})} F(t_1,\dots,t_n)}\right[,$$

*for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the system*

$$\begin{cases} (|u_i''|^{p_i-2}u_i'')'' = \lambda F_{u_i}(u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n) & \text{in } (0,1), \\ u_i(0) = u_i(1) = 0, \\ u_i''(0) = u_i''(1) = 0 \end{cases} \tag{14}$$

*for $1 \leq i \leq n$, admits at least three weak solutions in*

$$Y := \left(W^{2,p_1}(0,1) \cap W_0^{1,p_1}(0,1)\right) \times \cdots \times \left(W^{2,p_n}(0,1) \cap W_0^{1,p_n}(0,1)\right)$$

*whose norms are less than $\rho$.*

*Proof.* For each $u = (u_1,\dots,u_n) \in Y$, let

$$\Phi(u) := \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i},$$

$$\Psi(u) := \int_0^1 F(u_1(x),\dots,u_n(x))dx,$$

and

$$J(u) = \int_0^1 G(x,u_1(x),\dots,u_n(x))dx,$$

where for $1 \leq i \leq n$,

$$\|u_i\|_{p_i} := \left[\int_0^1 |\Delta u_i(x)|^{p_i} dx\right]^{\frac{1}{p_i}}.$$

Since the critical points of the functional $\Phi - \lambda\Psi - \mu J$ on $Y$ are exactly the weak solutions of system (14), our aim is to apply Lemma 1.1 to $\Phi$, $\Psi$ and $J$. As observed in Theorem 2.1, $\Phi$, $\Psi$ and $J$ satisfy the regularity assumptions in Lemma 1.1. Also, thanks to ($b_6$), for each $\lambda > 0$, the functional $\Phi - \lambda\Psi$ is coercive.

Now, put $r := \frac{\theta}{c\prod_{i=1}^n p_i}$ and $w(x) = (w_1(x), \dots, w_n(x))$ such that for $1 \leq i \leq n$,

$$w_i(x) := \begin{cases} \delta - 16\delta\left(\frac{1}{4} - |x - \frac{1}{2}|\right)^2 & x \in [0, \frac{1}{4}] \cup (\frac{3}{4}, 1], \\ \delta & x \in (\frac{1}{4}, \frac{3}{4}]. \end{cases}$$

It is easy to verify that $w = (w_1, \dots, w_n) \in Y$, and for $1 \leq i \leq n$,

$$\|w_i\|_{p_i}^{p_i} = \frac{(32\delta)^{p_i}}{2}.$$

Now, under the assumption of $\sum_{i=1}^n \frac{(32\delta)^{p_i}}{2cp_i} > \frac{\theta}{\prod_{i=1}^n p_i}$, we have

$$\Phi(w) = \sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i} > \frac{\theta}{c\prod_{i=1}^n p_i} = r > 0.$$

Since $0 \leq w_i(x) \leq \delta$ for each $x \in (0,1)$ for $1 \leq i \leq n$, it follows from ($b_4$) and ($b_{10}$) that

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) &= \sup_{\Phi(u)\leq r} \int_0^1 F(u_1(x), \dots, u_n(x))dx \\
&\leq \sup_{(t_1,\dots,t_n)\in K(\frac{\theta}{\prod_{i=1}^n p_i})} F(t_1, \dots, t_n) \\
&< \frac{\theta}{c} \frac{\int_0^1 F(w_1(x), \dots, w_n(x))dx}{\sum_{i=1}^n \big(\prod_{\substack{j=1 \\ j\neq i}}^n p_j\big)\|w_i\|_{p_i}^{p_i}} \\
&= \frac{\theta}{c\prod_{i=1}^n p_i} \frac{\int_0^1 F(w_1(x), \dots, w_n(x))dx}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} \\
&= r\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

Therefore, condition ($a_1$) of Lemma 1.1 is satisfied, and the proof is complete. □

## 3. **Conclusion**

Based on a recent three critical points theorem obtained by Ricceri [19], we established the existence of an open interval $]\lambda', \lambda''[$ and $\delta > 0$, such that for each $\lambda \in ]\lambda', \lambda''[$ and for each $\mu \in [0, \delta]$, a class of Navier doubly eigenvalue boundary value system involving the $(p_1, \dots, p_n)$-biharmonic operator and depending on parameters $\lambda$ and $\mu$ admits at least three weak solutions.

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