POWER SPECTRAL DENSITY ESTIMATION BY USING GROUPED *B*-SPLINE WINDOWS

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The reduced spectral leakage into the side-lobes of a window function is obtained with a more tapered window, with the same length, which has lower side lobes. The classical windows can realize these characteristics by using one parameter. To obtain a tradeoff between good frequency resolution and acceptable spectral leakage we can use a smoothing time window function. The window functions, with increased order of continuity can be obtained by using grouped B-spline functions. We optimize the power spectral density estimation by using three control parameters.

Keywords: optimum spline windows, spectral analysis, spectral leakage, high order of continuity, control parameters

1. Introduction

We are interested to determine the power spectral density (PSD) for different frequency components contained in a signal, by using a finite-length sequence. So, we obtain the power spectral estimation that is a process of estimating the different frequency components contained in a signal. The finite length of the available data record affects the quality of the estimate. The windowing of the input signal determines different spectral characteristics from the original continuous-time signal.

The advantage of windowing is to provide a trade-off between resolution and the spectral masking provided by the side lobes of the window functions. It is important the choice of the window to have a good precision for the estimated PSD. We can reduce the spectral leakage [1], [2] by using an increased order of continuity at the boundary of observation windows. The usual windows can control the resolution by one parameter. The performance parameters are used to compare the windows [3], [4], [5]. Windows are weighting functions applied to the finite observation data to reduce the spectral leakage.

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We often prefer variable parameter windows and we propose to use the grouped *B*-spline windows, with three control parameters. The proposed windows are a better choice for spectral analysis purposes because of the fact that the side-lobe attenuation as well as the resolution required can be simultaneously achieved, for a wide range of signals. This proposal is our original contribution.

2. Power spectral density estimation

The PSD is obtained by the Fourier transform for the autocorrelation function $\varphi_x(k)$ of the sequence $x(n)$:

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \varphi_x(k) e^{-jk\omega} \tag{1}$$

In practice we must use a finite length signal. The autocorrelation function is truncated and multiplied by a lag window. We calculate the Fourier transform of the autocorrelation estimate and the result is an estimate of the PSD, which is known as the periodogram. It is more convenient to write the periodogram directly, by using the windowed signal. The PSD of the periodogram can be computed as

$$P_{per}(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n) e^{-j\omega n}\right|^2 = \frac{1}{N}\left|\sum_{n=-\infty}^{\infty} f_R(n) x(n) e^{-j\omega n}\right|^2 \tag{2}$$

In (2) $f_R(n)$ is a rectangular window defined by:

$$f_R(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

In practice, the periodogram will get samples of the spectral estimate using a DFT as $P_{per}(e^{j2k\pi/N})$. The periodogram method estimates the power spectrum by using the fast Fourier transform.

The finite duration of $x(n)$ sets a limit to the quality of the spectral estimates we can obtain from the available data samples [6], [7]. Therefore, we can say that the periodogram method generates the PSD estimate of the given signal. The calculation process of this method is simple and efficient, because there is no need to calculate self-correlation functions.

3. Spectral analysis using grouped *B*-spline windows

Using more gradual windows with the same *N*, a lowering of the sidelobes can be obtained, but this improvement is accompanied by a widening of the mainlobe. Thus, a reduction of leakage, which is mainly due to sidelobes, is inevitably accompanied by a further loss of resolution, which is related to mainlobe width. We are interested for more gradual windows. We can use *B*-

spline functions for PSD estimation. The expression for the centered *B*-spline function of order k is:

$$S_k(t) = \sum_{j=0}^{k+1} \frac{(-1)^j}{k!} C_{k+1}^{j} \left(t + \frac{k+1}{2} - j\right)^k u\left(t + \frac{k+1}{2} - j\right) \quad (4)$$

where $u(t)$ is the unit step function.

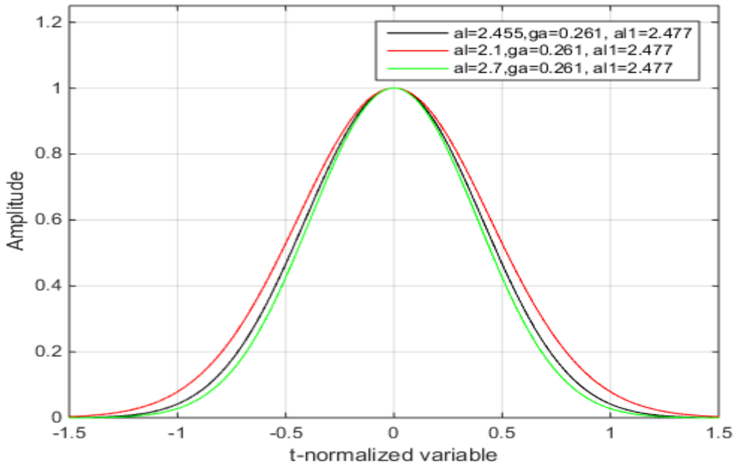

Fig. 1. Grouped tenth order *B*-spline for different values of a (a = al); g=0.261 (g=ga); a_1=2.477 (a_1 = al1).

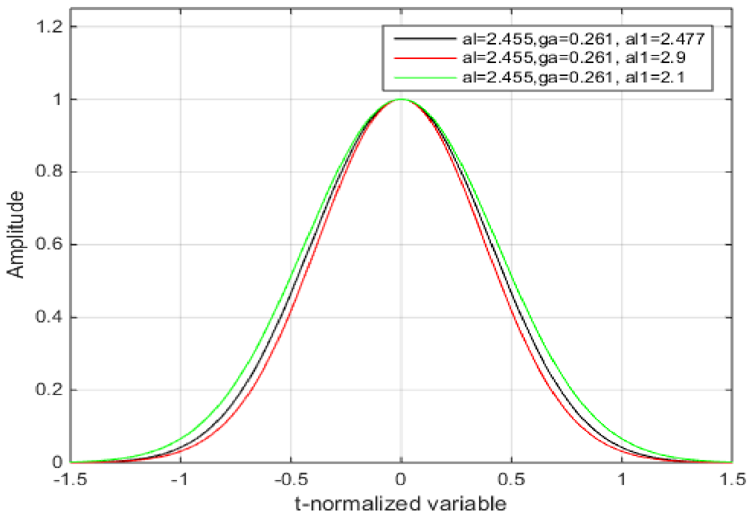

Fig. 2. Grouped tenth order B-spline for different values of a_1 (a_1= al1); a =2.455 (a = al); g=0.261 (g=ga).

We can use the next recurrence relation:

$$S_k(t) = [(\frac{k+1}{2} + t)S_{k-1}(t + \frac{1}{2}) + (\frac{k+1}{2} - t)S_{k-1}(t - \frac{1}{2})]/k \quad (5)$$

A *B*-spline function of order *k* can be obtained iteratively by convolution:

$$S_k(t) = S_{k-1}(t) * S_0(t) \quad (6)$$

where

$$S_0(t) = \begin{cases} 1, \text{ t} \in \text{(-1/2,1/2)} \\ 0, \text{ otherwise} \end{cases} \quad (7)$$

The frequency resolution can be increased by grouping three *B*-spline functions: one of order *k* and two of order *m*:

$$SB_{km}(t) = bS_m(at - g) + S_k(a_1 t) + bS_m(at + g) \quad (8)$$

where a, a_1, b and g are real coefficients.

We can detail the relation (8) for $SB_{km}(t)$:

$$SB_{km}(t) = \begin{cases} bS_m(at+g), \text{ for } -\frac{2g+\text{m}+1}{2a} < t \le -\frac{k+1}{2a_1} \\ bS_m(at+g) + S_k(a_1 t), \text{ for } -\frac{k+1}{2a_1} < t \le \frac{2g-\text{m}-1}{2a} \\ bS_m(at+g) + S_k(a_1 t) + bS_m(at-g), \\ \qquad \text{for } \frac{2g-\text{m}-1}{2a} < t \le \frac{-2g+\text{m}+1}{2a} \\ S_k(a_1 t) + bS_m(at-g), \text{ for } \frac{-2g+\text{m}+1}{2a} < t \le \frac{k+1}{2a_1} \\ bS_m(at-g), \text{ for } \frac{k+1}{2a_1} < t \le \frac{2g+\text{m}+1}{2a} \\ 0, \text{ otherwise} \end{cases} \quad (9)$$

Fig. 1 shows the grouped tenth order *B*-splines functions for different values of a. The parameters g and a_1 were fixed at g = 0.261, a_1=2.477.

Fig. 2 shows the dependence of the grouped tenth order *B*-spline functions on the parameter a_1. The parameters g and a were fixed at g = 0.261, a=2.455. We can see that the parameters a and a_1 give a fine control for an optimum window.

Most of the windows are the sampled versions of the continuous-time windows described. To generate grouped *B*-spline windows, the function given in (9) is uniformly sampled with N samples in the interval $t \in [-p,p)$, where p is $p = \frac{2g+\text{m}+1}{2a}$. So, we can obtain a discrete causal window for $n = \overline{0, N-1}$.

Table 1

Optimum parameters for grouped *B*-spline windows.

Type	a_1	a	g
Gr. Spl.7	2.37	2.35	0.312
Gr. Spl.8	2.674	2.65	0.282
Gr. Spl.9	2.475	2.45	0.263
Gr. Spl.10	2.477	2.455	0.261

The function $SB_{km}(t)$, given in (9), is optimized by minimizing the maximum relative error $|\varepsilon_{\max}|$, by using the parameters a, a_1 and g, for better estimation. The maximum relative error $\varepsilon_{\max}$ is [4]:

$$\varepsilon_{\max} = SL - 1 \tag{10}$$

where SL is the scalloping loss:

$$SL = \frac{\left|\sum_n f(nT)e^{-j\pi/N}\right|}{\sum_n f(nT)} \tag{11}$$

If we impose k=m=9, $BB_9(t) \neq 0$, for a=a_1=6, g=1 and x$\in$(-1,1). By decreasing $a = a_1$, in the first step, and g in the second step, the interferences are minimized. In the next step, a_1 is increased. The optimum values for the three parameters are a=2.45, g=0.263 and a_1=2.475.

In the situation k=m=10, $SB_{10}(t) \neq 0$, for a=a_1=6.5, g=1 and x$\in$(-1,1). In the first step, we decrease the parameters a=a_1 and g in the second step, to minimize the interferences. In the final step, a_1 is increased. Finally, we obtain the optimum values a=2.455, g=0.261 and a_1=2.477.

Table 1 gives the optimum parameters for grouped *B*-spline windows, of seventh to tenth order.

We want to present the advantages of the grouped *B*-spline windows to estimate the small signals and consider the signal:

$$x(n) = \sin(0.28\pi n) + 0.01\sin(0.32\pi n) + 0.001\sin(0.36\pi n) + 0.00001\sin(0.4\pi n) \tag{12}$$

Fig. 3 illustrates the spectrogram plot for the signal (12), by using optimal tenth order grouped *B*-spline window. The comparative presentation is made, for the optimum tenth order grouped *B*-spline window, in Fig. 4, respectively for

Hamming window in Fig. 5, for the spectrograms using chirp signal with the quadratic form variation of the signal frequency. We observe that the spectral leakage rapidly decreases for the optimum tenth order grouped *B*-spline windows.

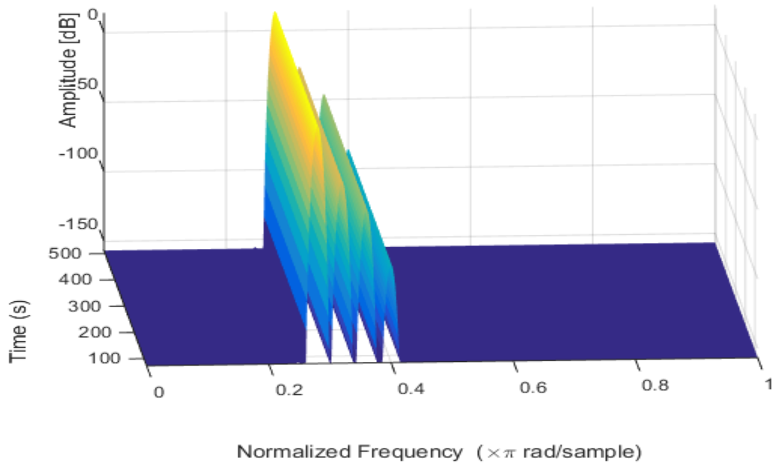

Fig. 3. Spectrogram plot of the signal (15) using the optimum tenth order *B*-spline window.

Fig. 6 presents a comparison between the classical windows and the optimum ninth order grouped B-spline window. Grouped B-spline window has superior performances, by comparison with classical windows, for the spectral estimation of the small amplitude signals. The resolution is 0.02 for the minimum amplitude of -175.6 dB, by using optimum ninth order grouped B-spline window.

Fig. 7 shows a complete comparison between the optimum seventh, eigth, ninth and tenth order grouped B-spline windows, for the spectral estimation of the small amplitude signals, with N = 1024.

We have used a reduced bandwidth test signal. This signal has 20 kHz bandwidth, centered on 0.6 GHz. Spectral estimation is realized by using many windows. The results are presented in table 2. The best SNR results by using the grouped ninth order B-spline window. The used sampling frequency is 1.3 GHz. We have used a large bandwidth test signal. This signal has 8 MHz bandwidth, centered on 0.75 GHz. Spectral estimation is realized by using many windows. The results are presented in table 3. The best SNR results by using the grouped ninth order B-spline window. The used sampling frequency is 1.6 GHz.

4. Welch periodogram using grouped *B*-spline windows

The Welch periodogram is based on creating an ensemble of sample sequences by dividing a long sequence of samples into a set of shorter segments. These shorter segments can be allowed to overlap with their neighbors for some portion of their length. So we can obtain a better estimate with reduced variance and the gradual consistent estimation. Furthermore, a data window $w(n)$, is applied to each segment before the segment's sample spectrum is computed.

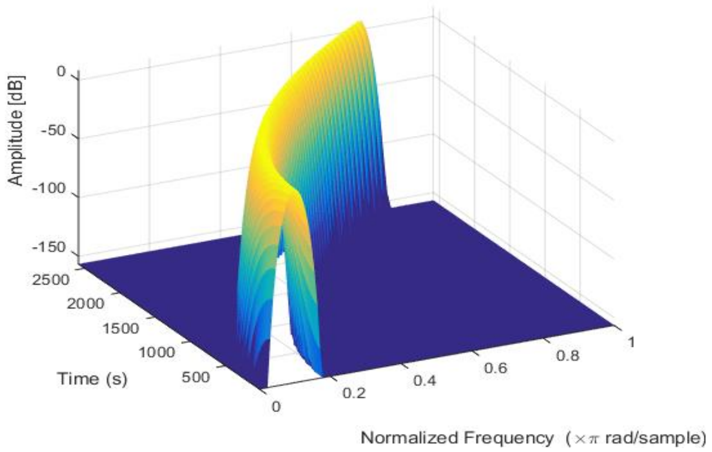

Fig. 4. Spectrogram plot of the chirp signal using the optimum tenth order grouped *B*-spline window.

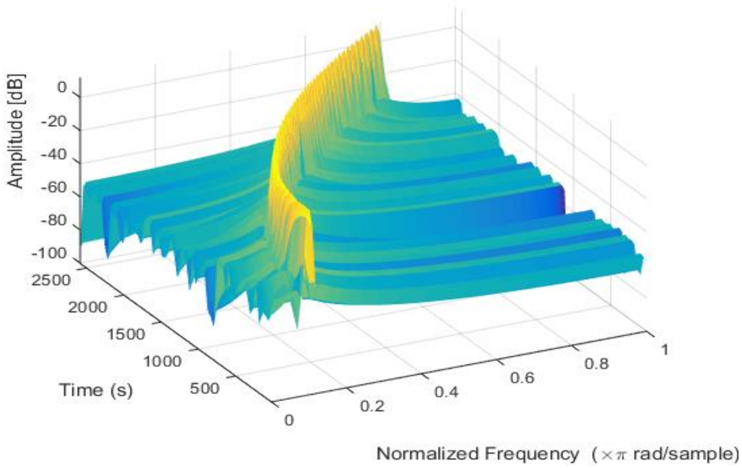

Fig. 5. Spectrogram plot of the chirp signal using Hamming window.

Welch method obtains the direct calculation of the PSD estimate, which is defined as the average of the modified periodograms. The steps for computing a Welch periodogram are:

1. Divide the available sample sequence into P overlapping segments of length D samples, where the parameter $S<D$ represents the number of overlap samples between consecutive segments. The p-th segment of the original sequence $x(k)$ is

$$x_p(n)= x(pS+n) \tag{13}$$

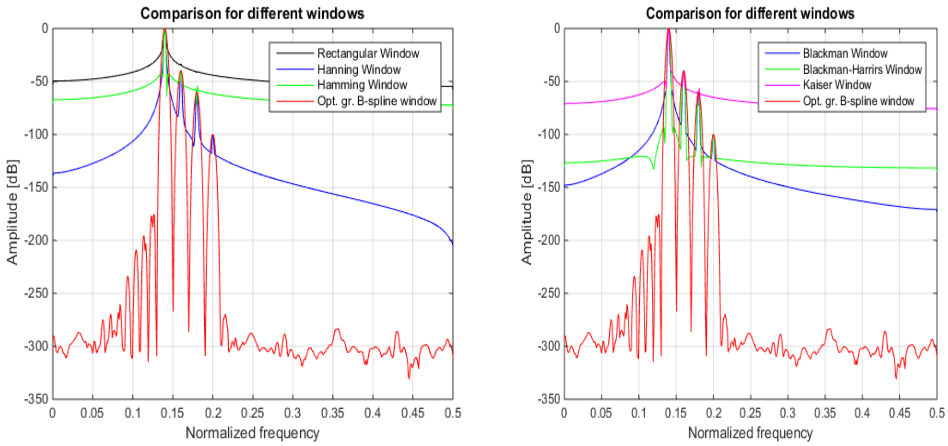

Fig. 6. Comparison between the classical windows and the optimum ninth order grouped *B*-spline window, for the spectral estimation of the small amplitude signals. For Kaiser window we have α_k =4.5.

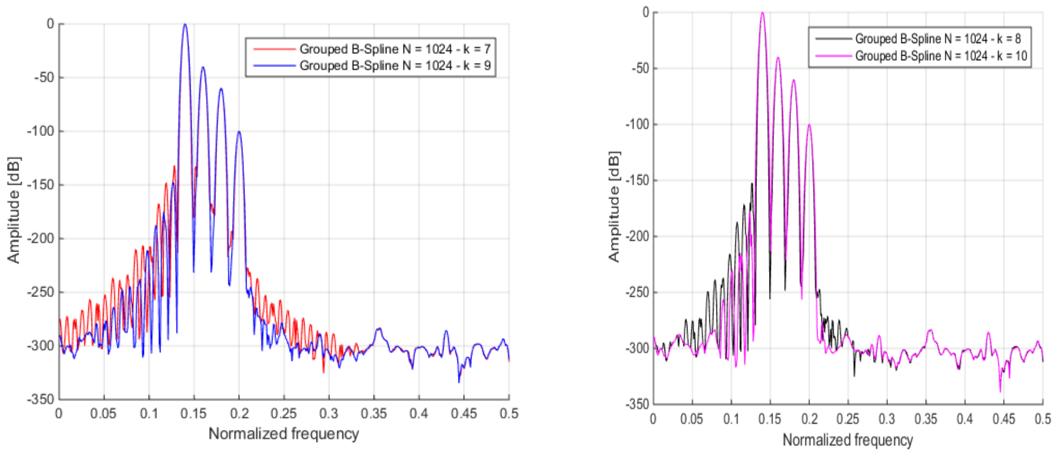

Fig. 7. Comparison of the realized performances, by using grouped *B*-spline windows of different orders, for the spectral estimation of the small amplitude signals, with N = 1024.

2. Each segment $x_p(n)$ is windowed by the function $w(n)$

$$y_p(n)=w(n)\, x_p(n),\ p=0,1,\ldots,P-1 \tag{14}$$

3. Compute the periodogram for each of the P windowed segments

$$S_p(m)=\frac{1}{UD}Y_p^2(m),\ m=0,1,\ldots,D-1 \tag{15}$$

where the normalization factor U is

$$U = \frac{1}{D}\sum_{n=0}^{D-1}|w(n)|^2 \qquad (16)$$

is the power in the window function. The spectrum of a finite-length signal typically exhibits side-lobes due to discontinuities at the endpoints. The window function *w(n)* eliminates the discontinuities and reduces the spread of the spectral energy into the side-lobes of the spectrum.

4. Compute the arithmetic average of the *P* periodograms at each frequency

$$S_w(m) = \frac{1}{P}\sum_{p=0}^{P-1} S_p(m)\,,\ m{=}0,1,\ldots\ldots,D\text{-}1 \qquad (17)$$

The result, S_w *(m)*, is the Welch power spectrum. The Welch power spectrum is the average of *P* periodograms obtained from overlapped and windowed segments of a signal.

Figs. 8 through 10 display the effect of applying different window functions for the Welch method [1].

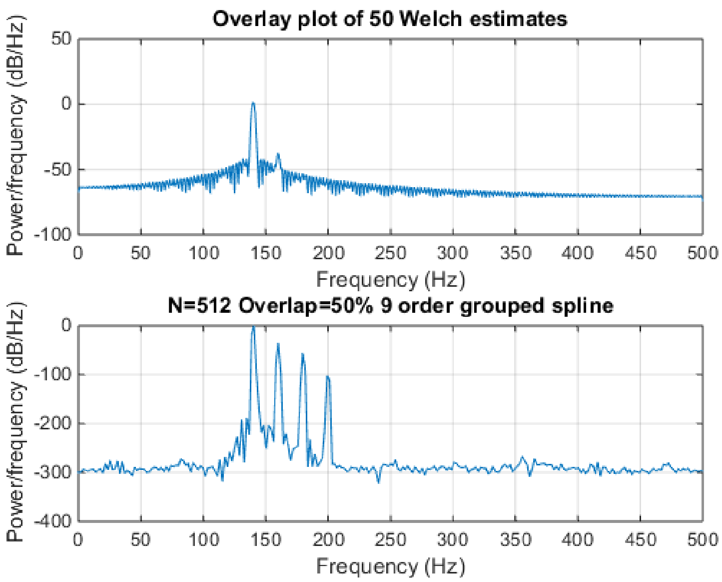

Fig. 8. Spectrum of the signal (12) using Welch method with an overlap of 50 percent and with the optimum ninth order grouped *B*-spline window.

Fig. 8 presents the spectrum of the signal (12) using Welch method with an overlap of 50 percent. Also the Welch method with an overlap of 50 percent is illustrated by using the optimum ninth order grouped *B*-spline window, for the signal (12). Sampling frequency is *Fs*=1000 Hz and the sequence length is *N*=512.

We can clearly see that the resolution of the Welch periodogram depends on the data window used.

Table 2.

Signal/noise ratio (SNR) before estimation and after spectral estimation, with the reduced bandwidth signal, for different SNR.

Signal/noise ratio [dB]							
SNR before estimation	Rect-angular window	Hanning window	Hamming window	Blackman window	Blackaman-Harris window	Kaiser window	Grouped *B*-Spline window k=9
-2,3729	-2,3729	-2,5943	-2,5543	-2,7271	-2.7334	-2,3719	6,1717
11,5309	11,5309	11,2935	11,2027	10,9567	10,8941	11,5366	18,1526
76,5722	76,5722	76,2938	76,2517	76,0120	75,9559	76,5685	82,9384

Table 3.

Signal/noise ratio (SNR) before estimation and after spectral estimation, with the large bandwidth signal, for different SNR.

Signal/noise ratio [dB]							
SNR before estimation	Rect-angular window	Hanning window	Hammin g window	Blackman window	Blackaman-Harris window	Kaiser window	Grouped *B*-Spline window k=9
-2,4421	-2,4421	1,8454	1,5926	2,7478	3,4684	-2,2663	5,5335
26,5579	26,5579	30,7933	30,5434	31,6905	32,4083	26,7313	34,4795
91,5872	91,5872	95,8722	95,6186	96,7804	97,5073	91,7623	99,5991

Fig. 9 plots the Welch method with an overlap of 50 percent by using the Blackman window for the signal (12). Sampling frequency is Fs=1000 Hz and the sequence length is N=512.

Fig. 10 presents the spectrum of the signal (12) using Welch method with an overlap of 50 percent. Also the Welch method with an overlap of 50 percent is illustrated by using the optimum tenth order grouped *B*-spline window for the signal (12). Sampling frequency is Fs=1000 Hz and the sequence length is N=512. We observe how flat is the peak-free part of the spectrum in the Welch's estimate with the optimum ninth and tenth order grouped *B*-spline windows.

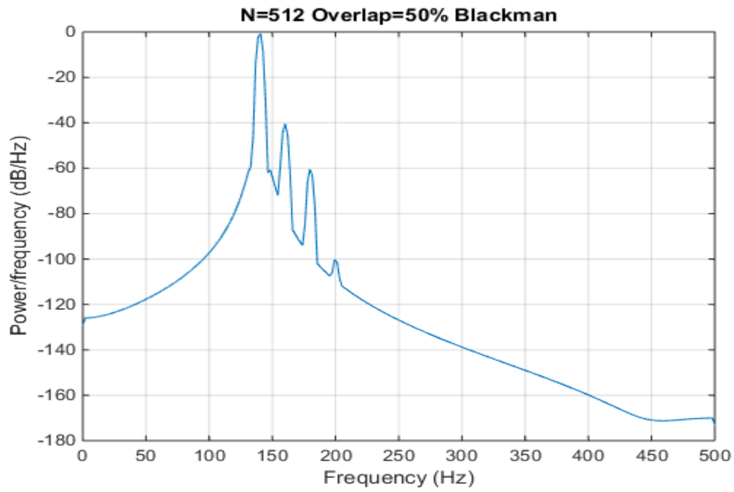

Fig. 9. Spectrum of the signal (12) using Welch method with an overlap of 50 percent and wth the Blackman window. Sampling frequency is *Fs*=1000 Hz and the sequence length is *N*=512.

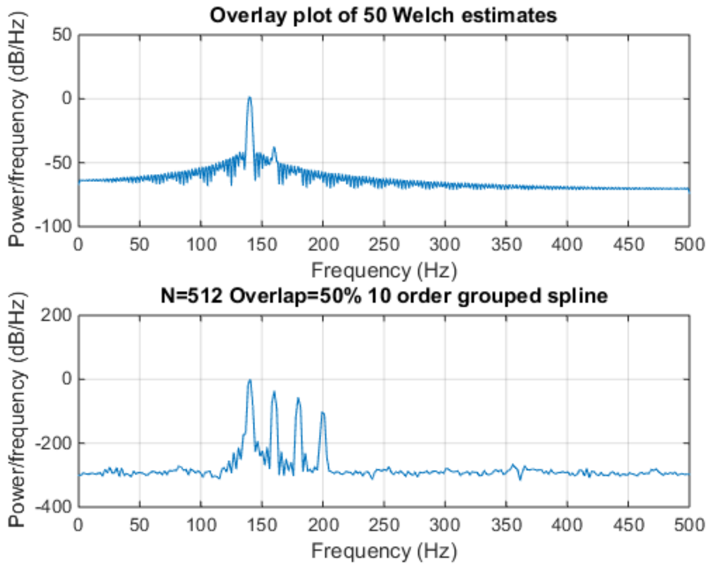

Fig. 10. Spectrum of the signal (12) using Welch method with an overlap of 50 percent and with the optimum tenth order grouped *B*-spline window. Sampling frequency is *Fs*=1000 Hz and the sequence length is *N*=512.

We have a reduction in height of the spectral window's side-lobes. The variance of the Welch estimator using the optimum ninth and tenth order grouped *B*-spline windows is reduced. In spectral analysis, the side lobes cause smearing or spreading of energy, while the main lobe is responsible for appropriate smoothing effects. These last results demonstrate the efficacy of the grouped *B*-

spline windows presented, in discerning the weak signal, by comparing with other windows.

5. Conclusions

The minimizing leakage is important and therefore large attenuation and rapid decrease of the sidelobes is desired. The finite duration of $x(n)$ sets a limit to the quality of the spectral estimates we can obtain from the available data samples. The shape and the width of the window are important for improving the accuracy of the harmonic amplitude. The resolution is sufficient, and the leakage is small, if the main lobe is narrow and the side lobes are low and decrease rapidly. In this paper, we presented an approach for the construction of the grouped *B*-spline windows. These new windows contain three variable parameters that can be controlled simultaneously, to have large attenuation and rapid decrease of the sidelobes. The reduction of leakage, which is mainly due to sidelobes, is inevitably accompanied by a further loss of resolution, which is related to mainlobe width. We have tested the spectral effects related to the use of more gradual window functions.

The presented applications confirm that for spectral analysis requiring high resolution and lower side-lobe levels, the grouped *B*-spline windows are the better choice.

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