STABILITY FOR SMALL DELAYS, METZLER MATRICES AND AN APPLICATION TO A FLIGHT CONTROLLER DESIGN

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In this paper a stability theorem for linear systems of delay differential equations with small delays is proved and an application to control design for flight control of an aircraft in a longitudinal flight is given.

Keywords: stability, delay differential equations.

1. Introduction

The study of stability for linear delay differential equations (DDE) with small delays originated in some papers in the last decades of the twentieth century. The papers [15], [10] contain the first results on asymptotic stability for some systems of linear DDEs using the hypothesis that the delays are small. For example, in [10], for the equation

$$x'(t) = L(t, x_t) \tag{1}$$

with $L : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ continuous, $L(t, \cdot)$ linear on $C([-\tau, 0], \mathbb{R}^n)$ that satisfies

$$\|L(t, \phi)\| \leq K sup\{\|\phi(s)\|, s \in [-\tau, 0]\} \tag{2}$$

the asymptotic behavior of solutions is described when

$$\tau K e < 1. \tag{3}$$

The approach related to small delays was further developed in [8], [1], [9].

For $A_k \in \mathcal{M}(\mathbb{R}^n), k = 0, \ldots, N$, $\tau_k > 0, k = 1, \ldots, N$, consider the DDEs system

$$x'(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \tau_k). \tag{4}$$

A condition of small delays is coupled, in [9], Proposition 2.3., with the condition that all A_k be Metzler to yield a criterion of asymptotic stability for (4). The basic property used in the proof is the positivity of the Cauchy matrix.

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We give a new proof of this, with more direct arguments, using a modified version of a theorem of Gyori: [13], Theorem 2.1.

The criterion in Proposition 2.3. in [9] will then be applied to give conditions for the design of a stabilizing controller for an aircraft in a longitudinal flight.

2. Metzler matrices and exponential stability

Proposition 2.1. *Let $a_0, ..., a_N$ be real numbers and $\tau_0, ..., \tau_N$ be positive real numbers. Define*

$$\tau = \max\{\tau_0, ..., \tau_N\} \tag{5}$$

and suppose that

$$0 < \tau \sum_{j=0}^{N} |a_j| < \frac{1}{e}. \tag{6}$$

Then the equation

$$x = \sum_{k=0}^{N} |a_k| e^{x\tau_k} \tag{7}$$

has a real root in the interval $(0, \frac{1}{\tau})$.

Proof. Define

$$f(x) = \sum_{k=0}^{N} |a_k| e^{x\tau_k} - x.$$

Then $f(0) = \sum_{k=0}^{N} |a_k|$ and

$$f(\frac{1}{\tau}) = \sum_{k=0}^{N} |a_k| e^{\frac{\tau_k}{\tau}} - \frac{1}{\tau} \leq \sum_{k=0}^{N} |a_k| e - \frac{1}{\tau} < 0$$

by (6) and the result follows. □

The next theorem is a slight modification of Th. 2.1. in [13]. It relies on a condition of small delays (that was not the case in the original one) and proves a property for the fundamental matrix of solutions for (4), another difference to the Th.2.1 in [13]where a continuity condition was supposed. Nevertheless the basic steps in the proof are preserved.

Theorem 2.1. *(see[13], Th 2.1) Let $A_k \in M_n(\mathbb{R})$(real $n \times n$ matrices), $k = 0, ..., N$ and $\tau_k \geq 0, k = 0, ..., N$ be given. Let $\tau > 0$ be defined in (5) and suppose*

$$\tau \sum_{k=0}^{N} \|A_k\| < \frac{1}{e}. \tag{8}$$

Then the equation

$$x = \sum_{k=0}^{N} \|A_k\| e^{x\tau_k} \tag{9}$$

has a real root in the interval $\left(0, \frac{1}{\tau}\right)$ *and the solution of the Cauchy problem*

$$\dot{X}(t) = \sum_{k=0}^{N} A_k X(t - \tau_k) \tag{10}$$

$$X(t) = 0, t \in [-\tau, 0), X(0) = I_n \tag{11}$$

has the property that $det\,(X(t)) > 0, \forall t \geqslant 0.$

Proof. We will give a slightly different proof from that in [13] using condition (8), thus avoiding an apparently flawed argument in the original proof.

We proceed now to the proof of the theorem. Since $X(0) = I_n$ and X is continuous on $[0, \infty)$ (see [13]), one has $det\,(X(t)) > 0$ for t sufficiently small. If $\sum_{k=0}^{N} \|A_k\|\tau_k = 0$, (10) becomes an ordinary differential equation with constant coefficients and $X(t) = e^{At}$ with A the sum of non zero matrices in (10) and the result follows.

Suppose then that $\sum_{k=0}^{N} \|A_k\|\tau_k > 0$ and denote by $x_0 \in \left(0, \frac{1}{\tau}\right)$ a solution of (9). Remark that $\sum_{k=0}^{N} \|A_k\| \leq x_0$ so, for small $t \geq 0$, one has

$$v(t) := \| \sum_{k=0}^{N} A_k X\left(t - \tau_k\right) X^{-1}(t)\| \leq x_0 \tag{12}$$

($X(t - \tau_k) = 0$ if $t - \tau_k < 0$). If the conclusion of the theorem is not valid it would exist $t_1 > 0$ so that

$$det(X(t)) > 0, \forall t \in [0, t_1), det(X(t_1)) = 0 \tag{13}$$

We prove now that (12) holds for $t \in [0, t_1)$. Suppose this is not true. Then there exists $\varepsilon > 0$ so that $v(t) < x_0 + \varepsilon, \forall t \in [0, t_2) \subset [0, t_1)$ and $v(t_2) = x_0 + \varepsilon$. Since v is continuous we can choose ε as small as we want. Rewrite (10) as

$$\dot{X}(t) = \sum_{k=0}^{N} A_k X(t - \tau_k) X^{-1}(t) X(t), t \in [0, t_1) \tag{14}$$

and introduce

$$Z(t) := \sum_{k=0}^{N} A_k X(t - \tau_k) X^{-1}(t), t \in [0, t_1) \tag{15}$$

$$Y(s) := X(t - s) X^{-1}(t), 0 \leq s \leq t < t_1 \tag{16}$$

and $Y(s) = 0$ if $t - s < 0$.

(14) implies that

$$\begin{aligned}\dot{Y}(s) &= -\dot{X}(t - s) X^{-1}(t) \\ &= -\sum_{k=0}^{N} A_k X(t - s - \tau_k) X^{-1}(t - s) \\ X(t - s) X^{-1}(t) &= -Z(t - s) X(t - s) X^{-1}(t) = -Z(t - s) Y(s)\end{aligned}$$

so

$$Y(s) = Y(0)e^{-\int_0^s Z(t-r)dr}, 0 \le s \le t < t_1 \tag{17}$$

When the Bellman-Gronwall inequality ([6]) is used, since, for a continuous $B : [0, \infty) \longrightarrow M_n(\mathbb{R})$, we have

$$\|e^{\int_0^s B(r)dr}\| = \| \int_0^s \frac{d}{dr}(e^{\int_0^r B(\eta)d\eta})dr + 1\| =$$
$$= \|1 + \int_0^s B(r)e^{\int_0^r B(\eta)d\eta}dr\| \le 1 + \int_0^s \|B(r)\|\|e^{\int_0^r B(\eta)d\eta}\|dr$$

it follows that

$$\|e^{\int_0^s B(r)dr}\| \le e^{\int_0^s \|B(r)\|dr}, s \ge 0.$$

Applied to (17) this gives, since $Y(0) = I_n$,

$$\|Y(s)\| \le e^{\int_0^s \|Z(t-r)\|dr} = e^{\int_{t-s}^t \|Z(\zeta)\|d\zeta}, 0 \le s \le t < t_1 \tag{18}$$

Remark that $\|A_k X(t-\tau_k)X^{-1}(t)\| = 0$ whenever $t < \tau_k$ and that, for $\tau_k > 0$ and $t \in [\tau_k, t_1), k = 0, ..., N$,

$$\|A_k X(t-\tau_k)X^{-1}(t)\| \le \|A_k\|\|X(t-\tau_k)X^{-1}(t)\| = \|A_k\|\|Y(\tau_k)\|$$

for $0 \le t < t_1$, hence,

$$v(t) \le \sum_{k=0}^N \|A_k X(t-\tau_k)X^{-1}(t)\| \le \sum_{k=0}^N \|A_k\|\|Y(\tau_k)\| \le$$
$$\le \sum_{k=0}^N \|A_k\| e^{\int_{t-\tau_k}^t \|Z(\zeta)\|d\zeta} = \sum_{k=0}^N \|A_k\| e^{\int_{t-\tau_k}^t v(\zeta)d\zeta} \le$$
$$\le \sum_{k=0}^N \|A_k\| e^{(x_0+\epsilon)\tau_k}, 0 \le t \le t_2$$

Once again, remark that $v(t) = 0$ if $t - \tau_k < 0, \forall k = 0, ..., N$. Then

$$v(t_2) = x_0 + \varepsilon \le \sum_{k=0}^N \|A_k\| e^{x_0\tau_k} \cdot e^{\varepsilon\tau_k} \tag{19}$$

By (9), $x_0 = \sum_{k=0}^N \|A_k\| e^{x_0\tau_k}$ so (19) gives

$$\sum_{k=0}^N \|A_k\| e^{x_0\tau_k} + \varepsilon \le \sum_{k=0}^N \|A_k\| e^{x_0\tau_k} e^{\varepsilon\tau_k} \Leftrightarrow$$

$$\Leftrightarrow \varepsilon \le \sum_{k=0}^N \|A_k\| e^{x_0\tau_k}(e^{\varepsilon\tau_k} - 1) = \sum_{k=0}^N \|A_k\| e^{x_0\tau_k} \varepsilon\tau_k h(\varepsilon\tau_k)$$

with $h(x) = \sum_{j=1}^{\infty} \frac{x^{j-1}}{j!}$ so $h(0) = 1$. Simplifying $\varepsilon, (\varepsilon > 0)$ one gets

$$\sum_{k=0}^{N} \|A_k\| e^{x_0 \tau_k} \tau_k h(\varepsilon \tau_k) \geq 1$$

and, for $\varepsilon \to 0$, this gives, by (5),

$$1 \leq \sum_{k=0}^{N} \|A_k\| e^{x_0 \tau_k} \tau_k \leq \tau \sum_{k=0}^{N} \|A_k\| e^{x_0 \tau_k} = \tau x_0$$

a contradiction to $\tau x_0 < 1$. It follows that $t_2 = t_1$ so (12) holds for all $t \in [0, t_1)$. Since

$$\lim_{\substack{t \to t_1 \\ t < t_1}} det(X^{-1}(t)) = \infty$$

it follows that

$$\lim_{\substack{t \to t_1 \\ t < t_1}} \|X^{-1}(t)\| = \lim_{\substack{t \to t_1 \\ t < t_1}} \|Y(t)\| = \infty$$

Consider now $Y(t) = X^{-1}(t), 0 \leq t < t_1$. Then

$$\dot{Y}(t) = -X^{-1}(t)\dot{X}(t)X^{-1}(t) = -Y(t) \sum_{k=0}^{N} A_k X(t - \tau_k) X^{-1}(t).$$

Since $Y(0) = I_n$ it follows that

$$Y(t) = e^{-\int_0^t \sum_{t=0}^{N} A_k X(s-\tau_k) X^{-1}(s) ds}.$$

As above, one gets

$$\|Y(t)\| \leq e^{\int_0^t \| \sum_{t=0}^{N} A_k X(s-\tau_k) X^{-1}(s) \| ds} = e^{\int_0^t v(s) ds} \leq e^{x_0 t_1}, 0 \leq t \leq t_1$$

so

$$\lim_{\substack{t \to t_1 \\ t < t_1}} \|Y(t)\| \leq e^{x_0 t_1} < \infty$$

a contradiction that proves $t_1 = \infty$ so that $det(X(t)) \neq 0, \forall t \geq 0$ and since $det(X(0)) > 0$ we have $det(X(t)) > 0, \forall t \geq 0$. □

Corollary 2.1. *(see*[13]*, Corollary 2.1) Consider the following scalar differential equation*

$$\dot{u}(t) = \sum_{k=0}^{N} a_k u(t - \tau_k), \tau_k \geq 0, k = \overline{0, N} \tag{20}$$

Define $\tau = \max\{\tau_0, ..., \tau_n\}$ *and suppose (6) if fulfilled. Then the solution* u_0 *of (20) with*

$$u(s) = 0, -\tau \leq s < 0, u(0) = 1 \tag{21}$$

is positive on $[0, \infty)$.

Proof. Proposition 2.1 and Theorem 2.1 ensures that the equation

$$x = \sum_{k=0}^{N} |a_k| e^{x\tau_k}$$

has a real root in the interval $(0, \frac{1}{\tau})$. Since $det(X(t)) = u_0(t)$ one has, by Theorem 2.1, that $u_0(t) > 0, \forall t \in [0, \infty)$. □

Consider again the problems (10), (11) and define, following [8] and [9], the Cauchy matrix as $C(t,s) = [c_{ij}(t,s)]_{1\leq i,j\leq n}$ so that, for fixed $s \geq 0$, it verifies

$$C'_t(t,s) = \sum_{k=0}^{n} A_k C(t-\tau_k, s), t \in [s, \infty)$$
$$C(\xi, s) = 0, \xi < s, C(s,s) = I_n$$

The solution of the non-homogeneous system

$$\dot{x} = \sum_{k=0}^{N} A_k x(t-\tau_k) + f(t), \;\; x(t) = 0, \;\; t < 0$$

is given by (see[9])

$$x(t) = \int_0^t C(t,s) f(s) ds + C(t,0)x(0),$$

If X_0 is a fundamental system of solutions for (10) (see[13])then $C(t,s) = X_0(t-s)$ is a Cauchy matrix.

Recall from [9], the followin definition.

Definition. A matrix $A \in \mathcal{M}(\mathbb{R}^n)$ is called a Metzler matrix if and only if $a_{ij} \geq 0, \forall i \neq j$.

Suppose, following [9], that the matrices $A_k, k = 0, \ldots, N$ in (10) are Metzler so $a^k_{ij} \geq 0, \forall i \neq j, \; \forall \; k = 0, \ldots, N$, where a^k_{ij} denotes the element a_{ij} in A_k.

The following Proposition is stated in [9].

Proposition 2.2. *(see*[9]*, Proposition 2.3). Suppose A_k is Metzler for every $k = \overline{0, N}$ and suppose that*

$$\tau \sum_{k=0}^{N} |a^k_{ii}| < \frac{1}{e}, \forall i = \overline{1, n}. \tag{22}$$

Then, if the matrix $A = A_0 + A_1 + ... + A_N$ is Hurwitz, (10) is exponentially stabile.

Proof. The proof of this proposition follows from Theorem 2 in [8] since, by Corrollary 1, the Cauchy functions of the scalar equations

$$\dot{x}_i = \sum_{k=0}^{N} a^k_{ii} x_i(t-\tau_k), x_i(s) = 0, s < 0, i = \overline{1, n}$$

are positive and then, since A is Hurwitz, the condition that equation (4) in [8] has a positive solution is satisfied and Theorem 1 in [8] concludes the exponential stability. It is to be remarked that the proof of Theorem 1 in [8] uses the Perron condition (see [14], Theorem 4.15.) □

3. **Flight control for ADMIRE**

The differential model used in this study is an aircraft model, ADMIRE (Aero-Data Model In a Research Environment) [12]. The Swedish Defence Research Agency built this model to provide a full aircraft model with six degrees of freedom to be utilized and widely disseminated by the scientific community.

The system of differential equations that governs the motion of ADMIRE [12], [5], [4] is:

$$
\begin{aligned}
\dot{\alpha} = {} & q - p\beta + \frac{g}{V} cos\theta cos\psi + z_\alpha \alpha + y_\beta \beta^2 + y_p(\alpha,\beta)p\beta + y_r(\beta) r\beta + \\
& + y_{\delta_a}\beta\delta_a + z_{\delta_e}\delta_e + y_{\delta_r}\beta\delta_r, \\
\dot{\beta} = {} & p\alpha - r + \frac{g}{V}\sin\phi\cos\theta - z_\alpha\alpha\beta + y_\beta - y_p(\alpha,\beta)p - y_r(\beta) r \\
& + y_{\delta_a}\delta_a - z_{\delta_e} + y_{\delta_r}\delta_r \\
\dot{p} = {} & -i_1 qr + l_\beta(\alpha)\beta + l_p p + l_r(\alpha) r + l_{\delta_\alpha}\delta_\alpha + l_{\delta_r}\delta_r \\
\dot{q} = {} & i_2 pr + m_\alpha \alpha + m_q q - \overline{m_{\dot{\alpha}}} p\beta + \overline{y_p} p\beta + \overline{y_\beta}\beta^2 + \overline{y_r}(\beta) r\beta + \\
& + \frac{g}{V}(\overline{m_{\dot{\alpha}}}\cos\theta\cos\phi - \frac{c_2}{\alpha}\alpha_2\sin\theta) + \overline{y_{\delta_\alpha}}\beta\delta_\alpha + m_{\delta_e}\delta_e + \overline{y_{\delta_r}}\beta\delta_r \\
\dot{r} = {} & -i_3 pq + n_\beta\beta + n_p(\alpha,\beta)p + n_r(\alpha,\beta) r + n_{\delta_\alpha}\delta_\alpha + n_{\delta_{c\alpha}}(\alpha)\delta_{c\alpha} + \\
& + n_{\delta_r}\delta_r \\
\dot{\phi} = {} & p + (q\sin\phi + r\cos\phi)\tan\theta \\
\dot{\theta} = {} & q\cos\phi - r\sin\phi \\
\dot{\psi} = {} & \frac{q\sin\phi + r\cos\phi}{\cos\theta}
\end{aligned}
\qquad (23)
$$

The system (23) is obtained from the general equations of motion [7] of an aircraft, replacing the general aerodynamic forces and moments with those related to ADMIRE [5], considering α and β small and having the next approximations [5].

$$\cos\beta \approx 1; \quad -p\cos\alpha\tan\beta \approx -p\beta; \quad -r\sin\alpha\tan\beta \approx 0; \quad \cos\alpha \approx 1; \quad \sin\alpha \approx 0$$

$$\cos\beta \approx 1$$

$$\frac{I_{xz}(I_x + I_z - I_y)}{I_x I_z - I_{xz}^2} \approx 0; \quad \frac{(I_y - I_z)I_z - I_{xz}^2}{I_x I_z - Ixz^2} \approx -i_1 \quad \frac{I_{xz}}{I_y} \approx 0;$$

$$\frac{I_{xz}^2 + (I_x - I_y)I_x}{I_x I_z - I_{xz}^2} \approx -i_3; \quad \frac{I_{xz}(I_y - I_z - I_{xz})}{I_x I_z - I_{xz}^2} \approx 0$$

Longitudinal flights are known as flights for which

$$\beta \equiv p \equiv r \equiv \phi \equiv \psi \equiv 0, \quad \delta_a = \delta_r = 0$$

From [4], we have the following result:

Proposition 3.1. *([4], Proposition 2.7) A longitudinal flight is possible if and only if* $Y = L = N = 0$ *for* $\beta = p = r = \phi = \psi = 0$ *and* $\delta_a = \delta_r = 0$.

According to [4], [3] and [5], the simplified system of differential equations that governs the motion around the center of mass in a longitudinal flight with a constant forward velocity of ADMIRE is given by ordinary differential equations:

$$\begin{aligned} \dot{\alpha} &= m_{11}\alpha + m_{12}q + c\cos\theta + b_1\delta_e \\ \dot{q} &= m_{21}\alpha + m_{22}q + cm_0\cos\theta - cc_1\sin\theta + b_2\delta_e \\ \dot{\theta} &= q \end{aligned} \tag{24}$$

Here, the state parameters are α - angle of attack, q - pitch rate, Euler pitch angle - θ and the control parameter is the elevator deflection δ_e. Let $(\alpha_0, 0, \theta_0)$ be an equilibrium point of the uncontrolled system (24) (i.e. when $\delta_e = 0$). Then it verifies

$$\begin{cases} m_{11}\alpha + ccos\theta = 0 \\ m_{21}\alpha + cm_0cos\theta - cc_1sin\theta = 0 \end{cases}$$

Perform a translation to zero introducing $x_1 = \alpha - \alpha_0, x_2 = q, x_3 = \theta - \theta_0$. To be able to study the effect of delays in this model, we will introduce two delays in the control parameter (see also [2]), and δ_e becomes a feedback control with delays:

$$\delta_e = k_1x_1(t-\tau_1) + k_2x_2(t-\tau_2) + k_3x_3(t-\tau_2).$$

Now, the longitudinal flight of ADMIRE can be described by a system of DDEs with a feedback control

$$\begin{aligned} \dot{x}_1 &= m_{11}(x_1+\alpha_0) + m_{12}x_2 + c\cos(x_3+\theta_0) + b_1k_1x_1(t-\tau_1) + \\ &\quad + b_1k_2x_2(t-\tau_2) + b_1k_3x_3(t-\tau_2) \\ \dot{x}_2 &= m_{21}(x_1+\alpha_0) + m_{22}x_2 + cm_0\cos(x_3+\theta_0) - cc_1\sin(x_3+\theta_0) + \\ &\quad + b_2k_1x_1(t-\tau_1) + b_2k_2x_2(t-\tau_2) + b_2k_3x_3(t-\tau_2) \\ \dot{x}_3 &= x_2. \end{aligned} \tag{25}$$

We next linearize the system around the equilibrium point and analyze its stability.

Define $A_0 = (a_{ij})_{i,j=\overline{1,3}}$ as the Jacobian matrix of the system with respect to x_1, x_2, x_3, $A_1 = (b_{ij})_{i,j=\overline{1,3}}$ the matrix of the derivatives with respect to $x_1(t-\tau_1)$ and $A_2 = (c_{ij})_{i,j=\overline{1,3}}$ the matrix of the derivatives with respect to $x_2(t-\tau_2)$ and $x_3(t-\tau_2)$.

We impose the conditions that A_0, A_1, A_2 are Metzler and the matrix $A = A_0 + A_1 + A_2$ is Hurwitz and apply then Proposition 2. Using also the Theorem of Stability by the First Approximation (see, e.g., [14], Th. 1.7.), we obtain the following stability result.

Proposition 3.2. *Suppose that A_0, A_1, A_2 are Metzler matrices and that the matrix $A = A_0 + A_1 + A_2$ is Hurwitz. Define $\tau = max(\tau_1, \tau_2)$. Suppose that the following conditions are satisfied*

$$\tau(|m_{11}| + |b_1 k_1|) < \frac{1}{e}, \tau(|m_{22} + |b_2 k_2|) < \frac{1}{e}. \tag{26}$$

Then, for system (24), with

$$\delta_e = k_1(\alpha(t - \tau_1) - \alpha_0) + k_2 q(t - \tau_2) + k_3(\theta(t - \tau_2) - \theta_0).$$

the equilibrium point $(\alpha_0, 0, \theta_0)$ is asymptotically stable.

4. Conclusions

This paper presents a stability theorem for linear systems of delay differential equations with small delays, representing an important result for systems where even the small delays are critical, such as the delays that can occur in the control of an aircraft.

The proof of the stability theorem uses the fact that, if all the matrices of partial derivatives with respect to undelayed state variables and to delayed state variables are Metzler, their sum is Hurwitz and the delays verify the condition (22), then the system is asymptotically stable.

The theory presented in this paper was applied to a model that describes the longitudinal flight of an aircraft. Conditions for the design of a stabilizing controller are deduced. The study underlines that we can get a stable longitudinal flight when the delays are not too large, imposing certain conditions on the parameters governing the motion of the aircraft.

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