A GENERALIZATION OF COMPLETE AND ELEMENTARY SYMMETRIC FUNCTIONS - PART II

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In this paper, we investigate new properties of the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$ which were introduced in the first part of our study. Combinatorial interpretations of these generalized symmetric functions are also introduced.

Keywords: symmetric functions, complete homogeneous symmetric functions, elementary symmetric functions, power sum symmetric functions

MSC2010: 05E05 11T06 05A10.

1. Introduction

The generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$ were introduced in the first part of our study [1] as follows:

$$\sum_{k=0}^{\infty} H_k^{(s)}(x_1, x_2, \ldots, x_n)\, t^k = \prod_{i=1}^{n} \big(1 - x_i t + \cdots + (-x_i t)^s\big)^{-1} \tag{1}$$

and

$$\sum_{k=0}^{\infty} E_k^{(s)}(x_1, x_2, \ldots, x_n)\, t^k = \prod_{i=1}^{n} \big(1 + x_i t + \cdots + (x_i t)^s\big), \tag{2}$$

where s is a positive integer. For the sake of brevity, we do not mention once again the notions and notations used in the first part of our study. So these remain valid and we will use them without further explanation.

In this paper, we investigate new properties of the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$. In Section 2, we show that the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$ can be expressed in terms of the complete and elementary symmetric functions. In Section 3, we consider some combinatorial interpretations for the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$.

2. Generalized symmetric functions in terms of the complete and elementary symmetric functions

It is well known that every symmetric function can be expressed as a sum of homogeneous symmetric functions. The homogeneous symmetric functions of degree k in n variables form a vector space, denoted Λ_n^k. There are several important bases for Λ_n^k, which are indexed by integer partitions of k. Proofs and details about these facts can be found in Macdonald's book [4]. In this section, we express the generalized symmetric functions $H_k^{(s)}$ and $E_k^{(s)}$ in terms of the complete and elementary symmetric functions. To do this, we

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consider the monomial symmetric function $m_\lambda(x_1, x_2, \dots, x_n)$ and for each partition λ we note

$$f_\lambda(x_1, x_2, \dots, x_n) = \prod_{i=1}^{\ell(\lambda)} f_{\lambda_i}(x_1, x_2, \dots, x_n),$$

where f is any of these complete or elementary symmetric functions.

Theorem 2.1. *Let k and s be two positive integers. Then*

$$H_k^{(s)} = (-1)^k \sum_{\substack{\lambda \vdash k \\ l(\lambda) \le s}} m_\lambda(\omega_{1,s+1}, \omega_{2,s+1}, \dots, \omega_{s,s+1}) h_\lambda,$$

$$E_k^{(s)} = (-1)^k \sum_{\substack{\lambda \vdash k \\ l(\lambda) \le s}} m_\lambda(\omega_{1,s+1}, \omega_{2,s+1}, \dots, \omega_{s,s+1}) e_\lambda,$$

where $\omega_{j,s+1} = e^{2j\pi i/(s+1)}$ with $j = 1, 2, \dots, s$.

Proof. Taking into account the generating functions [1, eq. (1)] and (1), we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k^{(s)}(x_1, x_2, \dots, x_n) t^k &= \prod_{i=1}^{n} \prod_{j=1}^{s} (1 + \omega_{j,s+1} x_i t)^{-1} = \prod_{j=1}^{s} \prod_{i=1}^{n} (1 + \omega_{j,s+1} x_i t)^{-1} \\
&= \prod_{j=1}^{s} \sum_{k=0}^{\infty} (-\omega_{j,s+1})^k h_k(x_1, x_2, \dots, x_n) t^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{\substack{j_1+j_2+\dots+j_s=k \\ j_i \ge 0}} (-1)^{j_1+j_2+\dots+j_s} \omega_{1,s+1}^{j_1} \omega_{2,s+1}^{j_2} \cdots \omega_{s,s+1}^{j_s} h_{j_1} h_{j_2} \dots h_{j_s} \right) t^k \\
&= \sum_{k=0}^{\infty} \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \le s}} (-1)^k m_\lambda (\omega_{1,s+1}, \omega_{2,s+1}, \dots, \omega_{s,s+1}) h_\lambda(x_1, x_2, \dots, x_n) t^k.
\end{aligned}$$

The second identity follows in a similar way, considering the generating functions [1, eq. (2)] and (2). □

The Ferrers diagram of a partition $[\lambda_1, \lambda_2, \dots, \lambda_k]$ is the k-row left-justified array of dots with λ_i dots in the i-th row. The conjugate of a partition into s parts, obtained by transposing the Ferrers diagram, is a partition with largest part s and vice versa. The action of conjugation establishes a 1–1 correspondence between partitions into s parts and partitions with largest part s. Considering [1, eq. (5)] and Theorem 2.1, we obtain a surprising identity involving this 1–1 correspondence between partitions into s parts and partitions with largest part s.

Corollary 2.1. *Let k, n and s be positive integers and let $x_1, x_2, \dots, x_n$ be independent variables. Then*

$$\sum_{\substack{\lambda \vdash k \\ \lambda_1 \le s}} m_\lambda(x_1, x_2, \dots, x_n) = (-1)^k \sum_{\substack{\lambda \vdash k \\ l(\lambda) \le s}} m_\lambda(\omega_{1,s+1}, \omega_{2,s+1}, \dots, \omega_{s,s+1}) e_\lambda(x_1, x_2, \dots, x_n).$$

where $\omega_{j,s+1} = e^{2j\pi i/(s+1)}$ with $j = 1, 2, \dots, s$.

The following result allows us to express the symmetric function $H_k^{(s)}$ and $E_k^{(s)}$ as convolutions of the complete and elementary symmetric functions.

Theorem 2.2. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$H_k^{(s-1)}(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{\lfloor k/s \rfloor} (-1)^{sj} h_j(x_1^s, x_2^s, \ldots, x_n^s) e_{k-sj}(x_1, x_2, \ldots, x_n),$$

$$E_k^{(s-1)}(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{\lfloor k/s \rfloor} (-1)^{j} e_j(x_1^s, x_2^s, \ldots, x_n^s) h_{k-sj}(x_1, x_2, \ldots, x_n).$$

Proof. According to (1), we have

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k^{(s-1)}(x_1, x_2, \ldots, x_n) t^k &= \left(\prod_{i=1}^{n} \frac{1}{1-(-x_i t)^s} \right) \left(\prod_{i=1}^{n} (1 + x_i t) \right) \\
&= \left(\sum_{j=0}^{\infty} h_j(x_1^s, x_2^s, \ldots, x_n^s)(-t)^{sj} \right) \left(\sum_{j=0}^{\infty} e_j(x_1, x_2, \ldots, x_n) t^j \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\lfloor k/s \rfloor} (-1)^{sj} h_j(x_1^s, x_2^s, \ldots, x_n^s) e_{k-sj}(x_1, x_2, \ldots, x_n) \right) t^k.
\end{aligned}$$

In a similar way, considering (2), we obtain the second identity. □

Corollary 2.2. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$\sum_{j=0}^{\lfloor k/s \rfloor} (-1)^{sj} h_j(x_1^s, x_2^s, \ldots, x_n^s) e_{k-sj}(x_1, x_2, \ldots, x_n) = (-1)^k \sum_{\substack{\lambda \vdash k \\ l(\lambda) < s}} m_\lambda(\omega_{1,s}, \omega_{2,s}, \ldots, \omega_{s-1,s}) h_\lambda(x_1, x_2, \ldots, x_n),$$

$$\sum_{j=0}^{\lfloor k/s \rfloor} (-1)^{j} e_j(x_1^s, x_2^s, \ldots, x_n^s) h_{k-sj}(x_1, x_2, \ldots, x_n) = (-1)^k \sum_{\substack{\lambda \vdash k \\ l(\lambda) < s}} m_\lambda(\omega_{1,s}, \omega_{2,s}, \ldots, \omega_{s-1,s}) e_\lambda(x_1, x_2, \ldots, x_n).$$

We remark that the second identity of this corollary is known and can be seen in a recent paper of Merca [6, Theorem 1.1].

Now, we are able to prove some formulas for the monomial symmetric function $m_\lambda(e^{2\pi i/(s+1)}, e^{4\pi i/(s+1)}, \ldots, e^{2s\pi i/(s+1)})$, when λ is a partition of k, $s \geq k-2$ and $\ell(\lambda) \leq s$.

Corollary 2.3. *Let k be a positive integer and let $\lambda = [1^{t_1} 2^{t_2} \ldots k^{t_k}]$ be a partition of k. Then*

$$m_\lambda(\omega_{1,k+1}, \omega_{2,k+1}, \ldots, \omega_{k,k+1}) = (-1)^{t_1+t_2+\cdots+t_k} \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \ldots, t_k},$$

where $\omega_{j,k+1} = e^{2j\pi i/(k+1)}$ with $j = 1, 2, \ldots, k$.

Proof. The case $s = k+1$ of Theorem 2.2 reads as $H_k^{(k)} = e_k$. By Theorem 2.1, we deduce that

$$e_k = \sum_{\lambda \vdash k} (-1)^k m_\lambda(\omega_{1,k+1}, \omega_{2,k+1}, \ldots, \omega_{k,k+1}) h_\lambda.$$

On the other hand, the relation

$$e_k = \sum_{\lambda\vdash k} (-1)^{k+\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda$$

can be found in [5, pp. 3-4]. It is clear that

$$\sum_{\lambda\vdash k} m_\lambda(\omega_{1,k+1}, \omega_{2,k+1}, \ldots, \omega_{k,k+1}) h_\lambda = \sum_{\lambda\vdash k} (-1)^{\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda.$$

The assertion of the corollary now follows by comparing coefficients of h_λ on both sides of this equation. □

Corollary 2.4. *Let $k > 1$ be a positive integer and let $\lambda = [1^{t_1} 2^{t_2} \ldots k^{t_k}]$ be a partition of k with $\ell(\lambda) < k$. Then*

$$m_\lambda(\omega_{1,k}, \omega_{2,k}, \ldots, \omega_{k-1,k}) = (-1)^{\ell(\lambda)} \left(1 - \frac{k}{\ell(\lambda)}\right) \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k},$$

where $\omega_{j,k} = e^{2j\pi i/k}$ with $j = 1, 2, \ldots, k-1$.

Proof. The case $s = k$ of Theorem 2.2 reads as $H_k^{(k-1)} = e_k + (-1)^k p_k$. By Theorem 2.1, we deduce that

$$e_k = \sum_{\substack{\lambda\vdash k \\ \ell(\lambda)<k}} (-1)^k m_\lambda(\omega_{1,k}, \omega_{2,k}, \ldots, \omega_{k-1,k}) h_\lambda - (-1)^k p_k.$$

On the other hand, we have

$$e_k = \sum_{\lambda\vdash k} (-1)^{k+\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda,$$

$$p_k = \sum_{\lambda\vdash k} \frac{(-1)^{1+\ell(\lambda)} \cdot k}{\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda.$$

We can write

$$\begin{aligned}
&\sum_{\substack{\lambda\vdash k \\ \ell(\lambda)<k}} (-1)^k m_\lambda(\omega_{1,k}, \omega_{2,k}, \ldots, \omega_{k-1,k}) h_\lambda \\
&\quad = \sum_{\lambda\vdash k} (-1)^{k+\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda - \sum_{\lambda\vdash k} \frac{(-1)^{k+\ell(\lambda)} \cdot k}{\ell(\lambda)} \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda \\
&\quad = \sum_{\lambda\vdash k} (-1)^{k+\ell(\lambda)} \left(1 - \frac{k}{\ell(\lambda)}\right) \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k} h_\lambda
\end{aligned}$$

and the proof is finished. □

Corollary 2.5. *Let $k > 2$ be a positive integer and let $\lambda = [1^{t_1} 2^{t_2} \ldots k^{t_k}]$ be a partition of k with $\ell(\lambda) \le k-2$. Then*

$$m_\lambda(\omega_{1,k-1}, \omega_{2,k-1}, \ldots, \omega_{k-2,k-1}) = (-1)^{\ell(\lambda)} \left(1 - \frac{t_1 \cdot (k-1)}{\ell(\lambda)^2 - \ell(\lambda)}\right) \binom{\ell(\lambda)}{t_1, t_2, \ldots, t_k},$$

where $\omega_{j,k-1} = e^{2j\pi i/(k-1)}$ with $j = 1, 2, \ldots, k-2$.

Proof. By Theorem 2.1, we deduce that

$$H_k^{(k-2)} = \sum_{\substack{\lambda\vdash k \\ \ell(\lambda)\le k-2}} (-1)^k m_\lambda(\omega_{1,k-1}, \omega_{2,k-1}, \ldots, \omega_{k-2,k-1}) h_\lambda.$$

On the other hand, taking into account the case $s = k-1$ of Theorem 2.2, we can write

$$\begin{aligned}
H_k^{(k-2)} &= e_k + (-1)^{k-1} p_{k-1} h_1 \\
&= \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k} \\
&+ \sum_{t_1+2t_2+\cdots+(k-1)t_{k-1}=k-1} \frac{(-1)^{k+t_1+\cdots+t_{k-1}}(k-1)}{t_1+\cdots+t_{k-1}} \binom{t_1+\cdots+t_{k-1}}{t_1,\ldots,t_{k-1}} h_1^{1+t_1} h_2^{t_2} \cdots h_{k-1}^{t_{k-1}} \\
&= \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k} \\
&- \sum_{\substack{t_1+2t_2+\cdots+kt_k=k \\ t_1>0}} \frac{(-1)^{k+t_1+t_2+\cdots+t_k}(k-1)}{t_1+t_2+\cdots+t_k-1} \binom{t_1+t_2+\cdots+t_k-1}{t_1-1,t_2,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k} \\
&= \sum_{t_1+2t_2+\cdots+kt_k=k} (-1)^{k+t_1+t_2+\cdots+t_k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k} \\
&- \sum_{\substack{t_1+2t_2+\cdots+kt_k=k \\ t_1>0}} \frac{(-1)^{k+t_1+t_2+\cdots+t_k} \cdot t_1 \cdot (k-1)}{(t_1+\cdots+t_k-1)(t_1+\cdots+t_k)} \binom{t_1+\cdots+t_k}{t_1,\ldots,t_k} h_1^{t_1} h_2^{t_2} \cdots h_k^{t_k} \\
&= (-1)^k \sum_{\lambda \vdash k} (-1)^{\ell(\lambda)} \left(1 - \frac{t_1 \cdot (k-1)}{\big(\ell(\lambda)-1\big)\ell(\lambda)} \right) \binom{\ell(\lambda)}{t_1,t_2,\ldots,t_k} h_\lambda .
\end{aligned}$$

This concludes the proof. □

Inspired by Theorem 2.2, we provide the following result.

Theorem 2.3. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$h_k(x_1^s, x_2^s, \ldots, x_n^s) = (-1)^{k(s+1)} \sum_{j=0}^{ks} (-1)^j h_j(x_1,x_2,\ldots,x_n) H_{ks-j}^{(s-1)}(x_1,x_2,\ldots,x_n),$$

$$e_k(x_1^s, x_2^s, \ldots, x_n^s) = (-1)^{k} \sum_{j=0}^{ks} (-1)^j e_j(x_1,x_2,\ldots,x_n) E_{ks-j}^{(s-1)}(x_1,x_2,\ldots,x_n).$$

If k is not congruent to 0 modulo s then

$$\sum_{j=0}^{k} (-1)^j h_j(x_1,x_2,\ldots,x_n) H_{k-j}^{(s-1)}(x_1,x_2,\ldots,x_n) = 0,$$

$$\sum_{j=0}^{k} (-1)^j e_j(x_1,x_2,\ldots,x_n) E_{k-j}^{(s-1)}(x_1,x_2,\ldots,x_n) = 0.$$

Proof. Considering that

$$\prod_{i=1}^{n} \frac{1}{1+x_i t} \sum_{k=0}^{\infty} H_k^{(s-1)}(x_1,x_2,\ldots,x_n) t^k = \prod_{i=1}^{n} \frac{1}{1+(-x_i t)^s},$$

$$\prod_{i=1}^{n} (1-x_i t) \sum_{k=0}^{\infty} E_k^{(s-1)}(x_1,x_2,\ldots,x_n) t^k = \prod_{i=1}^{n} \big(1-(x_i t)^s\big),$$

the proof follows easily. □

3. **Combinatorial interpretations**

Bazeniar et *al.* [2] showed that the generalized symmetric function $E_k^{(s)}$ is interpreted as weight-generating function of the lattice paths between the points $u = (0,0)$ and $v = (k, n-1)$ with at most s vertices in the eastern direction. For example, the paths from $(0,0)$ to $(3,2)$ associated to

$$E_3^{(2)}(x_1,x_2,x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3$$

can be seen in Figure 1.

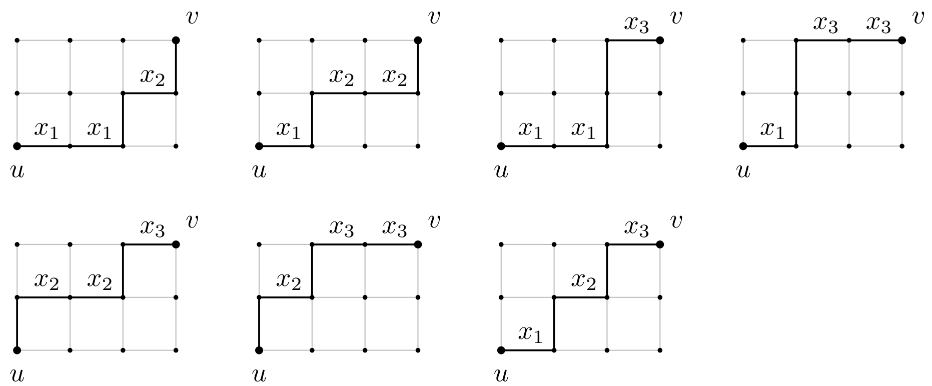

FIGURE 1. The seven paths from u to v associated to $E_3^{(2)}(x_1,x_2,x_3)$.

According to [2, Theorem 3.2], the number of lattice paths from $(0,0)$ to $(k, n-1)$ taking at most s vertices in the eastern direction is exactly the bi^{s}nomial coefficient, i.e.,

$$\binom{n}{k}_s = E_k^{(s)}(\underbrace{1,1,\ldots,1}_{n}).$$

By Theorem 2.2, we deduce that the bi^{s}nomial coefficient can be expressed in terms of the classical binomial coefficients, i.e.,

$$\binom{n}{k}_{s-1} = \sum_{j=0}^{\lfloor k/s \rfloor} (-1)^j \binom{n}{j}\binom{n+k-sj-1}{k-sj}.$$

We remark that this identity is given by Theorem 2.1 in [3]. In addition, by Theorem 2.2 we obtain the following analogs of this identity.

Corollary 3.1. *Let k, n and s be positive integers. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s-1)} = \sum_{j=0}^{\lfloor k/s \rfloor} (-1)^j q^{s\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{q^s} \begin{bmatrix} n+k-sj-1 \\ k-sj \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)} = E_k^{(s)}(1, q, \ldots, q^{n-1})$ is the q-bi^{s}nomial coefficient.

Corollary 3.2. *Let k, n and s be positive integers. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}^{(s-1)} = \sum_{j=0}^{\lfloor k/s \rfloor} (-1)^j p^{s\binom{n-j}{2}} q^{s\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p^s,q^s} \begin{bmatrix} n+k-sj-1 \\ k-sj \end{bmatrix}_{p,q},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}^{(s)} = E_k^{(s)}(p^{n-1}, p^{n-2}q, \ldots, q^{n-1})$ is the p, q-bi^{s}nomial coefficient.

These expressions of the q-bi^{s}nomial (rep. p,q-bi^{s}nomial) coefficient in terms of q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (resp. p,q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$) seem to be new.

And Theorem 2.3 allow us to express the binomial coefficient and its q^s-analogue in term of the bi^{s}niomial coefficient and its q-analogue, respectively.

Corollary 3.3. *Let k, n and s be positive integers. Then*

$$\binom{n}{k} = \sum_{j=0}^{ks} (-1)^{k+j} \binom{n}{j} \binom{n}{ks-j}_{s-1},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^s} = \sum_{j=0}^{ks} (-1)^{k+j} q^{\binom{j}{2} - s\binom{k}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ ks-j \end{bmatrix}_q^{(s-1)}.$$

Inspired by this interpretation of the generalized symmetric function $E_k^{(s)}$, we provide in this section a combinatorial interpretation for the generalized symmetric function $H_k^{(s)}$. To do this we consider the following result which allows us to express the generalized symmetric function $H_k^{(s)}$ in terms of the monomial symmetric functions m_λ considering all the partitions of k into parts congruent to 0 or 1 modulo $s+1$.

Theorem 3.1. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$H_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{\substack{\lambda \vdash k \\ \lambda_i \equiv \{0,1\} \bmod (s+1)}} (-1)^{k + \sum_{i=1}^{\ell(\lambda)} \lambda_i \bmod (s+1)} m_\lambda(x_1, x_2, \ldots, x_n).$$

Proof. According to (1), we can write

$$\sum_{k=0}^{\infty} H_k^{(s)}(x_1, x_2, \ldots, x_n) t^k = \prod_{i=1}^{n} \frac{1 + x_i t}{1 - (-x_i t)^{s+1}} = \prod_{i=1}^{n} (1 + x_i t) \sum_{j=0}^{\infty} (-x_i t)^{j(s+1)}$$

$$= \prod_{i=1}^{n} \left(\sum_{j=0}^{\infty} (-x_i t)^{j(s+1)} - \sum_{j=0}^{\infty} (-x_i t)^{j(s+1)+1} \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{\substack{\lambda \vdash k \\ \lambda_i \equiv \{0,1\} \bmod (s+1)}} (-1)^{\sum_{i=1}^{\ell(\lambda)} \lambda_i \bmod (s+1)} m_\lambda(x_1, x_2, \ldots, x_n) \right) (-t)^k$$

and the proof follows easily. □

Remark 3.1. *When s is odd, we have*

$$H_k^{(s)}(x_1, x_2, \ldots, x_n) = \sum_{\substack{\lambda \vdash k \\ \lambda_i \equiv \{0,1\} \bmod (s+1)}} m_\lambda(x_1, x_2, \ldots, x_n).$$

The following consequence of Theorem 3.1 is an analogy of Corollary 2.1 establishing a connection between all the partitions of k into parts congruent to 0 or 1 modulo $s+1$ and the partitions of k into at most s parts.

Corollary 3.4. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$\sum_{\substack{\lambda \vdash k \\ \lambda_i \equiv \{0,1\} \bmod (s+1)}} (-1)^{\sum_{i=1}^{\ell(\lambda)} \lambda_i \bmod (s+1)} m_\lambda(x_1, x_2, \ldots, x_n)$$
$$= \sum_{\substack{\lambda \vdash k \\ l(\lambda) \leq s}} m_\lambda(\omega_{1,s+1}, \omega_{2,s+1}, \ldots, \omega_{s,s+1}) h_\lambda(x_1, x_2, \ldots, x_n).$$

Let $\mathcal{P}^s_{n,k}$ be the set of the lattice paths between the points $u = (0,0)$ and $v = (k, n-1)$ where the number of the vertices in the eastern direction is congruent to 0 or 1 modulo $s+1$. For $P = (p_1, p_2, \ldots, p_{n+k-1}) \in \mathcal{P}^s_{n,k}$, we consider

$$n_i(P) := \text{ the number of the eastern step modulo } (s+1) \text{ in level } i.$$

and the $H^{(s)}$-labeling which assigns the label for each eastern step as follows

$$L(p_i) := (\text{the number of northern } p_j \text{ preceding } p_i) + 1.$$

Figure 2 shows the $H^{(s)}$-labeling.

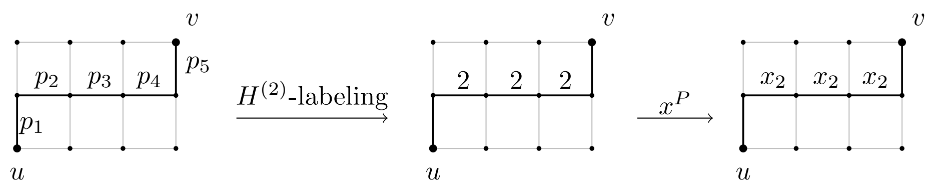

FIGURE 2. Illustration of x_2^3 by $H^{(2)}$-labeling.

Theorem 3.2. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then*

$$H_k^{(s)}(x_1, x_2, \ldots, x_n) = \begin{cases} \sum_{P \in \mathcal{P}^s_{n,k}} X^P, & \text{if } s \text{ odd,} \\ (-1)^k \sum_{P \in \mathcal{P}^s_{n,k}} (-1)^{P'} X^P, & \text{otherwise} \end{cases}$$

with $X^P = \prod_i x_{L(p_i)}$ and $P' = \sum_i n_i(P)$.

Proof. From Theorem 3.1, is easy to see that the generalized symmetric function $H_k^{(s)}(x_1, x_2, \ldots, x_n)$ is a weight-generating function of lattice paths between two points. For each unit variable x_i in this symmetric function we associate one unit horizontal (east) vertex, and if we suppose that each lattice path starting in $u = (0,0)$ then it ends in $v = (k, n-1)$ where the number of the vertices in the eastern direction equal to 0 or 1 modulo $(s+1)$. □

Figure 3 shows the lattice path interpretation for

$$H_3^{(2)}(x_1, x_2, x_3) = -x_1^3 - x_2^3 - x_3^3 + x_1 x_2 x_3.$$

By setting $s = k$ in Theorem 3.2, we will have the following result.

Corollary 3.5. *Let k, n be two positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then, the elementary symmetric function $e_k(x_1, x_2, \ldots, x_n)$ is a weight-generating function of the paths between the points $u = (0,0)$ and $v = (k, n-1)$ with at most one vertex in the eastern direction.*

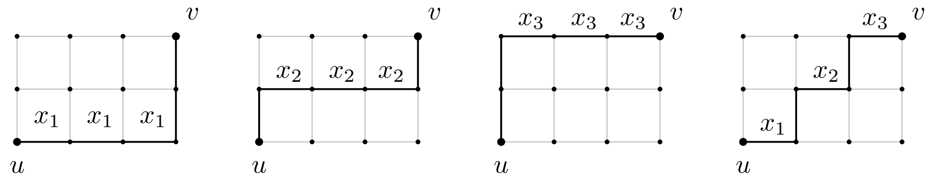

FIGURE 3. The four paths from u to v associated to $H_3^{(2)}(x_1, x_2, x_3)$.

As we can see in [2], the generalized symmetric functions $E_k^{(s)}$ can be interpreted considering the set of all tilings of an $(n + k - 1)$-board using exactly k black squares and $n - 1$ gray squares with at most s black squares successively. There is an obvious bijection between this tiling interpretation and the lattice path interpretation. This tiling interpretation for the generalized symmetric functions $E_k^{(s)}$ can be adapted to the generalized symmetric functions $H_k^{(s)}$ in the following way.

Let $\mathcal{T}_{n,k}^s$ be the set of all tilings of an $(n+k-1)$-board using exactly k black squares and $n-1$ gray squares such that the number of successive black squares is congruent to 0 or 1 modulo $s+1$. Also let $X^{w_T} = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$ be the weight of tiling T. For each $T \in \mathcal{T}_{n,k}^s$, we calculate $w_T = (w_1, w_2 \ldots, w_n)$ as follows:

(1) Assign a weight to each individual square in the tiling. A gray square always receives a weight of 1. A black square has weight x_{m+1} where m is equal to the number of gray squares to the left of that black square in the tiling.
(2) Calculate $w_T = (w_1, w_2 \ldots, w_n)$ by multiplying the weight x_{m+1} of all the black squares.

We also consider

$n_m(T)$:=the number of successive black squares modulo $(s + 1)$ after the m-th gray square to the left of these black squares in the tiling.

For example, the weight of the tiling bbgbg is $x_1^{1+1} x_2^1 = x_1^2 x_2$. Figure 4 shows this tiling and its lattice path.

FIGURE 4. A tiling of the weight $x_1^2 x_2$ and its associated lattice path.

Theorem 3.3. *Let k, n and s be positive integers and let $x_1, x_2, \ldots, x_n$ be independent variables. Then $H_k^{(s)}(x_1, x_2, \ldots, x_n)$ is created by summing the weights of all tilings of $\mathcal{T}_{n,k}^s$. That is,*

$$H_k^{(s)}(x_1, x_2, \ldots, x_n) = \begin{cases} \sum_{T \in \mathcal{T}_{n,k}^s} X^{w_T}, & \text{if } s \text{ odd,} \\ (-1)^k \sum_{T \in \mathcal{T}_{n,k}^s} (-1)^G X^{w_T}, & \text{otherwise} \end{cases}$$

with $G = \sum_{T \in \mathcal{T}_{n,k}^s} n_m(T)$.

Proof. Since the bijection between lattice paths and tiling is weight-preserving. Then, from Theorem 3.2 it is suffice to associate a lattice path to each $(n + k - 1)$-tiling using k black squares and $n - 1$ gray squares with the number of successive black squares congruent to 0 or 1 modulo $(s+1)$. This lattice path starts from in $u=(0,0)$ and ends in $v=(k,n-1)$ where the number of the vertices in the eastern direction is congruent to 0 or 1 modulo $(s+1)$ whose each gray tile represents a move one unit up and each black square represents a move one unit right. □

Figure 5 shows the tiling interpretation for $H_3^{(2)}(x_1,x_2,x_3)=-x_1^3-x_2^3-x_3^3+x_1x_2x_3$.

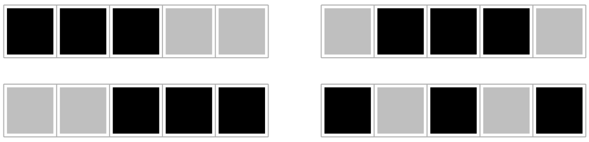

FIGURE 5. The four tilings associated to $H_3^{(2)}(x_1,x_2,x_3)$.

By Theorem 3.3, we can also interpret the elementary symmetric function as follows.

Corollary 3.6. *Let k,n be two positive integers and let $x_1,x_2,\ldots,x_n$ be independent variables. Then, the elementary symmetric function $e_k(x_1,x_2,\ldots,x_n)$ is a weight-generating function of all tilings of an $(n+k-1)$-board using exactly k black squares and $n-1$ gray squares with at most one black square successively.*

4. Concluding remarks

New properties of a pair of two symmetric functions which generalize the complete and elementary symmetric functions were investigated in this paper. These generalized symmetric functions satisfy many of the classical relations between complete and elementary symmetric functions. Most of these relationships have the same shape.

The Schur symmetric functions $s_\lambda(x_1,x_2,\ldots,x_n)$ for a partition λ can be extended in the same way. For example, we can define the generalized Schur symmetric function $s_\lambda^{(s)}=s_\lambda^{(s)}(x_1,x_2,\ldots,x_n)$ in terms of the generalized symmetric functions $H_k^{(s)}=H_k^{(s)}(x_1,x_2,\ldots,x_n)$ or $E_k^{(s)}=E_k^{(s)}(x_1,x_2,\ldots,x_n)$ as follows:

$$s_\lambda^{(s)}:=\det(H_{\lambda_i-i+j}^{(s)})_{1\le i,j\le n}\qquad \text{or}\qquad s_\lambda^{(s)}:=\det(E_{\lambda_i'-i+j}^{(s)})_{1\le i,j\le n},$$

where λ' is the conjugate of λ.

It would be very appealing to investigate the properties of the generalized Schur symmetric functions $s_\lambda^{(s)}$.

REFERENCES

[1] *M. Ahmia and M. Merca*, A generalization of complete and elementary symmetric functions - part I, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **85** (2023), No. 1, 119-128.

[2] *A. Bazeniar, M. Ahmia and H. Belbachir*, Connection between bi^{s}nomial coefficients with their analogs and symmetric functions, Turk J Math. **42**(2018) 807-818.

[3] *H. Belbachir, S. Bouroubi and A. Khelladi*, Connection between ordinary multinomials, Fibonacci numbers, Bell polynomials and discrete uniform distribution, Ann. Math. Inform. **35**(2008), 21-30.

[4] *I. Macdonald*, Symmetric functions and Hall polynomials, Oxford Univ Press, Oxford, 1979.

[5] *P. A. MacMahon*, Combinatory analysis, vol. 2, Chelsea Publishing Co., New York, 1960.

[6] *M. Merca*, Bernoulli numbers and symmetric functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM. **114**(2020), No. 1, 20.