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# APPROXIMATE CALCULUS BY DECONVOLUTION OF THE POLYNOMIAL ROOTS

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*În prezenta lucrare se dau algoritmi bazați pe deconvoluție discretă pentru calculul aproximativ al rădăcinilor polinoamelor. Aceştia sunt obținuți prin conexiunea dintre metoda lui Daniel Bernoulli de reducere a calculului aproximativ al rădăcinilor polinoamelor la rezolvarea unor probleme de valori inițiale pentru ecuații liniare omogene cu diferențe şi metoda, dată de autor, de rezolvare a unor astfel de probleme prin deconvoluție discretă. Luând în considerare diferite alegeri ale valorilor inițiale, se obțin diverse tipuri de formule pentru calculul rădăcinilor. Sunt de asemenea considerate unele cazuri în care metoda lui Bernoulli nu este direct aplicabilă. În particular, sunt date noi formule pentru calculul radicalilor.*

*Formulele de deconvoluție pentru calculul rădăcinilor unui polinom date aici sunt foarte simple şi adaptabile pe calculator. Simplitatea rezultă din faptul că nu sunt folosite funcții sau recurențe. Aceste metode numerice sunt în contrast cu metoda dată de B. Kalantari şi colaboratorii săi, bazată de asemenea pe metoda lui Bernoulli, care generalizează metodele Newton-Raphson şi Halley şi care utilizează anumite familii iterative de funcții.*

*In the present paper algorithms based on discrete deconvolution for approximate computation of the polynomials roots are given. These are obtained by the connection between the Daniel Bernoulli's method, which reduces the approximate computation of the polynomials roots to the solution of some initial valued proble for homogeneous linear difference equations and the method, given by the author, of solving such problems by discrete deconvolution Taking into account several choices of initial values, we obtain various kinds of formulae of th computation roots. Some cases in which Bernoulli's method is not directly applicable are also considered. Particularly, new formulae for computation of radicals are given.*

*The deconvolution formulae for the calculus of the roots of a polynomial given here are very simple and adaptable on computer. Their simplicity results from the fact that are not use functions or recurrences. These numerical methods are in contrast with the method given by B. Kalantari and his collaborators, also based on Bernoulli's method, that generalize the Newton-Raphson and Hilley methods and which uses some iteration function families.*

**Keywords:** polynomial equations, Bernoulli and deconvolution methods.

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## 1. Introduction

In the paper [1], we use the notions of discrete convolution (Cauchy product) and its inverse – the deconvolution (see [3]), to obtain the numerical solutions of the linear difference equations.

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On the other hand, the well known Bernoulli's method (see [4] or [5]), reduces the approximate calculus of the roots of a polynomial to an initial value problem for an associated homogeneous linear difference equation. Solving this problem by deconvolution, we obtain several new numerical algorithms for computation of the roots of a polynomial. If it is necessary, the convolution and deconvolution algorithms can be accelerated with the help of discrete fast Fourier transform (see [8] and [9]). In **Section 2** of the paper we give a short review of the notions of discrete convolution and deconvolution and in **Section 3** we present a formula, proved in [1], for the numerical calculus of the solution of the initial value problem for linear difference equations with constant coefficients. In the present paper we will apply this formula to the homogeneous linear difference equation to which is reduced the approximate calculus of the roots of a polynomial by the Bernoulli's method. We present the Bernoulli's method in **Section 4**, and our deconvolution method in **Section 5**. Several particular formulae, based on particular choices of the initial values of the solution of the mentioned difference equation, are given in **Section 6** and some special cases in **Section 7**. Several situations in which the Bernoulli's method is not directly applicable are analyzed in **Section 8**, some procedures being given to apply however the method. The computation of the radicals, considered in **Section 9**, is such a situation.

Other numerical method for solving nonhomogeneous linear differential equations with constant coefficients was given in the paper [2].

## 2. Discrete convolution and deconvolution

The sequence $c = a * b = (c_0, c_1, \ldots, c_k, \ldots)$ is named the *discrete convolution* (*Cauchy product*) of two sequences of real or complex numbers $a = (a_0, a_1, \ldots, a_k, \ldots)$ and $b = (b_0, b_1, \ldots, b_k, \ldots)$, if it has the components $c_k = \sum_{j=0}^{k} a_{k-j} b_j$ , $k = 0,1,2,\ldots$ . The convolution product is commutative, associative, distributive with respect to the addition of the sequences and has the unit $\delta = (1, 0, 0, ...)$

In case $a_0 \neq 0$, we can perform the inverse operation, $b = c/a$, named the *deconvolution* of the sequence $c$ by the sequence $a$, with the components $b_0 = \frac{c_0}{a_0}$, $b_k = \frac{1}{a_0}\left(c_k - \sum_{j=0}^{k-1} a_{k-j} b_j\right)$, $k = 1,2,\ldots$ , given by the algorithm

$$
\begin{array}{llll|llll}
c_0 & c_1 & c_2 & \dots & a_0 & a_1 & a_2 & \dots \\
\cline{5-7}
c_0 & a_1 b_0 & a_2 b_0 & \dots & b_0 = \dfrac{c_0}{a_0} & b_1 = \dfrac{c_1 - a_1 b_0}{a_0} & b_2 = \dfrac{c_2 - a_2 b_0 - a_1 b_1}{a_0} & \dots \\
\cline{1-4}
 & c_1 - a_1 b_0 & c_2 - a_2 b_0 & \dots & & & & \\
 & c_1 - a_1 b_0 & a_1 b_1 & \dots & & & & \\
\cline{2-4}
 & & c_2 - a_2 b_0 - a_1 b_1 & \dots & & & & \\
 & & c_2 - a_2 b_0 - a_1 b_1 & \dots & & & &
\end{array}
$$

Denoting by $a^{-1} = \delta / a$, the inverse of the sequence $a$, we have $c/a = c * a^{-1}$. We consider, with the same formulae, the notions of convolution and deconvolution also in the case when the sequences are finite, of the same length. If $a$ is a finite sequence and $b$ a finite or infinite sequence, we denote by $(a, b)$ the sequence obtained when we join the two sequences.

**3. Linear difference equations**

We consider the homogeneous linear difference equation (linear recurrence relations)

$$\sum_{j=0}^{n} a_{n-j} u_{j+k} = 0, k = 0,1,\dots \; , \tag{1}$$

with the coefficients $a_0 \neq 0, \; a_1, \dots, a_n$. We denote

$$a = (a_0, a_1, \dots, a_n, 0, 0, \dots) \; . \tag{2}$$

In [1] it was proved that the unique solution $u = (u_0, u_1, \dots; u_k, \dots)$ of the equation (1) with the known initial values $u_0, u_1, \dots, u_{n-1}$, is given by the formula

$$u = \tilde{u} / a = \tilde{u} * a^{-1} \; , \tag{3}$$

where

$$\tilde{u} = \left( a_0 u_0, a_1 u_0 + a_0 u_1, \cdots, \sum_{j=0}^{n-1} a_{n-1-j} u_j, 0, 0, \cdots \right) = $$
$$= \left( (a_0, a_1, \cdots, a_{n-1}) * (u_0, u_1, \cdots, u_{n-1}), 0, 0, \cdots \right) \tag{4}$$

**4. The Daniel Bernoulli's method**

We consider the polynomial equation

$$P_n(x) = \sum_{j=0}^{n} a_{n-j} x^j = 0 \tag{5}$$

with distinct real or complex roots $x_1,\dots,x_p$, where $p \le n$ , of multiplicity $m_1,\dots,m_p$, where $\sum_{i=1}^{p} m_i = n$. It is the characteristic equation of the linear difference equation (1), that has the general solution given by the formula (see [4])

$$u_k = \sum_{i=1}^{p}\sum_{j=0}^{m_i-1} C_{i,j} k^j x_i^k, k = 0,1,2,\dots , \tag{6}$$

where $C_{i,j}$ are arbitrary complex constants. For a choice of the initial values $u_0, u_1,\dots,u_{n-1}$ of the sequence $u$, the constants $C_{i,j}$ can be obtained from the first $n$ equations (6), hence by the linear algebraic system

$$\sum_{i=1}^{p}\sum_{j=0}^{m_i-1} k^j x_i^k C_{i,j} = u_k , \ k = 0,1,\dots,n-1 . \tag{7}$$

Replacing the values of $C_{i,j}$ determined from the system (7), in the equations (6), for $k = n, n+1,\dots$, we obtain the solution of the initial value problem formed by the difference equation (1) and the considered initial values. If the polynomial equation (5) has a dominant root $x_1$ , namely satisfying the conditions

$$|x_1| > |x_i| \quad , \ m_1 \ge m_i \ , \ i = 1,\dots,p \ , \tag{8}$$

then, according to (6), we obtain the formula

$$\lim_{k\to\infty}\frac{u_{k+1}}{u_k} = \lim_{k\to\infty}\frac{\sum_{i=1}^{p}\sum_{j=0}^{m_i-1} C_{i,j}(k+1)^j x_i^{k+1}}{\sum_{i=1}^{p}\sum_{j=0}^{m_i-1} C_{i,j} k^j x_i^k} =$$

$$= x_1 \lim_{k\to\infty}\frac{\sum_{j=0}^{m_1-1} C_{1,j}\frac{(k+1)^j}{k^{m_1-1}} + \sum_{i=2}^{p}\sum_{j=0}^{m_i-1} C_{i,j}\frac{(k+1)^j}{k^{m_1-1}}\left(\frac{x_i}{x_1}\right)^{k+1}}{\sum_{j=0}^{m_1-1} C_{1,j}\frac{k^j}{k^{m_1-1}} + \sum_{i=2}^{p}\sum_{j=0}^{m_i-1} C_{i,j}\frac{k^j}{k^{m_1-1}}\left(\frac{x_i}{x_1}\right)^{k}} = x_1\frac{C_{1,m_1-1}}{C_{1,m_1-1}} = x_1, \tag{9}$$

if from the system (7) it results $C_{1,m_1-1} \ne 0$. The computation of the dominant root $x_1$ by the formula (9) is named the Daniel Bernoulli's method (see for example [4] and [5]). By dividing the polynomial $P_n(x)$ to $(x - x_1)^{m_1}$, we can eventually apply again the Bernoulli's method, for the dominant root of the quotient.

The method is not directly applicable both if does not exist a dominant root of the equation (5) or in the case which such a root $x_1$ exists, but from the linear algebraic system (7) it results $C_{1,m_1-1} = 0$.

## 5. The deconvolution method for the computation of the roots of polynomial

The deconvolution method uses the formula (3) for solving the characteristic equation (1) associated to the polynomial equation (5), hence it consists by the following steps : By the deconvolution formula (3), we obtain the solution $u = (u_0, \dots, u_{n-1}, u_n, \dots, u_k, \dots)$ of the difference equation (1), associated to (5), for an arbitrary choice of the initial values $u_0, u_1, \dots, u_{n-1}$. According to formula (9), if $s_k = u_{k+1}/u_k$ and for given $\varepsilon > 0$ we have $|s_{k+1} - s_k| < \varepsilon$, then the approximate value of the dominant root of the polynomial equation (5) is $x_1 = s_k$. Otherwise, the method cannot be directly applied. Various initial valued choices, given in the **Section 5** of the paper, will give different forms for the deconvolution formula (3). The special case when all the roots of the polynomial are simple, and several situations that are reduced to this case are considered in the **Section 7**.

In some cases in which the Bernoulli's method is not directly applicable, there are several possibilities, as changes of variables, that make the method work. Some of such situations will be given in the **Sections 8** and **9**.

We observe that, for the approximate computation of the roots of polynomials, we use only the deconvolution formula (3), without explicitly appealing to the difference equation (1) or at other arguments, what gives an important simplification of the method.

### 6. The choice of the initial values.

**6.1.** Usually we take

$$u_0 = u_1 = \cdots = u_{n-2} = 0, u_{n-1} = 1 , \tag{10}$$

in which case the sequence $\tilde{u}$ given by the formula (4) has the form $\tilde{u} = ((a_0, a_1, \dots, a_{n-1}) * (0, \dots, 0, 1), 0, 0, \dots) = (0, \dots, 0, a_0, 0, 0, \dots)$. Since the first $n-1$ zeros of the sequence $\tilde{u}$ have no effect in the determination through deconvolution of the sequence $u = \tilde{u}/a$ given by (3), hence nor of the ratios $u_{k+1}/u_k$, we can take $\tilde{u} = (a_0, 0, 0, \dots) = a_0 \delta$, hence in this case the sequence $u$ is given by the formula

$$u = (1, u_n, u_{n+1}, \dots, u_k, \dots) = a_0\, \delta / a = a_0\, a^{-1} \tag{11}$$

*Exemple 1.* For the equation $x^3 + 9x^2 - 8x + 7 = 0$, the deconvolution algorithm (we present such an algorithm only here and in *Example 4*)

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | … | 1 | 9 | -8 | 7 | 0 | 0 | … |
| 1 | 9 | -8 | 7 | 0 | 0 | … | 1 | -9 | 89 | -880 | 8695 | -85918 | … |
| | -9 | 8 | -7 | 0 | 0 | … | | | | | | | |
| | -9 | -81 | 72 | -63 | 0 | … | | | | | | | |
| | | 89 | -79 | 63 | 0 | … | | | | | | | |
| | | 89 | 801 | -712 | 623 | … | | | | | | | |
| | | | -880 | 775 | -623 | … | | | | | | | |
| | | | -880 | -7920 | 7040 | … | | | | | | | |
| | | | | 8695 | -7663 | … | | | | | | | |
| | | | | 8695 | 78255 | … | | | | | | | |
| | | | | | -85918 | … | | | | | | | |
| | | | | | -85918 | … | | | | | | | |
| | | | | | | … | | | | | | | |

gives $u = a^{-1} = (1,9,-8,7,0,0,\ldots)^{-1} = (1,-9,89,-880,8695,-85918,\ldots)$, hence we obtain the approximate values of the dominant root, $\frac{89}{-9} \cong -9.888$, $\frac{-880}{89} \cong -9.8876$ , $\frac{8695}{-880} \cong -9.881$, $\frac{-85918}{8695} \cong -9.8813$, the last value having all decimals exact.

*Example 2.* For equation $P_7(x) = x^7 + x^6 - 2x^5 - 4x^4 - x^3 + 3x^2 + 3x + 1 = 0$, we have $u = a^{-1} = (1,1,-2,-4,-1,3,3,1,0,0,\ldots)^{-1} = (1,-1,3,\ldots, 318052,431134)$, the dominant root being $\frac{431134}{318052} \cong 1.35$.

While the process of convergence in *Example 1* is very fast, that from *Example 2* is slow, because the dominant root is multiple. We have the same situation when the modulus of the dominant root is almost equal with the modulus of other root. If the modulus of the dominant simple root is considerably greater than the other roots, the convergence process is very fast.

**6.2.** Another choice for the initial values of the sequence $u$ is

$$u_0 = 1, u_1 = \alpha, u_2 = \alpha^2, \ldots, u_{n-1} = \alpha^{n-1} \quad , \tag{12}$$

where $\alpha$ is an approximate known value of the dominant root. In this case the formula (3) takes the form

$$u = \left(a_0, a_1 + a_0\alpha, \ldots, \sum_{j=0}^{n-1} a_{n-1-j}\alpha^j, 0,0,\ldots\right)\Big/ a \tag{13}$$

For $\alpha = 0$, we obtain from (13) the formula

$$u = (a_0, a_1, \ldots, a_{n-1}, 0,0,\ldots)/(a_0, a_1, \ldots, a_n, 0,0,\ldots) \ . \tag{14}$$

*Example 3.* We consider again the equation from *Example 2*, for which $a = (1,1,-2,-4,-1,3,3,1,0,0,\ldots)$.

a) If $\alpha = 0$, in accord to (14), $u = (1,1,-2,-4,-1,3,3,0,0,\dots)/a$. Performing the deconvolution algorithm, a ratio $\frac{u_{k+1}}{u_k}$ that gives the approximate value of the dominant root is obtained to be $\frac{-1070665}{-791009} \cong 1.35$ .

b) If $\alpha = 1$ , according to formula (13) , $u = (1,2,0,-4,-5,-2,1,0,0,\dots)/a$ , and a ratio which gives the approximate value of the dominant root is $\frac{-8162885}{-6021555} \cong 1.35$. c) If $\alpha = 2$ , according to formula (13), $u = (1,3,4,4,7,17,37,0,0,\dots)/a$ , and a ratio which gives the approximate value of the dominant root is $\frac{880764}{652548} \cong 1.35$ .

**6.3.** A third choice of the initial values is

$$u_k = S_k \ , \ k = 0,1,\dots,n-1 \ , \tag{15}$$

where $S_k$ are the sums

$$S_k = \sum_{i=1}^{n} x_i^k \ , \ k = 0,\pm 1,\pm 2,\dots \ , \tag{16}$$

where $x_i$ , $i = 1,\dots,n$ are the roots, not necessary distinct, of the polynomial equation (5). The sums $S_k$ satisfies the Newton's formulae

$$\sum_{j=1}^{k} a_{k-j} S_j = -k \cdot a_k \ , \ k = 1,\dots,n-1, \tag{17}$$

$$\sum_{j=0}^{n} a_{n-j} S_{j+k} = 0 \ , \ k = 1,2,\dots \ . \tag{18}$$

In this case, from relation (4) we obtain

$$\begin{aligned}\tilde{u} &= ((a_0, a_1, \dots, a_k, \dots, a_{n-1}) * (S_0, S_1, \dots, S_k, \dots, S_{n-1}), 0,0,\dots) = \\ &= \left(a_0 S_0, a_0 S_1 + a_1 S_0, \dots, \sum_{j=0}^{k} a_{k-j} S_j, \dots, \sum_{j=0}^{n-1} a_{n-1-j} S_j, 0,0,\dots\right) = \\ &= (n a_0, (n-1) a_1, \dots, (n-k) a_k, \dots, 2a_{n-2}, a_{n-1}, 0,0,\dots) \ ,\end{aligned} \tag{19}$$

since in conformity with the Newton's formula (17), we have

$$\sum_{j=0}^{k} a_{k-j} S_j = \sum_{j=1}^{k} a_{k-j} S_j + a_k S_0 = (n-k) a_k \ , \ k = 0,1,\dots,n-1 \ .$$

From the relations (3) and (19), it results for the choice (15) of its initial values, that the sequence $u$ is given by the formula

$$u = (n a_0, (n-1) a_1, \dots, (n-k) a_k, \dots, 2a_{n-2}, a_{n-1}, 0,0,\dots)/(a_0, a_1, \dots, a_{n-1}, a_n, 0,0,\dots) \tag{20}$$

*Example 4.* For the equation $x^3 - x - 1 = 0$ , according to (20) , we have the following deconvolution algorithm

```
3 0 -1  0  0  0 ...  | 1  0 -1 -1  0  0 ...
3 0 -3 -3  0  0 ...  | 3  0  2  3  2  5 ...
-----------------
    2  3  0  0 ...
    2  0 -2 -2 ...
    -----------
       3  2  2 ...
       3  0 -3 ...
       --------
          2  5 ...
          2  0 ...
          -----
             5 ...
             5 ...
             -
               ...
```

which gives
$u = \tilde{u}/a = (3,0,-1,0,0,\ldots)/(1,0,-1,-1,0,0,\ldots) = (3,0,2,3,2,5,5,7,10,12,17,$
$22,29,39,51,68,90,\ldots)$ and we obtain the approximate values of the dominant real root $\frac{29}{22} \cong 1.31, \frac{90}{68} \cong 1.323$.

**7. Special cases.**

**7.1. Simple roots case.**

We now suppose that the polynomial equation (5) has only simple roots $x_1,\ldots,x_n$. The condition (8) for the root $x_1$ to be dominant, becomes

$$|x_1| > |x_i|, i = 2,\ldots,n, \tag{21}$$

the second condition from (8), about the multiplicity orders of the roots, being now obviously fulfil. In this case, the formula (6) has the form

$$u_k = \sum_{i=1}^{k} C_i \; x_i^k \;,\; k = 0,1,2,\ldots \;. \tag{22}$$

For an arbitrary choice of the initial values $u_0, u_1, \ldots, u_{n-1}$ of the sequence $u$, the constants $C_1,\ldots,C_n$ can be determined from the linear algebraic system (7), that now has the form

$$\begin{aligned}
&C_1 + C_2 + \cdots + C_n = u_0 \\
&x_1 \; C_1 + x_2 \; C_2 + \cdots + x_n \; C_n = u_1 \\
&x_1^2 \; C_1 + x_2^2 \; C_2 + \cdots + x_n^2 \; C_n = u_2 \\
&\qquad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\
&x_1^{n-1} \; C_1 + x_2^{n-1} \; C_2 + \cdots + x_n^{n-1} \; C_n = u_{n-1} \quad .
\end{aligned} \tag{23}$$

The system (23) is compatible, because its principal determinant is a non-null Vandermonde determinant and in the cases considered in **Sections 6.1-3**, its solutions $C_1,\ldots,C_n$ are different of zero. Consequently, the Bernoulli's method can be applied for every simple dominant root.

*Example 5.* For the equation $x^2-x-1=0$ one has $a=(1,-1,-1,0,\ldots)$, $u=a^{-1}=(1,1,2,3,5,8,13,21,34,55,89,\ldots)$ and $x_1 \cong 89/55 \cong 1.618$ is the approximate value with all exact decimals for the root $x_1=\frac{1+\sqrt{5}}{2}$ of the equation. In fact, the above obtained sequence $u$ of which the components are the *Fibonacci numbers* is the solution of the Cauchy problem formed by the difference equation $u_{k+2}-u_{k+1}-u_k=0, k=0,1,\ldots,$ and the initial values $u_0=0, u_1=1$, the root $x_1$ being the *golden number*.

### 7.2. Orthogonal polynomials case

We consider the sequence of polynomials $P_n(x)$ of degree $n=0,1,2,\ldots$, orthogonal on the interval $[a,b]$, with $-\infty \le a < b \le \infty$, with respect to a differentiable nonnegative weight function. In the well known conditions, every orthogonal polynomial $P_n(x)$ is given by the Rodrigues formula, it satisfies a linear differential equation with variable coefficients and has only simple real roots, situated in the interval $[a,b]$ and separated by the roots of the polynomial $P_{n-1}(x)$. By a suitable change of variables, the Jacobi, respective Laguerre polynomials can be considered on the interval $[0,1]$, respectively $[0,\infty)$. In these cases, their roots will be nonnegative distinct numbers, consequently the Bernoulli's method will be directly applicable. The roots of the polynomial $P_{n-1}(x)$ and the numbers $a$ and $b$ can be eventually taken as initial approximate values for the roots of the polynomial $P_n(x)$. For Hermite polynomials is indicated in some cases to make the change of variables $y=x^2$.

### 7.3. Multiple roots case

If the polynomial equation (5) has multiple roots, we replace in equation the polynomial $P_n(x)$ by its quotient with the great common divisor between $P_n(x)$ and its derivative $P_n'(x)$. All the distinct roots of the equation (5) are simple roots for the obtained equation, hence the dominant root in sense of (21) can be computed by Bernoulli's method.

*Example 6.* For the equation given in *Example 2*, the great common divisor of $P_7(x)$ and $P_7'(x)$ is $P_3(x) = x^3 - x - 1$, its zeros being the double roots for initial equation. The simple dominant real root of $P_3(x)$ was obtained in *Example 4* and can be also obtained by the inverse $u = a^{-1} = (1,0,-1,-1,0,0,\ldots)^{-1} = (1,0,1,1,1,2,3,4,5,7,9,12,\ldots, \quad 5842,7739,\ldots)$, namely $u_{k+1}/u_k$ is $7739/5842 \cong 1.324718$, this being the approximate value with all exact decimals, of the double real dominant root of the initial equation.

We observe how quickly is computed this value as simple root of the polynomial $P_3(x)$ in *Example 6* and how slowly as double root of the polynomial $P_7(x)$ in *Example 2*.

### 8. Cases in which the Bernoulli's method is not directly applicable

### 8.1. Minorant roots

If the roots of the equation (5) does not satisfy the condition (8), then the Bernoulli's method cannot be applied, because the sequence of ratios $u_{k+1}/u_k$ do not converge. In some of these situations, the Bernoulli's method can be eventually applied by a convenient change of variables. Thus, if a root $x_1$ is minorant, i.e. $|x_1| < |x_i|, m_1 \geq m_i$, $i = 2,\ldots,p$, by the change of variable $x = 1/y$, the Bernoulli method can be applied to the equation in $y$ and gives the dominant root $y_1 = 1/x_1$.

*Example 7.* For the equation $x^3 + 3x^2 + 2x - 1 = 0$ ,we have $u = a^{-1} = (1,3,2,-1,0,0,\ldots)^{-1} = (1,-3,7,-14,25,-40, 56,-63,37,71,-350,\ldots)$.Because $\frac{37}{-63} \cong -0.5$, $\frac{71}{37} \cong 1.9, \frac{-350}{71} \cong -4.9,\ldots$, the Bernoulli's method cannot be applied. Making the change of variables $x = 1/y$ , we obtain the equation $y^3 - 2y^2 - 3y - 1 = 0$ , with $\tilde{u} = \tilde{a}^{-1} = (1,-2,-3,-1,0,0,\ldots)^{-1} = (1,2,7,21,65,200,616, 1897,5842,17991,55405,\ldots)$ , $y_1 = \frac{55405}{17991} \cong 3.079595$ , so $x_1 = 1/y_1 \cong 0.324718$. From Viète relations we have $\mathrm{Re}(x_{2,3}) = -(x_1 + 3)/2 \cong -1.66238$, $|x_{2,3}| = \frac{1}{\sqrt{x_1}} \cong 1.754878$, $\mathrm{Im}(x_{2,3}) = \sqrt{|x_{2,3}|^2 - [\mathrm{Re}(x_{2,3})]^2} \cong 0.562218$, hence the initial equation has the dominant pair of conjugate complex roots $x_{2,3} \cong -1.66238 \pm 0.562218\, i$.

### 8.2. Changing the initial values

If, for a certain choice of the initial values $u_1,\ldots,u_{n-1}$, it results from the linear algebraic system (7), $C_{1,m_1-1}=0$ , the dominant root $x_1$ of the polynomial equation (5) cannot be determined by Bernoulli's method. In general, in this case, the initial values must be changed, for obtain $C_{1,m_1-1}\neq 0$ in the new choice. However, there are such cases in which, by Bernoulli's method we determine not the dominant root, but another root of the equation. For example, in case when all root of the equation (5) are simple, $x_1$ is its dominant root and there is an index $i_0>1$ such that from the algebraic system (7) it results

$$C_i=0, i=1,\ldots,i_0-1; C_{i_0}\neq 0, \quad (24)$$

while the roots fulfill the conditions

$$\left|x_{i_0}\right|>\left|x_i\right|, i=i_0+1,\ldots,n, \quad (25)$$

the formula (9) receives the form

$$\lim_{k\to\infty}\frac{u_{k+1}}{u_k}=x_{i_0}. \quad (26)$$

*Example 8.* Let us consider the equation $x^3-2x^2-5x+6=0$, with the roots $x_1=3$, $x_2=-2, x_3=1$. For the initial values $u_0=2, u_1=-1,$ $u_3=5$ of the sequence $u$, the system (23) takes the form $C_1+C_2+C_3=u_0=2$ , $3C_1-2C_2+C_3=u_1=-1$, $9C_1+4C_2+C_3=u_2=5$ and has the solutions $C_1=0, C_2=C_3=1$, hence the dominant root $x_1=3$ can not be obtained by Bernoulli`s method. However, we have

$a=(1,-2,-5,6,0,0,\ldots), \tilde{c}=((a_0,a_1,a_2)*(u_0,u_1,u_2),0,0,\ldots)=$
$((1,-2,-5)*(2,-1,5),0,0,\ldots) \;=(2,-5,3,0,0,\ldots),$
$u=\tilde{c}/a=(2,-1,5,-7,17,-31,65,-127,257,\ -511,1025,-2047,\ldots)$

and $\frac{1025}{-511}\cong -2.0058$ , $\frac{-2047}{1025}\cong -1.9970$ , hence in this situation we obtain by Bernoulli's method approximate values of the root $x_2=-2$, not of dominant root. If we change the initial values, namely take $u_0=u_1=0, u_2=1$, we have $u=a^{-1}=(1,2,9,22,77,\ 210,613,1934,5973,17578,53417,158886,\ldots)$.

Then $\frac{53417}{17578}\cong 3.03$, $\frac{158886}{53417}\cong 2.97$, hence by the Bernoulli's method one obtains approximate values of the dominant root $x_1=3$.

### 8.3. Complex roots of the polynomials with real coefficients.

If the equation (5) has all the coefficients real numbers, the Bernoulli method cannot be applied to the complex roots, because the conjugate of any such a root is also a root of the equation and the two conjugate roots have the same modulus, hence no such root is dominant. A possibility to use the Bernoulli's method even in this situation is to make a change of variables of the form $x = \alpha y + i\beta$, with $\alpha, \beta$ real numbers. The new equation has complex coefficients, but in some cases the corresponding roots are different in modulus hence the Bernoulli's method can be applied.

*Example 9.* For the equation from *Example 7*, we make the change of variables $x = z + i$ and obtain the new equation $z^3 + (3+3i)z^2 + (-1+6i)z - 4 + i = 0$. From $u = (1, 3+3i, -1+6i, -4+i, 0, 0, \ldots)^{-1} = (1, -3-3i, \ldots,$ $1842 - 18967i, -33186 + 28396i, \ldots)$ it results $z_3 \approx \dfrac{-33186 + 28396i}{1842 - 18967i} \approx -1{,}65 - 1.589i$, $x_3 = z_3 + i \cong -1.65 - 0.589i$, $x_2 = \bar{x}_3 \cong -1.65 + 0.589i$. From the Viète relation, we have $x_1 = -3 - 2\,\mathrm{Re}(x_{2,3}) \cong 0.3$.

## 9. Approximate computation of the radicals

**9.1.** Let $n, q > 1$ be natural numbers. The radical $x = \sqrt[n]{q}$ is a root of the polynomial equation $x^n - q = 0$ but the Bernoulli's method cannot be directly applied to this equation, because its (simple) roots are equal in modulus. Making the change of variables $x = y - 1$, we obtain the equation

$$(y-1)^n - q = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^k - q = 0, \tag{27}$$

having all roots distinct and its real root $y_1 > 1$ being dominant. Therefore the Bernoulli's method can be applied to the equation (27). In conformity with the formula (9), for an arbitrary choice of the numbers $u_0, u_1, \ldots, u_{n-1}$, the radical is given by the formula

$$\sqrt[n]{q} = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} - 1 \cong \frac{u_{k+1}}{u_k} - 1, \tag{28}$$

in which the numbers $u_k$ are the components of the sequence $u = \tilde{u}/a = \tilde{u} * a^{-1}$, given by the formula (3), where the formulae (2) and (4) that give the sequences $a$ and $\tilde{u}$ have now the form

$$a = \left(1, -\binom{n}{1}, \binom{n}{2}, \ldots, (-1)^{n-1}\binom{n}{n-1}, (-1)^n - q, 0, 0, \ldots\right), \tag{29}$$

$$\tilde{u} = \left(u_0, u_1 - \binom{n}{1}u_0, \ldots, \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}u_{n-j-1}, 0, 0, \ldots\right). \quad (30)$$

**9.2.** If $u_0 = u_1 = \cdots = u_{n-2} = 0, u_{n-1} = 1$, according to (11), we have

$$u = (1, u_n, u_{n+1}, u_k, \ldots) = a^{-1} = \left(1, -\binom{n}{1}, \binom{n}{2}, \ldots, (-1)^{n-1}\binom{n}{n-1}, (-1)^n - q, 0, 0, \ldots\right)^{-1}. \quad (31)$$

*Example 9.* If $n = q = 2$, then $u = a^{-1} = (1, -2, -1, 0, 0, \ldots)^{-1} =$ $= (1,2,3,8,19,46,111,278, 667,1612,\ldots)$, hence $\sqrt{2} \cong \frac{u_9}{u_8} - 1 = \frac{1612}{667} - 1 \cong 1.41$ .

*Example 10.* If n=3,q=2, then $u = a^{-1} = (1,-3,3,-3,0,0,\ldots)^{-1} = (1,3,6,12,27,63,$ $144,324,\ldots)$, so $\sqrt[3]{2} \cong \frac{u_7}{u_6} - 1 = \frac{324}{144} - 1 \cong 1.25$.

**9.3.** If we make the choice (12), where $\alpha$ is an approximate value of the number $\sqrt[n]{q} + 1$ , for example it is the greatest natural number so that $(\alpha - 1)^n \le q$ , there results from (13) that the sequence $\tilde{u}$ is given by the relation

$$\tilde{u} = \left(1, \alpha - \binom{n}{1}, \ldots, \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\alpha^{n-j-1}, 0, 0, \ldots\right). \quad (32)$$

*Example 11.* If $n = 5, q = 70, \alpha = 3$, then

$$a = \left(\binom{5}{5}, -\binom{5}{4}, \binom{5}{3}, -\binom{5}{2}, \binom{5}{1}, (-1)^5 - 70, 0, 0, \ldots\right) = (1,-5,10,-10,5,-71,0,0,\ldots),$$

$\tilde{u} = ((1,-5,10,-10,5) * (1,3,9,27,81), 0, 0, \ldots) = (1,-2,4,7,26,0,0,\ldots), u = \tilde{u}/a =$ $= (1,3,9,32,121,\ldots, 1892979,6297877,\ldots)$, hence $\sqrt[5]{70} \cong \frac{6297877}{1892979} - 1 \cong 2.33$ .

**9.4.** If $q$ is a complex number, we can compute the radical $\sqrt[n]{q}$ , solving by Bernoulli's method the equation $x^n - q = 0$ after a convenient change of variables.

*Example 12.* To compute $x = \sqrt{1+i}$ we make the change of variables $x = y - 1$ in equation $x^2 = 1 + i$, obtaining the equation $y^2 - 2y - i = 0$. Because $u = (1, -2, -i, 0, 0, \ldots)^{-1} = (1, 2, 4+i, 8+4i, 15+12i, 26+32i, \ldots)$, we obtain the approximate value of the radical $x_1 = \frac{26+32i}{15+12i} - 1 \cong 1.09 + 0.45i$ and $x_2 = -x_1$.

## 10. Conclusions

Among the advantages of the deconvolution method of polynomial roots calculus given here, we mention: 1) The method is entirely numerical, without to make use of functions; 2) The recurrence process given by the difference equation to which the Bernoulli's method reduces the polynomial roots calculus is solved directly, without recurrence; 3) The explicit utilization of the above mentioned difference equation is eliminated, instead it being obtained several direct formulae of calculus by deconvolution; 4) There are considered several cases in which Bernoulli's method is not directly applicable. Such a situation is the radicals calculus.

## REFERENCES

[1] *M.I.Cîrnu*, Solving difference and differential equations by discrete deconvolution, U.P.B. Sci. Bull, Series A, Vol. 69, No.1, 2007, pp. 13-26
[2] *M.I.Cîrnu*, Determination of particular solutions of nonhomogeneous linear differential equations by discrete deconvolution, U.P.B. Sci. Bull.,Series A, Vol. 69, No. 2, 2007, pp. 3-16
[3] *M.A.Cuenod*, Introduction a l'analyse impulsionelle. Principe et application, Dunod, Paris, 1970.
[4] *A.Guelfond*, Calcul des differences finies , Dunod , Paris , 1962.
[5] *F.B.Hildebrand*, Introduction to numerical analysis, Mc Graw-Hill ,New York-Toronto-London, 1956.
[6] *B. Kalantari, I. Kalantari, R. Zaare-Nahandi*, A basic family of iteration functions for polynomial roots finding and its characterizations, J. Comput. Appl. Math., 80 (1997) 209-226.
[7] *B. Kalantari*, Polynomiography and its applications in art, education and science, Computer and Graphics, 28 (2004) 417-430.
[8] *H.J. Nussbaumer*, Fast Fourier transform and convolution algorithms, Springer Verlag, Berlin-Heidelberg-New York, 1981.
[9] *R.Tolimieri, M. An, C. Lu*, Algorithms for discrete Fourier transform and convolution, Springer Verlag, Berlin-Heidelberg-New York, 1997.