THE NUMERICAL SOLVING OF A NON LINEAR INTEGRAL EQUATION OF HAMMERSTEIN TYPE

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Acest articol îşi propune să realizeze o trecere sumară în revistă a unora dintre cele mai des întâlnite metode numerice de rezolvare a ecuațiilor integrale.

Apoi de asemenea se vor aminti unele dintre elementele de bază ale teoriei siajului.

Partea originală a articolului o reprezintă abordarea unei integrale de tip Hammerstein, ce poate fi întâlnită în cadrul teoriei siajului şi despre care se va arăta că se poate rezolva prin metode numerice.

This article tries to achieve a summary of one of the most well known numerical methods for solving integral equations.

In the same time some elements about dead water-theory will be remind.

The original part of this article is represented by the solving of a Hammerstein equation, which can be found in the dead-water theory, and it will be demonstrated that it can be solved using numerical methods.

Key words: dead water, Hammerstein integrals.

Introduction

Among the founders of the linear integral equation's theory, we will mention,beside **Volterra** and **Fredholm**, also **David Hilbert** (1862-1943) and **Erhart Schmidt** (1876-1958). It is also important to remember the Romanian mathematician **Traian Lalescu** , who, in his doctorate thesis, entitled „Sur l'équation de Volterra" and sustained in Paris in 1908, used for the first time the successive approximation method for the integration of a Volterra equation. He also wrote the first book from the entire world about integral equations, published in Bucharest in Romanian language in 1911 and after that, also published in Paris, using French language, one year later, respectively in 1912.

Nonlinear integral equations are a kind of equations in which the unknown function y can be found under the sign of the integral in some complicated way. For example:

$$y(s)-\int_{0}^{l} g(s,t)[y(t)]^2\,dt = h(s)$$

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1. Hammerstein-type integrals

A. Hammerstein studied nonlinear integral equations looking like:

$$\psi(x)+\int_0^1 K(x,y)f[y,\psi(y)]dy=0 \qquad (1.1)$$

They can also be extended to on n-dimensional spaces, but this does not involve fundamental differences.

We will use the following important hypothesis

- Fredholm theorem is true for the linear integral equation having the kernel K
- the kernel K is symmetrical :

$$K(x,y)=K(y.x)$$

- the kernel K is positive, which means that all its eigen values are of the positive kind

If these conditions are fulfilled we can say that the integral equation really is of the Hammerstein type. Hammerstein used the fact that, according to relation (1.1):

$$\psi(x)=\int_0^1 K(x,y)g(y)dy \qquad \textit{with } g(y)=-f[y,\psi(y)]$$

if it exist also $g(y)\in L^2$ then $\psi(x)$ can be represented like an uniform convergent series having the form:

$$\psi(x)=\sum_{m=1}^{\infty} c_m\phi_m(x) \qquad (1.2)$$

using $\phi_1(x),\phi_2(x),\ldots$ like the ortonormalised eigen values for the kernel $K(x,y)$ corresponding to the eigen values $\lambda_1,\lambda_2,\ldots$ and $c_1,c_2,\ldots$ being unknown constants.

Then, because:

$$c_m=\int_0^1\psi(x)\phi_m(x)dx=-\int_0^1\phi_m(x)dx\int_0^1 K(x,y)f[y,\psi(y)]dy=$$

$$=-\int_0^1 f[y,\psi(y)]dy\int_0^1 K(x,y)\phi_m(x)dx=-\frac{1}{\lambda_m}\int_0^1 f[y,\psi(y)]\phi_m(y)dy$$

the problem of solving the given equation is equivalent with the one of solving an infinite system of equations having an infinite number of unknowns:

$$c_m=-\frac{1}{\lambda_m}\int_0^1 f\left[y,\sum_{h=1}^{\infty}c_h\phi_h(y)\right]\phi_m(y)dy \qquad m=1,2,3,\ldots \qquad (1.3)$$

It is normal now to consider the approximate solution:

$$\psi_n(x)=\sum_{m=1}^{n} c_{n,m}\phi_m(x) \tag{1.4}$$

with the constants $c_{n,1}, c_{n,2}, c_{n,3},\ldots$ having to verify the system with n equations and n unknowns:

$$c_{n,m}=-\frac{1}{\lambda_m}\int_0^1 f\left[y,\sum_{h=1}^{n} c_{n,h}\phi_h(y)\right]\phi_m(y)dy \qquad m=1,2,3,\ldots \tag{1.5}$$

We will ask about the existence of the solution for this system, if it exists or not. Hammerstein showed very nicely that the systems of the kind (1.5) have at least one solution, by demonstrating that the function $f(x,u)$is a continuous one and verifies a condition of the following type:

$$|f(x,u)|\le C_1|u|+C_2 \tag{1.6}$$

with C_1 and C_2 are two positive constants and C_1 is less the the first eigen value λ_1 of the positive kernel $K(x,y)$. Evan if the relation (1.6) can be relaxed, Hammerstein demonstrated that the condition $C_1<\lambda_1$ can not be generally enlarged.

For showing this Hammerstein used the continuous function:

$$H(x_1,x_2,\ldots,x_n)=\sum_{m=1}^{n}\lambda_m x_m^2+2\int_0^1 F\left[y,\sum_{h=1}^{n} x_h\phi_h(y)\right]dy \tag{1.7}$$

with

$$F(y,u)=\int_0^u f(y,v)dv$$

$H(x_1,x_2,\ldots,x_n)$ is a function having partial derivations closely related with the solutions for the system (1.5) because:

$$\frac{1}{2\lambda_m}\frac{\partial H}{\partial x_m}=x_m+\frac{1}{\lambda_m}\int_0^1 f\left[y,\sum_{h=1}^{n} x_h\phi_h(y)\right]\phi_m(y)dy\,. \tag{1.8}$$

It is easy to observe that the function H has a lower limit.

Using the relations (1.6) and (1.7) we will get that:

$$|F(x,u)|\le\frac{1}{2}C_1u^2+C_2|u| \tag{1.9}$$

and if C_1 is smaller than an arbitrary constant k, and also using the inequality:

$$ax-bx^2\le\frac{a^2}{4b} \qquad b>0 \tag{1.10}$$

identifying x with $|u|$, a with C_2 and b with $\frac{1}{2}(k-C_1)$ we get:

$$\frac{1}{2}C_1u^2 + C_2|u| \le \frac{1}{2}ku^2 + C_3$$

with

$$C_3 = \frac{C_2^2}{2(k-C_1)}.$$

Because:

$$C_1 < k < \lambda_1$$

we get:

$$F(x,u) \ge -\left(\frac{1}{2}ku^2 + C_3\right). \tag{1.11}$$

Then:

$$H(x_1, x_2, \ldots, x_n) \ge \sum_{m=1}^{n} \lambda_m x_m^2 - \int_0^1 \left[k\left(\sum_{h=1}^{n} x_h \phi_h(y)\right)^2 + 2C_3 \right] dy$$

because:

$$H \ge \sum_{m=1}^{n} \lambda_m x_m^2 - k\sum_{h=1}^{n} x_h^2 - 2C_3 = \sum_{m=1}^{n} (\lambda_m - k)x_m^2 - 2C_3 \tag{1.12}$$

Using that $k < \lambda_1 \le \lambda_2 \le \ldots$ the right side sum is not negative. So H has a lower limit in $-2C_3$.

Therefore it will exist at least one set of values $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ for the initial $x_1, x_2, \ldots, x_n$ so that the continuous function H reach it's absolute minimum value d_n. We can be sure by choosing, for example

$$c_{n,m} = x_m^{(0)} \qquad m = 1,2,\ldots,n.$$

Multiplying with the positive number $4b$, the relation (1.10) is equivalent with the statement that:

$$a^2 - 4abx + 4b^2x^2 = (a-2bx)^2 \ge 0.$$

The system (1.5) will be verified because, for:

$$x_m = x_m^{(0)} = c_{n,m} \qquad m = 1,2,\ldots,n \tag{1.13}$$

we will obtain:

$$\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial x_2} = \ldots = \frac{\partial H}{\partial x_n} = 0.$$

Now it is important to justify that, with this way of choosing the quantities $c_{n,m}$, the sum

$$S_n = \sum_{m=1}^{n} \lambda_m c_{n,m}^2 \tag{1.14}$$

has an upper limit independent by n.

In fact, with x_m having the values from (1.13), and using the relation (1.12) we get that:

$$\sum_{m=1}^{n} (\lambda_m - k) c_{n,m}^2 \le d_n + 2C_3$$

and then

$$\left(1 - \frac{k}{\lambda_1}\right) \sum_{m=1}^{n} \lambda_m c_{n,m}^2 = \sum_{m=1}^{n} \left(\lambda_m - k\frac{\lambda_m}{\lambda_1}\right) c_{n,m}^2 \le \sum_{m=1}^{n} (\lambda_m - k) c_{n,m}^2 \le d_n + 2C_3$$

meaning that

$$S_n \le \frac{d_n + 2C_3}{1 - k/\lambda_1}.$$

On the other hand, as far as

$$H_n(x_1, x_2, \ldots, x_n) = H_{n+1}(x_1, x_2, \ldots, x_n, 0)$$

we get that $d_{n+1} \le d_n \qquad n = 1,2,\ldots$

With $d_n \le d_1$ we obtain:

$$S_n \le D = \frac{d_1 + 2C_3}{1 - k/\lambda_1}. \tag{1.15}$$

This means that we are able to write:

$$\int_0^1 \psi_n^2(x)dx = \sum_{m=1}^{n} c_{n,m}^2 \le \frac{1}{\lambda_1} \sum_{m=1}^{n} \lambda_m c_{n,m}^2 \le \frac{D}{\lambda_1}. \tag{1.16}$$

Now we have to justify that, for $n \to \infty$, the family of functions $\psi_n(x)$ realize an approximation for the given equation.

For the beginning we will demonstrate that the function

$$\chi_n(x) = \psi_n(x) + \int_0^1 K(x,y) f[y, \psi_n(y)] dy \tag{1.17}$$

goes uniformly to 0 for $n \to \infty$.

In fact using Hilbert-Schmidt theorem we get:

$$\chi_n(x)=\sum_{m=1}^{n}c_{n,m}\phi_m(x)+\sum_{m=1}^{\infty}\phi_m(x)\int_0^1\phi_m(\xi)d\xi\int_0^1K(\xi,y)f[y,\psi_n(y)]dy=$$
$$=\sum_{m=1}^{n}c_{n,m}\phi_m(x)+\sum_{m=1}^{\infty}\frac{\phi_m(x)}{\lambda_m}\int_0^1f[y,\psi_n(y)]\phi_m(y)dy$$

Using the relation (1.5) we have that:

$$\frac{1}{\lambda_m}\int_0^1f[y,\psi_n(y)]\phi_m(y)dy=-c_{n,m}$$

and

$$\chi_n(x)=\sum_{m=n+1}^{\infty}\lambda_m^{-1}\phi_m(x)\int_0^1f[y,\psi_n(y)]\phi_m(y)dy\ .$$

Finally, we can say that:

$$\chi_n^2(x)\leq\sum_{m=n+1}^{\infty}\lambda_m^{-2}\phi_m^2(x)\sum_{m=n+1}^{\infty}\left[\int_0^1f[y,\psi_n(y)]\phi_m(y)dy\right]^2\ . \tag{1.18}$$

$$\left\{\int_0^1f[y,\psi_n(y)]\phi_m(y)dy\right\}^2\leq\int_0^1\phi_m^2(y)dy\int_0^1f^2[y,\psi_n(y)]dy=\int_0^1f^2[y,\psi_n(y)]dy\ .$$

$$f^2(x,u)\leq C_1^2u^2+2C_1C_2|u|+C_2^2=2C_1C_2|u|-(k-C_1^2)u^2+ku^2+C_2^2\ .$$

So, if $C_1^2<k$ and using (1.10) with $x=|u|, a=2C_1C_2, b=k-C_1^2$, we get that:

$$f^2(y,u)\leq ku^2+C_4\ .$$

Together with (1.16) now we have:

$$\int_0^1f^2[y,\psi_n(y)]dy\leq\int_0^1[k\psi_n^2(y)+C_4]dy\leq\frac{k}{\lambda_1}D+C_4=D^*\ . \tag{1.19}$$

Then the inequality (1.18) becomes:

$$\chi_n^2(x)\leq D^*\sum_{m=n+1}^{\infty}\lambda_m^{-2}\phi_m^2(x).$$

It is well known that the infinite series $\sum \lambda_m^{-2}\phi_m^2(x)$ goes uniformly to $K_2(x,x)$. Then $\chi_n(x)$ goes uniformly to zero for $n \to \infty$.

So we demonstrated that, if the sequence $\psi_1(x), \psi_2(x), \ldots$ goes to a limit function $\psi(x)$ and if the Lebesgue fundamental theorem can be applied for the evaluation of the limit in the situation that $n \to \infty$ for the integral:

$$\int_0^1 K(x,y) f[y, \psi_n(y)] dy$$

then the limit function $\psi(x)$ is a solution for the initial equation (1.1). We have now to study if the sequence $\psi_1(x), \psi_2(x), \ldots$ converge. We can easily justify that we can choose a subsequence $\psi_{n_1}(x), \psi_{n_2}(x), \ldots$ which is uniformly going to a limit function $\psi(x)$, and that will be even continuous .

For demonstration we will use:

$$\omega_n(x) = \chi_n(x) - \psi_n(x) = \int_0^1 K(x,y) f[y, \psi_n(y)] dy \qquad n = 1,2,\ldots$$

The sequence $\{\omega_n\}$ is equally bounded because, as a consequence of the relation(1.19), we get that:

$$\omega_n^2(x) \le \int_0^1 K^2(x,y) dy \int_0^1 f^2[y, \psi_n(y)] dy \le D^* \int_0^1 K^2(x,y) dy .$$

$$[\omega_n(x_1) - \omega_n(x_2)]^2 = \left\{ \int_0^1 [K(x_1,y) - K(x_2,y)] f[y, \psi_n(y)] dy \right\}^2 \le$$

$$\le \int_0^1 [K(x_1,y) - K(x_2,y)]^2 dy \int_0^1 f^2[y, \psi_n(y)] dy \le$$

$$\le D^* \int_0^1 \left[K^2(x_1,y) - 2K(x_1,y)K(x_2,y) + K^2(x_2,y)\right] dy =$$

$$= D^* [K_2(x_1,x_1) - 2K_2(x_1,x_2) + K_2(x_2,x_2)] =$$

$$= D^* [K_2(x_1,x_1) - K_2(x_1,x_2)] + D^* [K_2(x_2,x_2) - K_2(x_1,x_2)]$$

As far as $\lim \chi_n(x) = 0$ we can pass from $\{\omega_n\}$ to $\{\psi_n\}$.

So we justified the following:

<u>Existence Theorem</u>: If the kernel K satisfies (i), (ii) şi (iii), and the continuous function $f(y,u)$ is verifying the condition(1.6) then the nonlinear integral equation(1.1) has at least one solution (continuous one).

2. Dead water theory

The end of the XIX-th century and the beginning of the XX-th century represented for the flows mechanics periods of extreme intense investigations. These generated important works in this field of activity.

Among them is also the cavity theory, whom origin is in H. Helmholtz (1868) and G. Kirchhoff (1869) works, a theory elaborated with the purpose to explain the
D' Alembert paradox.

D' Alembert paradox represents the contradiction between the theoretical result saying that during a straight and uniform moving of a body through an ideal fluid will be no resistance coming from the fluid and the experimental observation that this resistance exist.

Helmholtz created a mathematic model and so he started a theory which became an important one, usually referred as dead water theory.

In this theory there are some nonlinear integral equations. One of those, of Hammerstein type is solved in this paper using some numerical methods.

3. Numerical results

Our purpose is to solve the following equation:

$$T(t)=\frac{\lambda}{\pi}\int_0^{\pi} e^{-T(\sigma)}\ln\left|\frac{\sin\frac{t+\sigma}{2}}{\sin\frac{t-\sigma}{2}}\right|(1+\sin\sigma)r(\sigma)\sin\sigma d\sigma$$

The function which must be integrated has a logarithmic singularity and this one is a week singularity.

We will rewrite the integral equation as:

$$T(t)=\frac{\lambda}{\pi}\int_0^{\pi} e^{-T(\sigma)}\ln\left|\frac{(t-\sigma)\sin\frac{t+\sigma}{2}}{\sin\frac{t-\sigma}{2}}\right|(1+\sin\sigma)r(\sigma)\sin\sigma d\sigma -$$

$$-\frac{\lambda}{\pi}\int_0^{\pi}\left[e^{-T(\sigma)}(1+\sin\sigma)r(\sigma)\sin\sigma - e^{-T(t)}(1+\sin t)r(t)\sin t\right]\ln|t-\sigma|d\sigma -$$

$$-\frac{\lambda}{\pi}e^{-T(t)}(1+\sin t)r(t)\sin t\int_0^{\pi}\ln|t-\sigma|d\sigma$$

The first and the second integral can be calculated using trapezia method and the third one will be analytically calculated.

We will consider in $[0,\pi]$ the nods $\{t_0, t_1, \ldots, t_n\}$ with $t_i = \frac{i}{n}\pi$, $i = 0, \ldots, n$.

Using trapezia method

$$\int_0^\pi f(\sigma)d\sigma = \frac{\pi}{2n}\left[f(t_0) + 2\sum_{i=1}^{n-1} f(t_i) + f(t_n)\right]$$

and, because $\sin t_0 = \sin t_n = 0$:

$$T(t_j) = \frac{\lambda}{n}\left(\begin{array}{l} \sum\limits_{i=1, i\neq j}^{n-1} e^{-T(t_i)} \ln\left|\dfrac{(t_j - t_i)\sin\dfrac{t_i + t_j}{2}}{\sin\dfrac{t_j - t_i}{2}}\right| \cdot \\ \cdot (1+\sin t_i) r(t_i)\sin t_i + e^{-T(t_j)}\ln(2\sin t_j)(1+\sin t_j) r(t_j)\sin t_j \end{array}\right) -$$

$$-\frac{\lambda}{n}\sum_{i=1, i\neq j}^{n-1}\left[e^{-T(t_j)}(1+\sin t_i) r(t_i)\sin t_i - e^{-T(t_j)}(1+\sin t_j) r(t_j)\sin t_j\right]\ln|t_i - t_j| -$$

$$-\frac{\lambda}{\pi} e^{-T(t_j)}(1+\sin t_j) r(t_j)\sin t_j\left[(\pi - t_j)\ln(\pi - t_j) + t_j\ln t_j - \pi\right]$$

Therefore, for $j = 1, \ldots, n-1$

$$T(t_j) = \frac{\lambda}{n}\sum_{i=1, i\neq j}^{n-1} e^{-T(t_i)}\ln\left|\frac{\sin\dfrac{t_i + t_j}{2}}{\sin\dfrac{t_j - t_i}{2}}\right|(1+\sin t_i) r(t_i)\sin t_i +$$

$$+\frac{\lambda}{n} e^{-T(t_j)}(1+\sin t_j) r(t_j)\sin t_j \cdot$$

$$\cdot\left[\frac{1}{n}\ln(2\sin t_j) + \sum_{i=1, i\neq j}^{n-1}\ln|t_i - t_j| - \frac{\pi - t_j}{\pi}\ln(\pi - t_j) - \frac{t_j}{\pi}\ln t_j + 1\right]$$

So finally we have to solve the algebric system:

$$T_j = \lambda\sum_{i=1}^{n-1} w_{ji} e^{-T_i} \qquad (*)$$

with $T_i = T(t_i) \qquad i = 1,\ldots,n$

and

$$w_{ji} = \frac{1}{n}\ln\left|\frac{\sin\frac{t_j + t_i}{2}}{\sin\frac{t_j - t_i}{2}}\right|(1+\sin t_i)r(t_i)\sin t_i \ge 0 \qquad\qquad i \ne j$$

Respectively

$$w_{jj} = (1+\sin t_j)r(t_j)\sin t_j\left[\frac{1}{n}\ln(2\sin t_j) + \frac{1}{n}\sum_{i=1,i\ne j}^{n-1}\ln|t_i - t_j| - \frac{\pi - t_j}{\pi}\ln(\pi - t_j) - \frac{t_j}{\pi}\ln t_j + 1\right] \ge 0$$

We intend to estimate the differences:

$$\left|\lambda\sum_{i=1}^{n-1} w_{ji}e^{-T_i} - \lambda\sum_{i=1}^{n-1} w_{ji}e^{-S_i}\right| \le \lambda\sum_{i=1}^{n-1} w_{ji}\left|e^{-T_i} - e^{-S_i}\right| \le \lambda\sum_{i=1}^{n-1} w_{ji}|T_i - S_i|$$

Because of the definition of w_{ji} we say that:

$$\lambda\sum_{i=1}^{n-1} w_{ji} \cong \frac{1}{\pi}\int_0^{\pi}\ln\left|\frac{\sin\frac{t_j + \sigma}{2}}{\sin\frac{t_j - \sigma}{2}}\right|(1+\sin\sigma)r(\sigma)\sin\sigma d\sigma \le$$

$$\le \frac{\lambda}{\pi}\int_0^{\pi}\ln\left|\frac{\sin\frac{t_j + \sigma}{2}}{\sin\frac{t_j - \sigma}{2}}\right|(1+\sin\sigma)\sin\sigma d\sigma =$$

$$= \frac{\lambda}{\pi}\int_0^{\pi}\sum_{m\ge1}\frac{1}{m}\sin mt_j \sin m\sigma(2\sin\sigma + 1 - \cos 2\sigma)d\sigma =$$

$$= \frac{\lambda}{\pi}\sin t_j\int_0^{\pi} 2\sin\sigma d\sigma + \frac{\lambda}{\pi}\int_0^{\pi}\sum_{m\ge1}\frac{1}{m}\sin mt_j \sin m\sigma d\sigma =$$

$$= \lambda \sin t_j + \frac{\lambda}{\pi} \sum_{m \geq 1} \frac{1}{m^2} \sin m t_j \cos m\sigma \Big|_{\pi}^{0} =$$

$$= \lambda \sin t_j + \frac{\lambda}{\pi} \sum_{m \geq 1} \frac{1}{m^2} \sin m t_j \cdot \left(1 - (-1)^m\right) \leq$$

$$\leq \lambda + \frac{\lambda}{\pi} \sum_{n \geq 0} \frac{2}{(2n + L)^2} \leq \lambda + \frac{\lambda}{\pi} \left(2 + \sum_{n \geq 1} \frac{2}{(2n+1)(2n-1)} \right) =$$
$$= \lambda + \frac{\lambda}{\pi} \left(2 + \sum_{n \geq 1} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right) = \frac{\lambda(3+\pi)}{\pi}$$

So we can say that the system (*) has only one solution for $0 < \lambda < \dfrac{\pi}{3+\pi}$, and this solution can be found using the successive approximation method.

Conclusions

Some fields of activities, for example the one regarding the flows studies can create, after being modeled in a mathematical way, some nonlinear integrals and their solving is of equal interest for mathematicians and also for the ones working in more practical aspects.

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