U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 3, 2014 ISSN 1223-7027

ON MULTI-TIME HAMILTON-JACOBI THEORY VIA SECOND ORDER LAGRANGIANS

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This paper aims to study some aspects of Hamilton-Jacobi theory based on multi-time second order Lagrangians, namely: Hamilton-Jacobi divergence PDE, Hamilton-Jacobi system of PDEs, generating function, change of variables in Hamiltonian, gauge transformation.

Keywords: Hamilton-Jacobi system of PDEs; multi-time second order Lagrangian; generating function; Hamilton-Jacobi divergence PDE; Legendrian duality; canonical multi-momenta.
MSC2010: 35F21, 70H20, 49L99.

1. Introduction

The classical single-time Hamilton-Jacobi theory appeared in mechanics from the desire to describe the motion of a particle by a wave. In this direction, the Euler-Lagrange equations or the associated Hamilton equations are replaced by partial differential equations that describe the generating function. This theory is related to Legendre transformation, the change of variables in Hamiltonian and the generating function.

In the sequel, we shall consider the *multi-index notation* introduced by D. J. Saunders (see [10]). He defines a *multi-index* as an m-tuple I of natural numbers and the components of I are denoted $I(\alpha)$, where α is an ordinary index, $1 \leq \alpha \leq m$. The multi-index 1_α (defined by $1_\alpha(\alpha) := 1,\ 1_\alpha(\beta) := 0$) will be frequently used in this work. The addition and the substraction of multi-indexes are defined componentwise (although the result of a substraction might not be a multi-index!). The *length* of a multi-index is defined as $\mid I \mid := \sum_{\alpha=1}^{m} I(\alpha)$ and its *factorial* is $I! := \prod_{\alpha=1}^{m} (I(\alpha))!$. The number of distinct indices represented by $\{\alpha_1, \alpha_2, ..., \alpha_k\},\ \alpha_j \in \{1, 2, ..., m\}$, $j = \overline{1, k}$, is

$$n(\alpha_1, \alpha_2, ..., \alpha_k) = \frac{\mid 1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k} \mid !}{(1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k})!}.$$

For a better understanding of the previous notion, let consider the following particular cases:

- $k = 1$ involves: $n(\alpha_1) = 1$
- $k = 2$ involves:

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$n(\alpha_1, \alpha_2) = 1$, for $\alpha_1 = \alpha_2$
$n(\alpha_1, \alpha_2) = 2$, for $\alpha_1 \neq \alpha_2$
• $k = 3$ involves:
$n(\alpha_1, \alpha_2, \alpha_3) = 1$, for $\alpha_1 = \alpha_2 = \alpha_3$
$n(\alpha_1, \alpha_2, \alpha_3) = 3$, for $\alpha_1 = \alpha_2 \neq \alpha_3$
$n(\alpha_1, \alpha_2, \alpha_3) = 6$, for $\alpha_1 \neq \alpha_2 \neq \alpha_3$.
Also, let assume the following notations:

$$x_\alpha(t) := \frac{\partial x}{\partial t^\alpha}(t), \quad x_{\alpha\beta}(t) := \frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t) \quad \alpha, \beta \in \{1, ..., m\}.$$

For other different ideas but connected to this subject, the reader is addressed to [1]-[9] and [13]. In [11], higher order PDEs of Hamilton-Jacobi and parabolic type are solved relying on the characteristic system method combined with a fundamental system of solutions in the kernel of the corresponding linear operator. The analysis encompasses those cases for which the iterated linear operator includes first and second order derivations with respect to some given smooth vector fields.

2. **Hamilton-Jacobi system of PDEs based on second order Lagrangians**

The single-time case (via second order Lagrangians) was already studied in another research paper by C. Udrişte and A. Pitea. For consulting it, the reader is addressed to [12].

The multi-time case. Let consider the function $S : R^n \times R^{nm} \times R^m \to R$ and the constant level sets $\Sigma_c : S(x, x_\gamma, t) = c, \;\; \gamma \in \{1, 2, ..., m\}$. We suppose that these sets are submanifolds in R^{n+m+nm}, that is the normal vector field $\left(\frac{\partial S}{\partial x^i}, \frac{\partial S}{\partial x^i_\gamma}, \frac{\partial S}{\partial t^\gamma}\right)$ is nowhere zero. Let $\tilde{\Gamma} : \left(x^i(t), x^i_\gamma(t), t\right), \;\; t \in R^m$, be an m-sheet transversal to the submanifolds Σ_c. Then, the real function $c(t) = S(x(t), x_\gamma(t), t)$ has nonzero partial derivatives (the summation over the repeated indices is assumed!)

$$\frac{\partial c}{\partial t^\zeta} = \frac{\partial S}{\partial x^i}(x(t), x_\gamma(t), t) \frac{\partial x^i}{\partial t^\zeta}(t) + \frac{\partial S}{\partial x^i_\gamma}(x(t), x_\gamma(t), t) \frac{\partial x^i_\gamma}{\partial t^\zeta}(t)$$

$$+ \frac{\partial S}{\partial t^\zeta}(x(t), x_\gamma(t), t) := \Delta_\zeta(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t) \neq 0.$$

We admit that $L_\zeta(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t) = \Delta_\zeta(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t)$, that is, the Lagrange 1-form L_ζ is the total derivative of the function $c(\cdot)$. Let suppose it has independent variables (for instance, the variable $x_{12}(t)$ is the same with the variable $x_{21}(t)$ and, consequently, only one appears as variable of L_ζ). By computation, we obtain the *(generalized) canonical multi-momenta*

$$p^\gamma_{\zeta,i} := \frac{\partial L_\zeta}{\partial x^i_\gamma} = \frac{\partial \Delta_\zeta}{\partial x^i_\gamma} = \frac{\partial S}{\partial x^i}\delta^\gamma_\zeta,$$

or, explicitly, $p^{\gamma}_{\zeta,i}=0$, for $\gamma\neq\zeta$ and $p^{\gamma}_{\zeta,i}=\dfrac{\partial S}{\partial x^i}$, for $\gamma=\zeta$, and

$$q^{\gamma\zeta}_{\zeta,i}:=\frac{1}{n(\gamma,\zeta)}\frac{\partial L_\zeta}{\partial x^i_{\gamma\zeta}}=\frac{1}{n(\gamma,\zeta)}\frac{\partial \Delta_\zeta}{\partial x^i_{\gamma\zeta}}=\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^i_{\gamma}}.$$

So, the link $H_\zeta=x^i_\gamma p^\gamma_{\zeta,i}+x^i_{\gamma\zeta}q^{\gamma\zeta}_{\zeta,i}-L_\zeta$ becomes $H_\zeta=x^i_\zeta\dfrac{\partial S}{\partial x^i}+x^i_{\gamma\zeta}\dfrac{\partial S}{\partial x^i_\gamma}-L_\zeta$.

Denote $p=(p^\zeta_{\zeta,i})$, without summation relative to ζ. We accept that the previous relations define a Legendre duality. In these conditions, the relations

$$x^i_\zeta(t)=x^i_\zeta\left(x^j(t),p^\zeta_{\zeta,j}(t),q^{\gamma\zeta}_{\zeta,j}(t),t\right)$$
$$x^i_{\gamma\zeta}(t)=x^i_{\gamma\zeta}\left(x^j(t),p^\zeta_{\zeta,j}(t),q^{\gamma\zeta}_{\zeta,j}(t),t\right)$$

become

$$x^i_\zeta(t)=x^i_\zeta\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)$$
$$x^i_{\gamma\zeta}(t)=x^i_{\gamma\zeta}\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right),$$

where $i,\ j\in\{1,2,...,n\}$ and $\zeta,\ \gamma\in\{1,2,...,m\}$.

Also, the relation

$$L_\zeta\left(x^i(t),x^i_\gamma(t),x^i_{\gamma\zeta}(t),t\right)=\frac{\partial S}{\partial x^i}\left(x(t),x_\gamma(t),t\right)$$
$$\cdot\frac{\partial x^i}{\partial t^\zeta}\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)$$
$$+\frac{\partial S}{\partial x^i_\gamma}\left(x(t),x_\gamma(t),t\right)$$
$$\cdot\frac{\partial x^i_\gamma}{\partial t^\zeta}\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)$$
$$+\frac{\partial S}{\partial t^\zeta}\left(x(t),x_\gamma(t),t\right)$$

can be rewritten as

$$-\frac{\partial S}{\partial t^\zeta}\left(x(t),x_\gamma(t),t\right)=\frac{\partial S}{\partial x^i}\left(x(t),x_\gamma(t),t\right)$$
$$\cdot\frac{\partial x^i}{\partial t^\zeta}\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)$$
$$+\frac{\partial S}{\partial x^i_\gamma}\left(x(t),x_\gamma(t),t\right)$$
$$\cdot\frac{\partial x^i_\gamma}{\partial t^\zeta}\left(x^j(t),\frac{\partial S}{\partial x^j}\left(x(t),x_\gamma(t),t\right),\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^j_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)$$
$$-L_\zeta\left(x^i(t),x^i_\gamma(t),x^i_{\gamma\zeta}(t),t\right),$$

or as *Hamilton-Jacobi system of PDEs based on second order Lagrangians*

$$(H-J)\qquad \frac{\partial S}{\partial t^\zeta}+H_\zeta\left(x,\frac{\partial S}{\partial x},\frac{\partial S}{\partial x_\gamma},t\right)=0,\ \ \zeta\in\{1,2,...,m\}.$$

Usually, the Hamilton-Jacobi system of PDEs based on second order Lagrangians is accompanied by the initial condition $S(x,x_\gamma,0)=S_0(x,x_\gamma)$ and by completely integrability conditions. The solution $S(x,x_\gamma,t)$ is called the *generating function* of the canonical multi-momenta.

Remark 2.1 Conversely, let $S(x,x_\gamma,t)$ be a solution of the Hamilton-Jacobi system of PDEs based on second order Lagrangians. We define

$$p^\zeta_{\zeta,i}(t)=\frac{\partial S}{\partial x^i}\left(x(t),x_\gamma(t),t\right);\quad p^\gamma_{\zeta,i}(t)=0,\ (\gamma\neq\zeta)$$

$$q^{\gamma\zeta}_{\zeta,i}(t)=\frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x^i_\gamma}\left(x(t),x_\gamma(t),t\right).$$

Then, the following link appears (see Γ_{t_0,t_1} as being the curve joining the points $t_0,t_1\in R^m$)

$$\int_{\Gamma_{t_0,t_1}}L_\zeta\left(x(t),x_\gamma(t),x_{\gamma\zeta}(t),t\right)dt^\zeta$$

$$=\int_{\Gamma_{t_0,t_1}}[\frac{\partial S}{\partial x^i}\left(x(t),x_\gamma(t),t\right)x^i_\zeta(t)+\frac{\partial S}{\partial x^i_\gamma}\left(x(t),x_\gamma(t),t\right)x^i_{\gamma\zeta}(t)$$

$$-H_\zeta\left(x(t),\frac{\partial S}{\partial x^i}\left(x(t),x_\gamma(t),t\right),\frac{\partial S}{\partial x^i_\gamma}\left(x(t),x_\gamma(t),t\right),t\right)]dt^\zeta$$

$$=\int_\Gamma\frac{\partial S}{\partial x^i}dx^i+\frac{\partial S}{\partial x^i_\gamma}dx^i_\gamma+\frac{\partial S}{\partial t^\zeta}dt^\zeta=\int_\Gamma dS.$$

The last formula shows that the action integral can be written as a path independent curvilinear integral.

Theorem 2.2 *The generating function of the canonical multi-momenta is solution of the Cauchy problem*

$$\frac{\partial S}{\partial t^\zeta}+H_\zeta\left(x,\frac{\partial S}{\partial x},\frac{\partial S}{\partial x_\gamma},t\right)=0,\ \ S(x,x_\gamma,0)=S_0(x,x_\gamma),\ \ \zeta\in\{1,2,...,m\}.$$

3. Gauge transformation and moments for second order Lagrangians

The single-time case. Let us suppose that two second order Lagrangians, $L^1\left(x(t),\dot{x}(t),\ddot{x}(t),t\right)$ and $L^2\left(x(t),\dot{x}(t),\ddot{x}(t),t\right),\ t\in[t_0,t_1]\subset R,\ x(\cdot)\in R^n$, are joined by a transformation of *gauge* type, i.e.,

$$L^2=L^1+\frac{d}{dt}f(x,\dot{x},t)=L^1+\frac{\partial f}{\partial x^i}\dot{x}^i+\frac{\partial f}{\partial \dot{x}^i}\ddot{x}^i+\frac{\partial f}{\partial t},\quad i=\overline{1,n}.$$

The summation over the repeated indices is assumed. Then, the corresponding moments $p^1_i,\ p^2_i,\ q^1_i,\ q^2_i$ satisfy the following relations

$$p^2_i:=\frac{\partial L^2}{\partial \dot{x}^i}=\frac{\partial L^1}{\partial \dot{x}^i}+\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i}+\frac{\partial f}{\partial x^i}=p^1_i+\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i}+\frac{\partial f}{\partial x^i}$$

$$q_i^2 := \frac{\partial L^2}{\partial \ddot{x}^i} = \frac{\partial L^1}{\partial \ddot{x}^i} + \frac{\partial f}{\partial \dot{x}^i} = q_i^1 + \frac{\partial f}{\partial \dot{x}^i}, \quad i = \overline{1,n}.$$

By a direct computation, we get

$$\frac{\partial L^2}{\partial x^i} = \frac{\partial L^1}{\partial x^i} + \frac{d}{dt}\frac{\partial f}{\partial x^i}, \quad \frac{\partial L^2}{\partial \dot{x}^i} = \frac{\partial L^1}{\partial \dot{x}^i} + \frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i} + \frac{\partial f}{\partial x^i}$$

$$\frac{\partial L^2}{\partial \ddot{x}^i} = \frac{\partial L^1}{\partial \ddot{x}^i} + \frac{\partial f}{\partial \dot{x}^i}, \quad i = \overline{1,n}.$$

The previous reasonings and computations lead us to the following result.

Proposition 3.1 *Two single-time second order Lagrangians satisfying* $L^2 = L^1 + \frac{d}{dt} f(x, \dot{x}, t)$*, where* L^2*,* L^1 *and* f *are considered* C^3*-class functions, produce the same Euler-Lagrange ODEs, i.e.,*

$$\frac{\partial L^2}{\partial x^i} - \frac{d}{dt}\frac{\partial L^2}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L^2}{\partial \ddot{x}^i} = \frac{\partial L^1}{\partial x^i} - \frac{d}{dt}\frac{\partial L^1}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L^1}{\partial \ddot{x}^i}, \quad i = \overline{1,n}.$$

The multi-time case. Consider two multi-time second order Lagrangians, $L^1(x(t), x_\gamma(t), x_{\alpha\beta}(t), t)$ and $L^2(x(t), x_\gamma(t), x_{\alpha\beta}(t), t)$, where $t \in \Omega_{t_0,t_1} \subset R^m$, $\alpha, \beta, \gamma \in \{1, 2, ..., m\}$, $x(\cdot) \in R^n$. Denote $x_\gamma(t) := \frac{\partial x}{\partial t^\gamma}(t)$, $x_{\alpha\beta}(t) := \frac{\partial^2 x}{\partial t^\alpha \partial t^\beta}(t)$. Like any function, the previous Lagrangians have independent variables. Suppose they are joined by a transformation of *gauge* type, i.e.,

$$L^2 = L^1 + D_\zeta f^\zeta(x, x_\gamma, t) = L^1 + \frac{\partial f^\zeta}{\partial t^\zeta} + \frac{\partial f^\zeta}{\partial x^i}\frac{\partial x^i}{\partial t^\zeta} + \frac{\partial f^\zeta}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\zeta}$$

$$\zeta = \overline{1,m}, \quad i = \overline{1,n}.$$

The summation over the repeated indices is assumed. The associated moments p^γ_{i1}, p^γ_{i2}, $q^{\alpha\beta}_{i1}$, $q^{\alpha\beta}_{i2}$ satisfy the following relations

$$p^\gamma_{i2} := \frac{\partial L^2}{\partial x^i_\gamma} = \frac{\partial L^1}{\partial x^i_\gamma} + D_\zeta \frac{\partial f^\zeta}{\partial x^i_\gamma} + \frac{\partial f^\gamma}{\partial x^i} = p^\gamma_{i1} + D_\zeta \frac{\partial f^\zeta}{\partial x^i_\gamma} + \frac{\partial f^\gamma}{\partial x^i}$$

$$q^{\alpha\beta}_{i2} := \frac{\partial L^2}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1}{\partial x^i_{\alpha\beta}} + \frac{\partial f^\zeta}{\partial x^i_\gamma}\delta^\gamma_\alpha \delta^\zeta_\beta = q^{\alpha\beta}_{i1} + \frac{\partial f^\beta}{\partial x^i_\alpha},$$

and a direct calculation gives us

$$\frac{\partial L^2}{\partial x^i} = \frac{\partial L^1}{\partial x^i} + D_\zeta \frac{\partial f^\zeta}{\partial x^i}, \quad \frac{\partial L^2}{\partial x^i_\gamma} = \frac{\partial L^1}{\partial x^i_\gamma} + D_\zeta \frac{\partial f^\zeta}{\partial x^i_\gamma} + \frac{\partial f^\gamma}{\partial x^i}$$

$$\frac{\partial L^2}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1}{\partial x^i_{\alpha\beta}} + \frac{\partial f^\beta}{\partial x^i_\alpha}.$$

Proposition 3.2 *Two multi-time second order Lagrangians, satisfying* $L^2 = L^1 + D_\zeta f^\zeta(x, x_\gamma, t)$*, where* L^2*,* L^1 *and* f *are considered* C^3*-class functions, produce the same PDEs, i.e.,*

$$\frac{\partial L^2}{\partial x^i} - D_\gamma \frac{\partial L^2}{\partial x^i_\gamma} + D^2_{\alpha\beta}\frac{\partial L^2}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1}{\partial x^i} - D_\gamma \frac{\partial L^1}{\partial x^i_\gamma} + D^2_{\alpha\beta}\frac{\partial L^1}{\partial x^i_{\alpha\beta}}, \quad i = \overline{1,n}.$$

Proof. Direct computation.

Remark 3.3 The PDEs in Proposition 3.2 are similar to the multi-time Euler-Lagrange PDEs,

$$\frac{\partial L}{\partial x^i} - D_\gamma \frac{\partial L}{\partial x^i_\gamma} + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial L}{\partial x^i_{\alpha\beta}} = 0, \quad i = \overline{1,n},$$

where $n(\alpha,\beta) = 1$ for $\alpha = \beta$, and $n(\alpha,\beta) = 2$, for $\alpha \neq \beta$.

Corollary 3.4 *If* $\dfrac{\partial f^\beta}{\partial x^i_\alpha} = 0$ *for* $\alpha \neq \beta$*, then the PDEs which appear in Proposition 3.2 can be rewritten as multi-time Euler-Lagrange PDEs*

$$\frac{\partial L^2}{\partial x^i} - D_\gamma \frac{\partial L^2}{\partial x^i_\gamma} + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial L^2}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1}{\partial x^i} - D_\gamma \frac{\partial L^1}{\partial x^i_\gamma} + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial L^1}{\partial x^i_{\alpha\beta}}.$$

Remark 3.5 The above multi-time case takes into account the total divergence of f. As well, we can consider multi-time second order Lagrangian 1-forms, $L^j_\zeta\,(x(t), x_\gamma(t), x_{\alpha\beta}(t), t)\, dt^\zeta$, $j = 1, 2$. In this situation, the transformation of *gauge* type becomes $L^2_\zeta = L^1_\zeta + D_\zeta f$ and the corresponding moments $p^\gamma_{i\zeta,1}$, $p^\gamma_{i\zeta,2}$, $q^{\alpha\beta}_{i\zeta,1}$, $q^{\alpha\beta}_{i\zeta,2}$ satisfy the following relations

$$p^\gamma_{i\zeta,2} := \frac{\partial L^2_\zeta}{\partial x^i_\gamma} = \frac{\partial L^1_\zeta}{\partial x^i_\gamma} + D_\zeta \frac{\partial f}{\partial x^i_\gamma} + \frac{\partial f}{\partial x^i} \delta^\gamma_\zeta = p^\gamma_{i\zeta,1} + D_\zeta \frac{\partial f}{\partial x^i_\gamma} + \frac{\partial f}{\partial x^i} \delta^\gamma_\zeta$$

$$q^{\alpha\beta}_{i\zeta,2} := \frac{\partial L^2_\zeta}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1_\zeta}{\partial x^i_{\alpha\beta}} + \frac{\partial f}{\partial x^i_\gamma} \delta^\gamma_\alpha \delta^\zeta_\beta = q^{\alpha\beta}_{i\zeta,1} + \frac{\partial f}{\partial x^i_\alpha} \delta^\beta_\zeta.$$

Knowing that

$$\frac{\partial L^2_\zeta}{\partial x^i_\gamma} = \frac{\partial L^1_\zeta}{\partial x^i_\gamma} + D_\zeta \frac{\partial f}{\partial x^i_\gamma} + \frac{\partial f}{\partial x^i} \delta^\gamma_\zeta, \quad \frac{\partial L^2_\zeta}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1_\zeta}{\partial x^i_{\alpha\beta}} + \frac{\partial f}{\partial x^i_\alpha} \delta^\beta_\zeta$$

$$\frac{\partial L^2_\zeta}{\partial x^i} = \frac{\partial L^1_\zeta}{\partial x^i} + D_\zeta \frac{\partial f}{\partial x^i}$$

we establish the following result.

Proposition 3.6 *Two multi-time second order Lagrangian* 1*-forms, satisfying* $L^2_\zeta = L^1_\zeta + D_\zeta f(x, x_\gamma, t)$*, where* L^2_ζ*,* L^1_ζ *and* f *are considered* C^3*-class functions, produce the same PDEs, i.e.,*

$$\frac{\partial L^2_\zeta}{\partial x^i} - D_\gamma \frac{\partial L^2_\zeta}{\partial x^i_\gamma} + D^2_{\alpha\beta} \frac{\partial L^2_\zeta}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1_\zeta}{\partial x^i} - D_\gamma \frac{\partial L^1_\zeta}{\partial x^i_\gamma} + D^2_{\alpha\beta} \frac{\partial L^1_\zeta}{\partial x^i_{\alpha\beta}}, \quad i = \overline{1,n}.$$

Corollary 3.7 *If* $D^2_{\gamma\zeta} \dfrac{\partial f}{\partial x^i_\gamma} = 0$ *for* $\gamma \neq \zeta$*, then the PDEs in Proposition 3.6 can be rewritten as multi-time Euler-Lagrange PDEs*

$$\frac{\partial L^2_\zeta}{\partial x^i} - D_\gamma \frac{\partial L^2_\zeta}{\partial x^i_\gamma} + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial L^2_\zeta}{\partial x^i_{\alpha\beta}} = \frac{\partial L^1_\zeta}{\partial x^i} - D_\gamma \frac{\partial L^1_\zeta}{\partial x^i_\gamma} + \frac{1}{n(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial L^1_\zeta}{\partial x^i_{\alpha\beta}}.$$

4. **The change of variables in Hamiltonian and the generating function for second order Lagrangians**

The single-time case. Let $H = \dot{x}^i p_i + \ddot{x}^i q_i - L$ be the Hamiltonian and

$$\frac{dp_i}{dt}(t) - \frac{d^2 q_i}{dt^2}(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t), q(t), t)$$

$$\frac{dx^i}{dt}(t) = \frac{\partial H}{\partial p_i}(x(t), p(t), q(t), t), \quad \frac{d^2 x^i}{dt^2}(t) = \frac{\partial H}{\partial q_i}(x(t), p(t), q(t), t)$$

the associated ODEs. Let suppose that we want to pass from our coordinates (x^i, p_i, q_i, t) to the coordinates (X^i, P_i, Q_i, t) with the following change of variables (difeomorphism)

$$X^k = X^k(x^i, p_i, q_i, t), \;\; P_k = P_k(x^i, p_i, q_i, t),$$

$$Q_k = Q_k(x^i, p_i, q_i, t), \;\; k \in \{1, 2, ..., n\}.$$

Then, the Hamiltonian $H(x, p, q, t)$ changes in $K(X, P, Q, t)$. The above change of variables is called *canonical transformation* if there is a Hamiltonian, $K(X, P, Q, t)$, such that the associated ODEs,

$$\frac{dP_i}{dt}(t) - \frac{d^2 Q_i}{dt^2}(t) = -\frac{\partial K}{\partial X^i}(X(t), P(t), Q(t), t)$$

$$\frac{dX^i}{dt}(t) = \frac{\partial K}{\partial P_i}(X(t), P(t), Q(t), t), \quad \frac{d^2 X^i}{dt^2}(t) = \frac{\partial K}{\partial Q_i}(X(t), P(t), Q(t), t),$$

and the ODEs

$$\frac{dp_i}{dt}(t) - \frac{d^2 q_i}{dt^2}(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t), q(t), t)$$

$$\frac{dx^i}{dt}(t) = \frac{\partial H}{\partial p_i}(x(t), p(t), q(t), t), \quad \frac{d^2 x^i}{dt^2}(t) = \frac{\partial H}{\partial q_i}(x(t), p(t), q(t), t)$$

take place simultaneously. This thing is possible if the functions

$$\dot{x}^i(t) p_i(t) + \ddot{x}^i(t) q_i(t) - H(x(t), p(t), q(t), t)$$

and

$$\dot{X}^i(t) P_i(t) + \ddot{X}^i(t) Q_i(t) - K(X(t), P(t), Q(t), t)$$

differ by a total derivative $\dfrac{dW}{dt}(x(t), \dot{x}(t), t)$.

Lemma 4.1 *If the Lagrangians*

$$L_1 := \dot{x}^i p_i + \ddot{x}^i q_i - H$$

$$L_2 := \dot{X}^i P_i + \ddot{X}^i Q_i - K$$

produce the same Euler-Lagrange ODEs, then the change of variables

$$(x^i, p_i, q_i, t) \hookrightarrow (X^i, P_i, Q_i, t), \quad i = \overline{1, n}$$

is a canonical transformation.

Proof. Using Proposition 3.1 the result is obvious.

The function W is called the *generating function* of the canonical transformation.

The multi-time case. Let $H = x^i_\gamma p^\gamma_i + x^i_{\alpha\beta} q^{\alpha\beta}_i - L$ be the Hamiltonian and

$$D_\gamma p^\gamma_i(t) - D^2_{\alpha\beta} q^{\alpha\beta}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t), q(t), t)$$

$$\frac{\partial x^i}{\partial t^\gamma}(t) = \frac{\partial H}{\partial p^\gamma_i}(x(t), p(t), q(t), t), \quad \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}(t) = \frac{\partial H}{\partial q^{\alpha\beta}_i}(x(t), p(t), q(t), t)$$

the associated PDEs (see $p^\gamma_i := \frac{\partial L}{\partial x^i_\gamma}$, $q^{\alpha\beta}_i := \frac{1}{n(\alpha,\beta)} \frac{\partial L}{\partial x^i_{\alpha\beta}}$). The summation over the repeated indices is assumed. Let suppose that we want to pass from our coordinates (x^i, p_i, q_i, t) to the coordinates (X^i, P_i, Q_i, t) with the following change of variables (difeomorphism)

$$X^k = X^k(x^i, p_i, q_i, t), \ \ P_k = P_k(x^i, p_i, q_i, t),$$

$$Q_k = Q_k(x^i, p_i, q_i, t), \ \ k \in \{1, 2, ..., n\}.$$

Then, the Hamiltonian $H(x, p, q, t)$ changes in $K(X, P, Q, t)$. The above change of variables is called *canonical transformation* if there is a Hamiltonian, $K(X, P, Q, t)$, such that the associated PDEs,

$$D_\gamma P^\gamma_i(t) - D^2_{\alpha\beta} Q^{\alpha\beta}_i(t) = -\frac{\partial K}{\partial X^i}\left(X(t), P(t), Q(t), t\right)$$

$$\frac{\partial X^i}{\partial t^\gamma}(t) = \frac{\partial K}{\partial P^\gamma_i}\left(X(t), P(t), Q(t), t\right)$$

$$\frac{\partial^2 X^i}{\partial t^\alpha \partial t^\beta}(t) = \frac{\partial K}{\partial Q^{\alpha\beta}_i}\left(X(t), P(t), Q(t), t\right),$$

and the PDEs

$$D_\gamma p^\gamma_i(t) - D^2_{\alpha\beta} q^{\alpha\beta}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t), q(t), t)$$

$$\frac{\partial x^i}{\partial t^\gamma}(t) = \frac{\partial H}{\partial p^\gamma_i}(x(t), p(t), q(t), t), \quad \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta}(t) = \frac{\partial H}{\partial q^{\alpha\beta}_i}(x(t), p(t), q(t), t)$$

take place simultaneously. But, according to Corollary 3.4, this thing is possible if the functions

$$x^i_\gamma(t) p^\gamma_i(t) + x^i_{\alpha\beta}(t) q^{\alpha\beta}_i(t) - H\left(x(t), p(t), q(t), t\right)$$

and

$$X^i_\gamma(t) P^\gamma_i(t) + X^i_{\alpha\beta}(t) Q^{\alpha\beta}_i(t) - K\left(X(t), P(t), Q(t), t\right)$$

differ by a total divergence $D_\zeta W^\zeta\left(x(t), x_\gamma(t), t\right)$ and $\frac{\partial W^\beta}{\partial x^i_\alpha} = 0$ for $\alpha \neq \beta$.

Lemma 4.2 *If the Lagrangians*

$$L_1 := x^i_\gamma p^\gamma_i + x^i_{\alpha\beta} q^{\alpha\beta}_i - H, \quad L_2 := X^i_\gamma P^\gamma_i + X^i_{\alpha\beta} Q^{\alpha\beta}_i - K$$

produce the same multi-time Euler-Lagrange PDEs (see $\frac{\partial W^\beta}{\partial x^i_\alpha} = 0$ *for* $\alpha \neq \beta$*, where* $L_2 = L_1 + D_\zeta W^\zeta$*), then the change of variables*

$$(x^i, p_i, q_i, t) \hookrightarrow (X^i, P_i, Q_i, t), \quad i = \overline{1, n}$$

is a canonical transformation.

Proof. Using Corollary 3.4 the result is obvious.

The function W is called the *multi-time generating function* of the canonical transformation.

5. **Hamilton-Jacobi divergence PDE based on second order Lagrangians**

Further, following the same algorithm as in the second section, let consider the function $S : R^n \times R^{nm} \times R^m \to R^m$ to whom we attach the constant level sets $\Sigma_c : S(x, x_\gamma, t) = c$, or $\Sigma_c : S^\zeta(x, x_\gamma, t) = c^\zeta$, for $\gamma,\ \zeta \in \{1, 2, ..., m\}$. Assuming that these sets are submanifolds in R^{n+m+nm}, the normal vector fields $\left(\frac{\partial S^\zeta}{\partial x^i}, \frac{\partial S^\zeta}{\partial x^i_\gamma}, \frac{\partial S^\zeta}{\partial t^\gamma}\right)$ are nowhere zero. Let $\Gamma : \left(x^i(t), x^i_\gamma(t), t\right),\ \ t \in R^m$, be an m-sheet transversal to the submanifolds Σ_c. Then, the vectorial function $c(t) = S(x(t), x_\gamma(t), t)$ has nonzero total divergence

$$Div(c) = \frac{\partial c^\zeta}{\partial t^\zeta}(t) = \frac{\partial S^\zeta}{\partial x^i}(x(t), x_\gamma(t), t)\,\frac{\partial x^i}{\partial t^\zeta}(t) + \frac{\partial S^\zeta}{\partial x^i_\gamma}(x(t), x_\gamma(t), t)\,\frac{\partial x^i_\gamma}{\partial t^\zeta}(t)$$

$$+\frac{\partial S^\zeta}{\partial t^\zeta}(x(t), x_\gamma(t), t) := \Delta(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t) \neq 0.$$

Now, let us introduce the Lagrangian

$$L(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t) := \Delta(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t)\,.$$

By computation, we obtain the *generalized (canonical) multi-momenta* $p = (p^\gamma_i),\ \ q = (q^{\gamma\zeta}_i),\ \ i \in \{1, 2, ..., n\}$, as

$$p^\gamma_i := \frac{\partial L}{\partial x^i_\gamma} = \frac{\partial \Delta}{\partial x^i_\gamma} = \frac{\partial S^\zeta}{\partial x^i}\delta^\gamma_\zeta,$$

or, explicitly, $p^\gamma_i = 0$, for $\gamma \neq \zeta$ and $p^\gamma_i = \frac{\partial S^\zeta}{\partial x^i}$, for $\gamma = \zeta$, and

$$q^{\gamma\zeta}_i := \frac{1}{n(\gamma,\zeta)}\frac{\partial L}{\partial x^i_{\gamma\zeta}} = \frac{1}{n(\gamma,\zeta)}\frac{\partial \Delta}{\partial x^i_{\gamma\zeta}} = \frac{1}{n(\gamma,\zeta)}\frac{\partial S^\zeta}{\partial x^i_\gamma}.$$

We accept that these relations define a Legendre duality. In these conditions, the relations

$$x^i_\zeta(t) = x^i_\zeta(x(t), p(t), q(t), t)$$

$$x^i_{\gamma\zeta}(t) = x^i_{\gamma\zeta}(x(t), p(t), q(t), t)$$

become

$$x^i_\zeta(t) = x^i_\zeta\left(x(t), \frac{\partial S}{\partial x}(x(t), x_\gamma(t), t)\,, \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}(x(t), x_\gamma(t), t)\,, t\right)$$

$$x^i_{\gamma\zeta}(t) = x^i_{\gamma\zeta}\left(x(t), \frac{\partial S}{\partial x}(x(t), x_\gamma(t), t)\,, \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}(x(t), x_\gamma(t), t)\,, t\right),$$

where $i \in \{1, 2, ..., n\}$ and $\zeta,\ \gamma \in \{1, 2, ..., m\}$. On the other hand, the relation

$$L\left(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t\right) = \frac{\partial S^\zeta}{\partial x^i}\left(x(t), x_\gamma(t), t\right)$$

$$\cdot \frac{\partial x^i}{\partial t^\zeta}\left(x(t), \frac{\partial S}{\partial x}\left(x(t), x_\gamma(t), t\right), \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}\left(x(t), x_\gamma(t), t\right), t\right)$$

$$+ \frac{\partial S^\zeta}{\partial x^i_\gamma}\left(x(t), x_\gamma(t), t\right)$$

$$\cdot \frac{\partial x^i_\gamma}{\partial t^\zeta}\left(x(t), \frac{\partial S}{\partial x}\left(x(t), x_\gamma(t), t\right), \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}\left(x(t), x_\gamma(t), t\right), t\right)$$

$$+ \frac{\partial S^\zeta}{\partial t^\zeta}\left(x(t), x_\gamma(t), t\right)$$

can be rewritten as

$$-\frac{\partial S^\zeta}{\partial t^\zeta}\left(x(t), x_\gamma(t), t\right) = \frac{\partial S^\zeta}{\partial x^i}\left(x(t), x_\gamma(t), t\right)$$

$$\cdot \frac{\partial x^i}{\partial t^\zeta}\left(x(t), \frac{\partial S}{\partial x}\left(x(t), x_\gamma(t), t\right), \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}\left(x(t), x_\gamma(t), t\right), t\right)$$

$$+ \frac{\partial S^\zeta}{\partial x^i_\gamma}\left(x(t), x_\gamma(t), t\right)$$

$$\cdot \frac{\partial x^i_\gamma}{\partial t^\zeta}\left(x(t), \frac{\partial S}{\partial x}\left(x(t), x_\gamma(t), t\right), \frac{1}{n(\gamma,\zeta)}\frac{\partial S}{\partial x_\gamma}\left(x(t), x_\gamma(t), t\right), t\right)$$

$$- L\left(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t\right),$$

or as *Hamilton-Jacobi divergence PDE based on second order Lagrangians*

$$(H - J - div.) \qquad \frac{\partial S^\zeta}{\partial t^\zeta} + H\left(x, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial x_\gamma}, t\right) = 0, \quad \zeta,\ \gamma \in \{1, 2, ..., m\}.$$

As a rule, the Hamilton-Jacobi divergence PDE based on second order Lagrangians is accompanied by the initial conditions $S^\zeta(x, x_\gamma, 0) = S^\zeta_0(x, x_\gamma)$. The solution $\left(S^\zeta(x, x_\gamma, t)\right)$ is called the *generating function* of the canonical multi-momenta.

Remark 5.1 Conversely, let $\left(S^\zeta(x, x_\gamma, t)\right)$ be a solution of the Hamilton-Jacobi divergence PDE based on second order Lagrangians. We define

$$p^\zeta_i(t) = \frac{\partial S^\zeta}{\partial x^i}\left(x(t), x_\gamma(t), t\right)$$

$$q^{\gamma\zeta}_i(t) = \frac{1}{n(\gamma,\zeta)}\frac{\partial S^\zeta}{\partial x^i_\gamma}\left(x(t), x_\gamma(t), t\right).$$

It appears the following link

$$\int_\Omega L\left(x(t), x_\gamma(t), x_{\gamma\zeta}(t), t\right) dt^1 \cdots dt^m$$

$$= \int_\Omega \left[\frac{\partial S^\zeta}{\partial x^i}\left(x(t), x_\gamma(t), t\right) x^i_\zeta(t) + \frac{\partial S^\zeta}{\partial x^i_\gamma}\left(x(t), x_\gamma(t), t\right) x^i_{\gamma\zeta}(t) - H\right] dt^1 \cdots dt^m$$

$$= \int_{\Omega} [\frac{\partial S^{\zeta}}{\partial x^i} (x(t), x_{\gamma}(t), t)\, x^i_{\zeta}(t) + \frac{\partial S^{\zeta}}{\partial x^i_{\gamma}} (x(t), x_{\gamma}(t), t)\, x^i_{\gamma\zeta}(t)$$

$$+ \frac{\partial S^{\zeta}}{\partial t^{\zeta}} (x(t), x_{\gamma}(t), t)] dt^1 \cdots dt^m = \int_{\Omega} \frac{\partial c^{\zeta}}{\partial t^{\zeta}}(t) dt^1 \cdots dt^m = \int_{\partial\Omega} \delta_{\zeta\eta} c^{\zeta}(t) n^{\eta}(t) d\sigma.$$

The last formula shows that the action multiple integral depends only on the boundary values of $c(t)$.

Theorem 5.2 *The generating function of the canonical multi-momenta is solution of the Cauchy problem*

$$\frac{\partial S^{\zeta}}{\partial t^{\zeta}} + H\left(x, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial x_{\gamma}}, t\right) = 0, \quad S^{\zeta}(x, x_{\gamma}, 0) = S^{\zeta}_0(x, x_{\gamma})$$

for $\zeta, \gamma \in \{1, 2, ..., m\}$.

6. Conclusion and further development

We managed to obtain Hamilton-Jacobi system of PDEs based on multi-time second order Lagrangians (see $H-J$) and, as well, Hamilton-Jacobi divergence PDE via multi-time second order Lagrangians (see $H-J-div.$). Also, there are investigated the relations between two Lagrangians which are joined by a transformation of gauge type.

We shall direct our future research to the development of (single-time and multi-time) Hamilton-Jacobi theory via higher order Lagrangians.

Acknowledgements. Thanks to referees for their precise remarks. Also, I would like to thank Professor Constantin Udrişte for valuable suggestions and discussions during the preparation of this work.

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