U. P. B. Sci. Bull., Series D, Vol. 73, Iss.4, 2011 ISSN 1454-2358

ON THE MAXIMUM BRAKING CAPABILITY OF AUTOMOBILES

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În lucrare se prezintă studiul regimului optimal de frânare a unui automobil, care conduce la performanțe maxime de frânare. Pentru aceasta se aplică principiul de maxim al lui Pontreaghin, arătându-se că există o soluție singulară a problemei. În cazul general, comanda optimală include comenzi de tip releu şi comanda singulară. Pentru un sistem uzual de frânare realizarea acestor comenzi este dificilă, astfel că decelerația maximă posibilă limitată de aderență este practic imposibil de obținut.

The paper presents the study of the optimal braking duty of an automobile that leads to the maximum braking performance. With this end in view one applies the Pontryagin's maximum principle. It is proved that there exists a singular solution of the problem. In the general case, the optimal control consists of relay type controls and singular control. This control is difficult to achieve by an usual braking system. Therefore, the maximum deceleration limited by adhesion is impossible of execution in practice.

Key words: automobile, braking torque, deceleration, optimal control, slip ratio

1. Introduction

The maximum braking capability of an automobile is characterized by one of the following quantities: the minimum braking distance, the minimum braking time and the maximum deceleration. The maximum deceleration is often used because it is theoretically determined by a more direct method.

The maximum possible braking capability is given by the maximum deceleration that is limited by the adhesion between tyre and road, d_{maxp}. Its expression is [1-5]

$$d_{\max p} = g(\varphi_x \cos\alpha + \sin\alpha) + 0.5\rho\, c_x A v^2 / m \quad (1)$$

where:

- A[m^2] is automobile frontal area;
- c_x[-]-aerodynamic resistance coefficient;
- g[m/s^2]-acceleration due to gravity;
- m[kg]-automobile mass;
- v[m/s]-automobile velocity;
- α[rad]-slope angle;

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-φ_x[-]-longitudinal road adhesion coefficient (maximum braking effort coefficient);

-ρ[kg/m^3]-air density.

Essentially, the relationship (1) is established by use of the rectilineal motion equation for an automobile assuming that the wheels reach the adhesion limits simultaneously. Therefore, the problem is dealed with from quasistatic point of view and so it is mathematically reduced to an algebraical problem. In fact, even if is assumed that a braking system could suddenly generate the necessary brake torques, it should be noticed that during process the inertia of the braked wheels is involved and that the braking forces are produced by the agency of the wheels and the tyres, which are elastic in longitudinal direction. The last feature is strongly connected with the rolling characteristic of a tyre.

In the present paper the maximum braking capability of an automobile is investigated taking into account the above mentioned facts.To solve the problem we will apply the Pontryagin's maximum principle [6,7] and make evident the difference relating to the classical approach and resort to suitable numerical examples.

2. State equations

It is assumed that a two-axle automobile moves on a road with the longitudinal slope of angle α (when the automobile climbes uphill $\alpha>0$ and when it moves downhill $\alpha<0$). The Newton's equation for the automobile motion is:

$$m\dot{v} = -(X_{b1} + X_{b2}) - mg\sin\alpha - 0.5\rho c_x A v^2 \tag{2}$$

where X_{b1} and X_{b2} are the longitudinal braking reactions acting on the front and the rear wheels, respectively ($X_{b1}>0$, $X_{b2}>0$).

We consider the general case when the automobile braking is performed with coupled engine. If it is supposed, for instance, that the front wheels are driving, then one should take into account for the engine brake torque and the inertia of the rotating parts of the engine and the driveline. By applying theorem on the angular momentum about the front wheel rotation axis (for details see, for instance, [1,8]) we obtain:

$$I_t\dot{\omega}_{w1} = -M_{be} i_t \eta_t^{-1} - M_{b1} + X_{b1}.r_{d1} - fZ_1 r_{d1}, \tag{3}$$

$$2I_{w2}\dot{\omega}_{w2} = -M_{b2} + X_{b2}.r_{d2} - fZ_2.r_{d2} \tag{4}$$

where

$$I_t = I_e i_t^2 \eta_t^{-1} + I_{gb} i_0^2 \eta_0^{-1} + I_0 + 2I_{w1} \tag{5}$$

with the following notations:

-f [-]-coefficient of rolling resistance (it is assumed that the coefficients of rolling resistance of the all wheels are same) ;

-i_0, i_t [-]-final drive gear ratio, overall gear ratio (axle and transmission);

-I_e, I_{gb}, I_0, I_{w1},I_{w2} [kg.m^2]-mass moment inertia of the rotating parts of the engine, gear box, final drive, front and rear wheel, respectively;
-M_{be}, M_{b1}, M_{b2} [N.m]-braking torque produced by engine and brake system (for front and rear axle), respectively;
-r_{d1}, r_{d2} [m]-dynamic loaded radius of front and rear tyres, respectively;
-Z_1, Z_2 [N]-normal reactions on the front and rear axles, respectively;
-η_0, η_t [-]-efficiency of the final drive and overall transmission efficiency, respectively;
-ω_{w1}, ω_{w2} [rad/s]-angular speeds of the front and rear wheels, respectively.

The longitudinal braking reactions can be expressed as

$$X_{bj} = \xi_{bj} . Z_j , \; j = 1,2 \tag{6}$$

where ξ_{bj} (j=1,2) are the specific longitudinal braking reactions. The normal reactions on axles are functions of the specific longitudinal braking reactions [8]:

$$Z_1 = mg\cos\alpha \frac{1 - a_l + \chi[1 + F_{az}/(mg\cos\alpha)]\xi_{b2} + F_{az1}/(mg\cos\alpha)}{1 + \chi(-\xi_{b1} + \xi_2)}, \tag{7}$$

$$Z_2 = mg\cos\alpha \frac{a_l - \chi[1 + F_{az}/(mg\cos\alpha)]\xi_{b1} + F_{az2}/(mg\cos\alpha)}{1 + \chi(-\xi_{b1} + \xi_{b2})} \tag{8}$$

where a_l=a/L, χ=h_g/L with the following notations:
-a [m]-distance between the gravity center and the normal plane on the road that passes trough the front axle;
-F_{az}, F_{az1}, F_{az2} [N]-aerodynamic lift force, front and rear aerodynamic lift forces;
-h_g [m]-height of center of gravity of the automobile;
-L [m]- wheelbase.

Generally, the specific braking reaction is depended on the longitudinal slip ratio, normal reaction acting on tyre, wheel center velocity and type road surface. On a given type road surface in certain conditions the effect of the normal reaction and velocity are not important, so that, in this paper, we suppose that ξ_{bj} (j=1, 2) are depended on the longitudinal slip ratio only, which is defined as

$$s_j = 1 - \frac{r_{r0j} . \omega_{wj}}{v} \tag{9}$$

where r_{r0j} is rolling radius of the freely rolling wheel corresponding to the wheels of the j axle.

Taking into account the relationship (6), (7) and (8), Equations (2), (3) and (4) may be written in the general form:

$$\begin{aligned} \dot{v} &= f_1(v, \omega_{w1}, \omega_{w2}), \\ \dot{\omega}_{w1} &= f_2(v, \omega_{w1}, \omega_{w2}) - u_1, \\ \dot{\omega}_{w2} &= f_3(v, \omega_{w1}, \omega_{w2}) - u_2, \end{aligned} \tag{10}$$

the functions f_1, f_2 and f_3 being defined by the expressions:

$$f_1 = -g\cos\alpha \frac{(1-a_l)\xi_{b1} + a_l\xi_{b2} + [F_{az1}/(mg\cos\alpha)]\xi_{b1} + [F_{az2}/(mg\cos\alpha)]\xi_{b2}}{1+\chi(-\xi_{b1}+\xi_{b2})} -$$

$$- g\sin\alpha - 0.5\rho c_x A v^2/m, \tag{11}$$

$$f_2 = -\frac{M_{be}i_t}{I_t\eta_t} + \frac{r_{d1}}{I_t}(\xi_{b1} - f)Z_1(\xi_{b1},\xi_{b2},v), \tag{12}$$

$$f_3 = \frac{r_{d2}}{2I_{w2}}(\xi_{b2} - f)Z_2(\xi_{b1},\xi_{b2},v). \tag{13}$$

Because F_{az} and F_{azj} (j=1, 2) are depended on vehicle speed and, consequently, the normal reactions are depended on the vehicle velocity explicitly. In the above expressions ξ_{bj} are substituted for the functions $\xi_{bj}(s_j)$ (j=1,2), which are known. Finally, the state equations contain the control variables u_1 and u_2 defined by relations

$$u_1 = M_{b1}/I_t, u_2 = M_{b2}/(2I_{w2}). \tag{14}$$

3. Optimal duty of automobile motion during braking

The dynamic system being defined by the state equations (10) it raises the problem to establish the control variables u_1 and u_2 that ensure the maximum braking capability of an automobile. It can be expressed as the minimum braking time. In this way we reach the optimal control problem. The considered criterion is given by

$$J = \int_0^{t_f} dt \tag{15}$$

where t_f is the final time. The optimization problem consists in the minimization of J taking into account the constraints (10). The initial and final conditions are:

$$\begin{aligned} &v(0) = v_0,\ \omega_{w1}(0) = \omega_{w10},\ \omega_{w2}(0) = \omega_{w20}, \\ &v(t_f) = v_f,\ \omega_{w1}(t_f) = \omega_{w1f},\ \omega_{w2}(t_f) = \omega_{w2f}. \end{aligned} \tag{16}$$

First, one considers the stopping braking time that correspond to v_f =0. However, it is necessary to take into account that at v=0 the tire rolling characteristic is not defined. For this reason, further we consider $v_f \neq 0$, but small enough (for instance, 1.5 m/s). It is not absolute by necessary to specify the values of the wheel final angular speeds, thus leaving them as free variables at the final moment.

The Hamiltonian of the present problem will be:

$$H = \lambda_0 + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 - \lambda_2 u_1 - \lambda_3 u_2 \tag{17}$$

where λ_0, λ_1(t), λ_2(t), λ_3(t) are the components of the adjoint vector (costate vector). Obviously, the braking torques on the two-axle is limited to the maximum values M_{b1max} and M_{b2max}, to which correspond to the maximum of the control variables u_{1max} and u_{2max}, namely

$$0 \le u_j \le u_{j\,\max},\ j = 1, 2. \tag{18}$$

The adjoint system of equations is written as:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial v} = -\left(\lambda_1 \frac{\partial f_1}{\partial v} + \lambda_2 \frac{\partial f_2}{\partial v} + \lambda_3 \frac{\partial f_3}{\partial v} \right), \tag{19$_1$}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial \omega_{w1}} = -\left(\lambda_1 \frac{\partial f_1}{\partial \omega_{w1}} + \lambda_2 \frac{\partial f_2}{\partial \omega_{w1}} + \lambda_3 \frac{\partial f_3}{\partial \omega_{w1}} \right), \tag{19$_2$}$$

$$\lambda_3 = -\frac{\partial H}{\partial \omega_{w2}} = -\left(\lambda_1 \frac{\partial f_1}{\partial \omega_{w2}} + \lambda_2 \frac{\partial f_2}{\partial \omega_{w2}} + \lambda_3 \frac{\partial f_3}{\partial \omega_{w2}} \right). \tag{19$_3$}$$

If $\mathbf{x} = (v, \omega_{w1}, \omega_{w2})$ is the state vector and $\mathbf{x}^* = (v^*, \omega_{w1}{}^*, \omega_{w2}{}^*)$ represents the state vector corresponding to the optimal braking duty, the Pontryagin's maximum principle yields

$$\sup_{u_1, u_2} H\left(v^*, \omega_{w1}^*, \omega_{w2}^*, u_1, u_2\right) = H\left(v^*, \omega_{w1}^*, \omega_{w2}^*, u_1^*, u_2^*\right), \tag{20}$$

$$H\left(v^*, \omega_{w1}^*, \omega_{w2}^*, u_1^*, u_2^*\right) = 0,\ \forall t \in \left[t_0, t_f\right] \tag{21}$$

where u_1^* and u_2^* are the extremal control variables. Taking into account (17) we get

$$u_1^* = \begin{cases} u_{1\max} & if\ \lambda_2 < 0 \\ 0 & if\ \lambda_2 > 0 \end{cases},\quad u_2^* = \begin{cases} u_{2\max} & if\ \lambda_3 < 0 \\ 0 & if\ \lambda_3 > 0. \end{cases} \tag{22}$$

But, because the Hamiltonian depends linearly on the control variables it is necessary to study the existence of a singular solution. Therefore it is necessary to study the possibility of the existence of the relationships $\lambda_2(t) \equiv 0$, $\lambda_3(t) \equiv 0$ for $t \in [t_i, t_e] \subseteq [0, t_f]$. Taking into account (17) and (21) we get

$$\lambda_0 + \lambda_1 f_1 = 0. \tag{23}$$

According to the maximum principle, λ_0 is a constant so that $\lambda_0 \le 0$. If $\lambda_0 = 0$, then $\lambda_1 = 0$ because always $f_1 \ne 0$. This would mean that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$ which is contradictory to the requirements of the maximum principle. Thus, $\lambda_0 < 0$, $\lambda_1 \ne 0$. With imposed conditions $d\lambda_2/dt \equiv 0$ and $d\lambda_3/dt \equiv 0$, from the two last relationships (19) we get

$$\frac{\partial f_1}{\partial \omega_{w1}} = 0, \frac{\partial f_1}{\partial \omega_{w2}} = 0. \tag{24}$$

Taking into account the expression of f_1 given by (11) one obtains

$$\frac{\partial f_1}{\partial \omega_{w1}} = -\frac{g\cos\alpha}{[1+\chi(-\xi_{b1}+\xi_{b2})]^2}\left[1-a_l+\frac{F_{az1}}{mg\cos\alpha}+\chi(1+\frac{F_{az}}{mg\cos\alpha})\xi_{b2}\right]\cdot\frac{d\xi_{b1}}{d\omega_{w1}}, \tag{25}$$

$$\frac{\partial f_1}{\partial \omega_{w2}} = -\frac{g\cos\alpha}{[1+\chi(-\xi_{b1}+\xi_{b2})]^2}\left[a_l+\frac{F_{az2}}{mg\cos\alpha}-\chi(1+\frac{F_{az}}{mg\cos\alpha})\xi_{b1}\right]\cdot\frac{d\xi_{b2}}{d\omega_{w2}}. \tag{26}$$

As a result, the conditions (24) lead to

$$\frac{d\xi_{bj}}{da_{rj}} = 0,\ j = 1,2 \tag{27}$$

which shows that the specific longitudinal reactions should get the maximum values. These values correspond to the known optimal slip ratios s_{mj}, j=1, 2. It means that there exists a singular solution of the optimal problem. The state variables corresponding to the singular solution satisfy the relationships

$$S_1(v,\omega_{w1}) = v(1-s_{m1}) - r_{r01}\omega_{w1} = 0, S_2(v,\omega_{w2}) = v(1-s_{m2}) - r_{r02}\omega_{w2} = 0. \tag{28}$$

The tire rolling characteristic is so that $d\xi_{bj}/da_{rj}>0$ for $S_j\ (v,\ \omega_{rj})>0$ and $d\xi_{bj}/da_{rj}<0$ for $S_j\ (v,\ \omega_{rj})<0$.

In Figure 1 the straight lines $S_1=0$ and $S_2=0$ corresponding to the singular solution are shown. Each straight line divides into two domains the suitable plane $(v,\ \omega_{wj})$. In the first domain $S_j<0$ and in the second domain $S_j>0$ (j=1, 2). Let $\|_1(v,\ \omega_{w1})$ and $\|_2(v,\ \omega_{w2})$ be the representative points corresponding to a certain state. Because $f_1<0$, from (23) it results that $\lambda_1<0$. Taking into consideration the two last equations (19), we get the following features:

1) $\left.\frac{d\lambda_2}{dt}\right|_{\lambda_2(t)=0} = 0$ if $\|_1\in(S_1=0)$; 2) $\left.\frac{d\lambda_2}{dt}\right|_{\lambda_2(t)=0} < 0$ if $\|_1\in I_2$;

3) $\left.\frac{d\lambda_2}{dt}\right|_{\lambda_2(t)=0} > 0$ if $\|_1\in I_1$; 4) $\left.\frac{d\lambda_3}{dt}\right|_{\lambda_3(t)=0} = 0$ if $\|_2\in(S_2=0)$; 5) $\left.\frac{d\lambda_3}{dt}\right|_{\lambda_3(t)=0} < 0$ if $\|_2\in II_2$; 6) $\left.\frac{d\lambda_3}{dt}\right|_{\lambda_3(t)=0} > 0$ if $\|_2\in II_1$.

If, at a given moment, $\lambda_2(t)=0$ and $\lambda_3(t)=0$, we can establish the sequence of the control depending on the positions of the points $\|_1$ and $\|_2$. But, in the general case, it is difficult to establish this sequence. In any case, at the braking beginning we have $\|_1\in I_1$ and $\|_2\in II_1$ in a way that is shown in Fig. 1

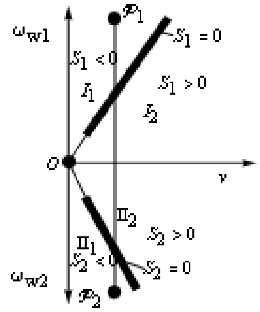

Fig. 1. The lines corresponding to the singular solution

The singular control corresponds to the singular solution and it can be determined in the following way. Using the relationships (10), we can write equalities

$$\frac{d\omega_{w1}}{dv}=\frac{f_2(v,\omega_{w1},\omega_{w2})-u_1}{f_1(v,\omega_{w1},\omega_{w2})},\ \frac{d\omega_{w2}}{dv}=\frac{f_3(v,\omega_{w1},\omega_{w2})-u_2}{f_1(v,\omega_{w1},\omega_{w2})}. \tag{29}$$

Taking into account the relationships (28), we get

$$\frac{d\omega_{wj}}{dv}=\frac{1-s_{mj}}{r_{r0j}},\ j=1,2. \tag{30}$$

If φ_{xj} is the longitudinal adhesion coefficient of the tyres of the axle *j*, one can write $\xi_{bj}=\varphi_{xj}$. Taking into account the relationships (29), (30) and $\xi_{bj}=\xi_{bjmax}$ we get

$$u_{1sg}=\frac{r_{d1}}{I_t}(\varphi_{x1}-f)\frac{1-a_l+\chi[1+F_{az}/(mg\cos\alpha)]\varphi_{x2+F_{az1}/(mg\cos\alpha)}}{1+\chi(\varphi_{x2}-\varphi_{x1})}.mg\cos\alpha+$$
$$+\frac{1-s_{m1}}{r_{r01}}.d_{cs}-\frac{M_{be}i_t}{I_t\eta_t}, \tag{31}$$

$$u_{2sg}=\frac{r_{d2}}{2I_{w2}}(\varphi_{x2}-f)\frac{a_l-\chi[1+F_{az}/(mg\cos\alpha)]\varphi_{x1}+F_{az2}/(mg\cos\alpha)}{1+\chi(\varphi_{x2}-\varphi_{x1})}.mg\cos\alpha+$$
$$+\frac{1-s_{m2}}{r_{r02}}.d_{cs} \tag{32}$$

where d_{cs} represents the automobile deceleration in the singular duty of braking:

$$d_{cs}=g\cos\alpha.\frac{(1-a_l)\varphi_{x1}+a_l\varphi_{x2}+(F_{az1}\varphi_{x1}+F_{az2}\varphi_{x2})/(mg\cos\alpha)}{1+\chi(\varphi_{x2}-\varphi_{x1})}+g\sin\alpha+$$
$$+\frac{\rho c_x A v^2}{m}. \tag{33}$$

If $\varphi_{x1}=\varphi_{x2}=\varphi_x$, then d_{cs} represents even the maximum possible deceleration given by (1).

Knowing u_{jsg}, j=1, 2, we can easily determine the brake torques by means of relationships (14).

Obviously, to achieve the singular motion the maximum brake torques should be large enough. Thus

$$u_{jsg} \leq u_{j\max}, \, j = 1, 2. \tag{34}$$

If the brake system cannot achieve the maximum brake torques large enough, namely the conditions (34) are not achieved, the optimal braking is performed by the control given by relationships (22). As we mentioned already the establishment of the sequence of the optimal controls is difficult. In such cases it is necessary to use the numerical methods. At braking beginning, obviously, the controls u_{1max} and u_{2max} are suddenly applied and further, if such is the case, the brakes are completely released. After that this process is repeated. If the maximum brake torques are close by the brake torque values corresponding to the singular solution but smaller than the preceding mentioned values there may exist the tendency of the locking of some wheels in the way the numerical simulations show. But, if the maximum brake torques are not too large, the wheels do not have tendency of locking and, consequently, it is not necessary to cancel the brake torques.

Obviously, in the case of the optimal braking strategy the braking time decreases when the maximum brake torque increases. Also, when the relationship (34) is satisfied, at the braking beginning the control (22) is applied until the singular braking duty is attained, after that the braking is achieved by applying the singular brake torques until the vehicle velocity becomes small enough. We can choose u_{1max} and u_{2max} in such a way as to reach the singular braking duty after a short time without other controls. It is noticed that the reaching of the singular duty means that the slip ratios of the wheels should get the optimum values at the same time.

As it has been found, generally, the optimal braking strategy is intricate enough and it is difficult to put in practice. In addition to all this the brake torques cannot suddenly increase to the requested values. Consequently, the possible maximum deceleration cannot be achieved in practice. This issue should be considered by the evaluation of the braking performances of an automobile.

4. Numerical exemplification

Taking into account the above mentioned considerations, we consider a passenger car with the following features: A=1.80m^2, c_x=0.35, f=0.012, I_e=0.180kg.m^2, I_{w1}=1.5kg.m^2, I_{w2}=1.5 kg.m^2, L=2.69m, m=1600kg, r_{r01}=0.30m, r_{r02}=0.30m, r_{d1}=0.29m, r_{d2}=0.29m, χ=0.20. The lift aerodynamic effects are negligible.

The braking torque produced by engine changes depending on the mean piston velocity or the engine speed by the linear law [3, 8, 9]:

$$M_{be} = V_h(\alpha_1 . n + \beta_1) = V_h\left(\frac{30}{\pi} i_t \alpha_1 \omega_{w1} + \beta_1\right) \quad (35)$$

where V_h [l] is engine displacement, n [1/min] represents the crankshaft speed and α_1 and β_1 are certain constants depended on the engine type. In present paper it is assumed the following values: $\alpha_1 = 10^{-8}$ N.m/(l.1/min), $\beta_1 = -0.15$ N.m/l.

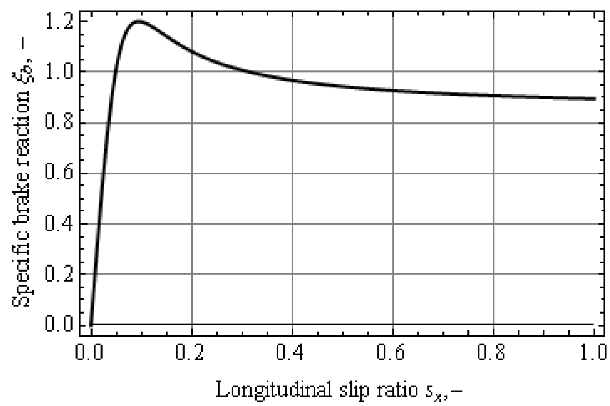

Fig.2. Rolling characteristic of a tyre

The tyre rolling characteristic is expressed by "*Magic Formula*" [10] (one considers the same characteristic for the all wheels):

$$\xi_b = \varphi_x \sin[C_x \arctan\{B_x s_x - E_x (B_x s_x - \arctan(B_x s_x))\}] \quad (36)$$

where s_x is the longitudinal slip ratio. The choice values of the constants are: B_x=14, C_x=1.5, E_x=-1. For φ_x=1.2 the tyre rolling characteristic is shown in Figure 2. It is typical for the actual tyres while rolling on the dry surface of a modern road.

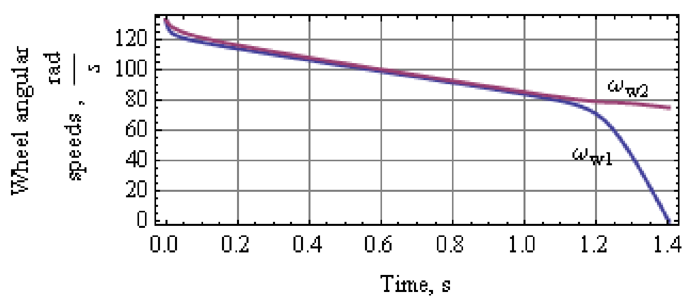

Fig.3. Variation of wheel angular speeds with time during braking

We have conceived a computer program in *Mathematica* for the integration of the differential equation system (10) as for well as the determination of different quantities which are of interest to the present study. First one treats the case of the braking with a shut-off engine. If we consider the singular controls and the initial conditions corresponding to these controls one obtains by numerical integration the maximum deceleration exactly.

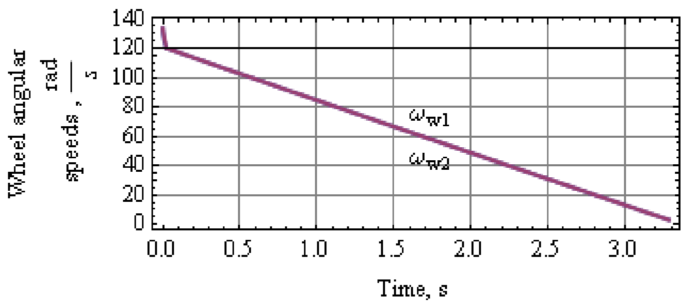

Fig.4. Variation of wheel angular speeds with time during optimal braking

But, if the initial angular speeds of the wheels have the real values corresponding to the situation when the brake torques are not yet applied, that is $\omega_{wj0}=v_0/r_{r0j}$ (j=1, 2), the front wheels are locked at time of 1.4 s (Fig. 3, v_0=40 m/s) before the vehicle speed becomes zero. Therefore, one confirms the theory and it is necessary to begin with the extremal controls.

By means of a computer program the constant brake torques have been determined, which ensure the wheel angular speeds corresponding to the singular duty after a very short time. These torques are $M_{b1}=1.2M_{b1max}$, $M_{b2}=1.86M_{b2max}$ where M_{b1max} and M_{b2max} represent the singular brake torques corresponding to vehicle speed of 40 m/s. It can be noticed that these torques have the large enough values. The numerical integration of the motion equations shows that the optimal braking duty is indeed obtained (Fig. 4). The time necessary to get the singular braking duty is 0.02 s.

Let Φ_b be the coefficient of the braking longitudinal reaction distribution in the case of a two-axle automobile:

$$\Phi_b = X_1 /(X_{b1} + X_{b2}). \tag{37}$$

In the case of the braking with a shut-off engine this ratio may be approximated with enough accuracy by the ratio of the corresponding brake torques. A simple analysis of the automobile braking and wheel locking by taking into account the brake torques in a direct adequate way has been achieved in [11]. According to the classical theory, the optimal braking distribution coefficient that ensures the carrying out the possible maximum deceleration is given by relation [1-5]:

$$\Phi_{bop} = 1 - a_l + \chi \cdot \varphi_x . \tag{38}$$

During braking, the brake torques change with respect to time according to the following expressions:

$$M_{b1}(t)=\begin{cases}\Phi_b \dfrac{M_{b\max}}{t_0}t & for\ t\le t_0,\\ \Phi_b M_{b\max} & for\ t>t_0\end{cases};M_{b2}(t)=\begin{cases}(1-\Phi_b)\dfrac{M_{b\max}}{t_0}t & for\ t\le t_0,\\ (1-\Phi_b)M_{b\max} & for\ t>t_0\end{cases} \quad (39)$$

where M_{bmax} [N.m] is the total brake torque corresponding to a certain braking operation (for a certain control given by driver) and t_0 represents the increase time of the brake torque.

If it is assumed that Φ_b has the optimal value given by (38), namely 0.74, and the total brake torque corresponds to the limit of the adhesion, by numerical integration of the motion equations the obtained results are depicted in Figs. 5 and 6 (one considers that t_0=0.15 s). The designed computer program ensures the complete numerical integration of the system even in the case when the wheels of one axle or two axles are being locked –up.

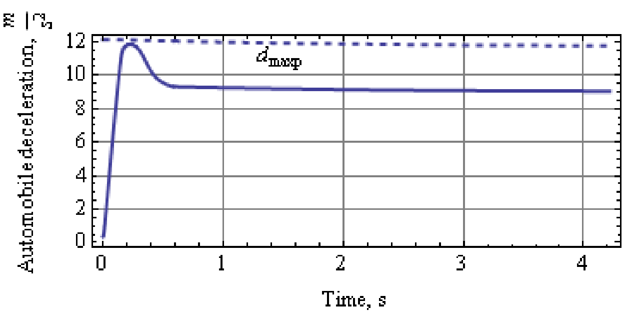

Fig.5. Variation of deceleration with time (Φ_b=Φ_{bop}, t_0==0.15 s)

It is found that because the optimality requirements established before are not fulfilled, the deceleration is small enough in comparison with the possible maximum deceleration and the front wheels are being locked-up which alters the steering control.

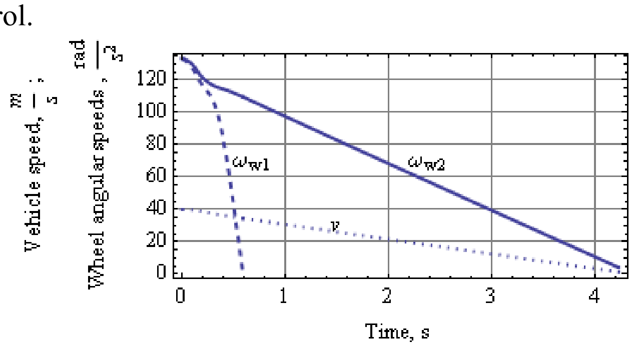

Fig.6. Variation of vehicle speed and wheel speeds with time (Φ_b=Φ_{bop}, t_0=0.15 s)

Therefore, taking into account that always there is an increased time t_0, in the case of a classical braking system we may formulate the question of the determination of the optimal brake torque distribution coefficient and the total torque M_{bmax} which ensure the maximum braking performance (e.g. the minimum braking time).

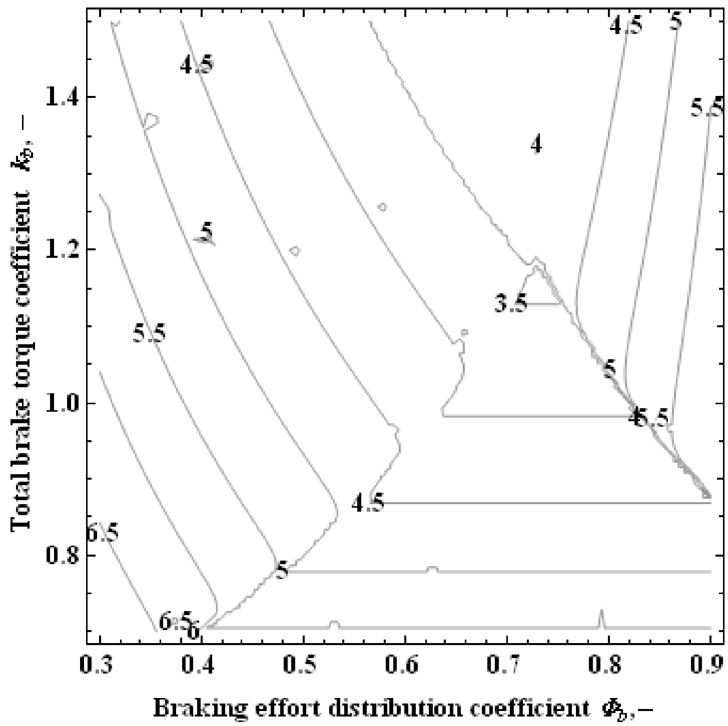

Fig.7. Variation of braking time with k_b and Φ_b

For this purpose, on the basis of the mentioned program we have defined a function that expresses the dependence of the braking time corresponding to a very small vehicle velocity on M_{bmax} and Φ_b. After that, this function has been depicted by contour lines (see Fig. 7). The maximum total brake is defined as to:

$$M_{b\max} = mgr_{d1}k_b \tag{40}$$

where k_b [-] represents the brake coefficient.

By means of the mentioned plot and several successive trials by using the mentioned program it has been established that the minimum braking time of 3.439 s is attained for k_b=1.15 and Φ_b=0.726.This means that the maximum brake torque is less than the brake torque corresponding to the adherence limit ($k_b<\varphi_x$).

Also, the coefficient Φ_b is less than the value given by (38), namely 0.74. It is noticed that for the smaller values of the brake torque there is a variation interval of Φ_b for which the braking time remains constant. Besides it is noticed that for certain values of Φ_b there are two different values of M_{bmax} that lead to the same braking time. At the mentioned optimum point the braking is closed to the ideal braking (see Fig. 8).

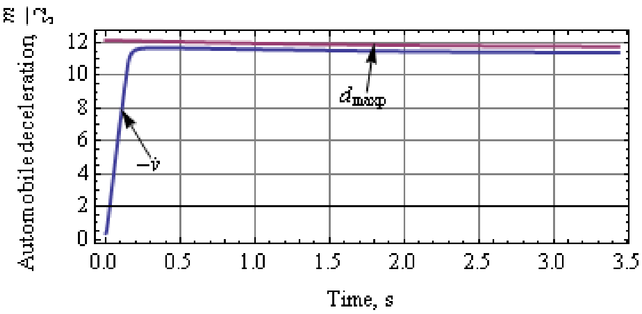

Fig.8. Variation of deceleration with time (k_b=1.15, Φ_b=0.726)

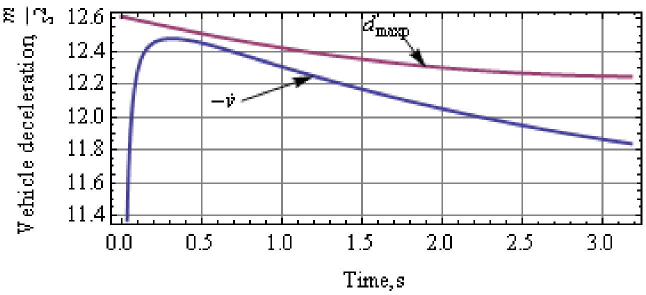

Fig.9. Variation of deceleration with time during combined braking (see text)

It is found that both the brake torque distribution coefficient and the proper values of the brake torques are of importance. But, in real conditions, the maintaining of the brake torques at the mentioned values is difficult.

In the case of the combined braking with coupled engine one finds that if the torques corresponding to the singular solution are applied even if the wheel angular speeds are $\omega_{w0j}=v_0/r_{r0j}$ (j=1, 2), the deceleration is very closed to the ideal deceleration without some wheel locking up. If the constant torques corresponding to the singular torques at vehicle velocity of 40 m/s are applied and it is considered the increase time t_0, the wheel blocking-up is not produced (variation of deceleration is represented in Figure 9). Therefore, the results of the combined braking are better than those obtained by the braking with the engine shut off. In exchange, if it is assumed that Φ_b=0.74 (according to relationship (38)) and the maximum total brake torque corresponds to the adherence limit, then the front wheels are locked up even if later than when using the brake system only. During braking with a coupled engine, the coefficient Φ_b corresponding to the singular braking duty changes with time; in the present case the variation range is [0.726, 0.734]. Anywise, these values are smaller than those corresponding to (38). Therefore, even if, generally, the braking performance achieved by combined braking may be nearer to the ideal performance, the coefficient Φ_b is not much different from its optimum value and the inadequate maximum brake torques may lead to the diminution of the braking performance and the wheel locking up.

5. Conclusions

At the moment of automobile braking, the maximum deceleration limited by adherence (ideal deceleration) can be theoretically achieved by means of a brake system control that consists of extreme controls (rely type) and the singular control which are studied in the paper.

In the case of a classical brake system the ideal deceleration cannot be achieved. However, there is a distribution coefficient of the brake torques and suitable values of torques which ensure the maximum deceleration closed to the ideal deceleration. The mentioned coefficient does not coincide with the optimal coefficient considered in literature.

Generally, during combined braking with coupled engine, the maximum deceleration may be nearer to the ideal deceleration. Nevertheless, if the value of the distribution coefficient does not differ much from the optimum value, the wheel may become locked.

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