U.P.B. Sci. Bull., Series A, Vol. 74, Iss. A, 2012 ISSN 1223-7027

SYSTEMS OF DIFFERENTIAL EQUATIONS, ASSOCIATED PARABOLAS AND GENERALIZATIONS

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Se propune o metoda de explicitare a solutiei aproximative a unui sistem diferential neomogen,neliniar. Apoi se considera sisteme mai generale de ecuati, rezolvabile pe baza unor contractii pe intervale mici. Se studiaza pe scurt unele aspecte privind norma operatorului de derivare si sisteme ortogonale cu proprietati speciale adecvate acestei problematici. Se propune o forma explicita a solutiei unei probleme legate de miscarea unui fluid perfect. In fine, fiind dat un operator liniar marginit, se construieste un subspatiu asociat, pe care operatorul de derivare coincide cu cel dat.

A method of finding explicit approximate solution for a nonlinear differential system is proposed. Then one considers similar more general systems, which can be solved by using contractions on small intervals. Some aspects concerning the continuity and the norm of the derivation operator, as well as related orthogonal systems are briefly discussed.

An explicit form of the solution related to the movement of a perfect fluid is proposed. Finally, to any linear bounded operator, one associates a subspace on which the derivation operator equals the given operator.

Keywords: stick-slip solutions, local contractions, derivation operators, analyticity.

1 Introduction

There are several methods of solving "exactly" or approximating the solutions of linear and nonlinear systems of differential, partial differential equations and integral equations: [1]-[3], [5], [7], [10], [11],[13], [16], [18], [19], [21]. Some of these systems are motivated by movement equations, vibrations or other phenomena, having applications in several fields: [1], [3], [7], [8], [10], [11], [13], [18], [19], [21]. In most of the cases, it is difficult to find expressions of exact solutions. That is the reason of the development of general approximating and variation-calculus methods: [1], [8], [10], [11], [13], [18], [19], [21]. The approximation is local or "global". In both of these cases, the successive approximation and similar iterative methods remains one of the most important

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tools. Methods based on integral transforms, distribution theory and functional analysis are developed and frequently used: [1], [4], [5], [7], [8], [11], [13], [19], [21]. In studding stability at equilibrium points, even in the case of affine systems of first order equations, difficulties may occur because of the non-constant "free term". That is why special methods could be useful in both problems. Complex functions, probabilistic, algebraic, and nonstandard methods for similar problems are used: [2], [7], [8], [16]. There are functional equations, discrete-type problems and constructive problems, optimization problems, which can be reduced to differential or integral equations, or which can be used in solving similar problems: [1], [5], [6], [9], [12], [14], [15], [19], [20]. In the first part of this work, we study some aspects of the following nonlinear system of equations describing the motion during the stick-slip phenomenon [18]:

$$
\begin{aligned}
& I\ddot{\varphi}_1 + c_1(\varphi_1 - \varphi_2) = -M_{t1}(\dot{\varphi}_1)\,; \\
& I'\ddot{\varphi}_2 - c_1(\varphi_1 - \varphi_2) + c_r u(u\varphi_2 - \varphi_r) = -M_{t2}(\dot{\varphi}_2)\,; \\
& I_r\ddot{\varphi}_r + c_r(\varphi_r - u\varphi_2) = M_m(\dot{\varphi}_r) \\
& I = I_1 = I_2\,, \quad I' = I + I_3 u^2
\end{aligned}
\tag{1}
$$

Here $I = I_j,\, j = 1{,}2$ is the inertia moment of each wheel.

It is easy to see that even in the case of linear moments (see the first case (2)), a direct computation of the explicit solution seems to be difficult. Therefore, special methods can be useful. This is the aim of the first part of this work. For the first part, the meaning of the notations, details and numerical values used are those from [18]. The inertia friction-moments of the wheels, respectively the traction motor moment are:

$$
\begin{aligned}
& M_{tj}(\dot{\varphi}_j) = \frac{\mu}{\dot{\varphi}_p}\left(\dot{\varphi}_j - \frac{v}{r}\right)\cdot Qr, \quad for \left|\dot{\varphi}_j - \frac{v}{r}\right| \le \dot{\varphi}_p, \\
& M_{tj}(\dot{\varphi}_j) = \left(\frac{a}{r}\cdot\frac{1}{\left|\dot{\varphi}_j - v/r\right|} + b\right)\cdot Qr\cdot \operatorname{sgn}(\dot{\varphi}_j - v/r) \quad for \left|\dot{\varphi}_j - \frac{v}{r}\right| > \dot{\varphi}_p \\
& M_m(\dot{\varphi}_r) = M_0 + K_m(\dot{\varphi}_r - vu/r)
\end{aligned}
\tag{2}
$$

The inertia moments I, I', I_r are constants. Some other symbols appearing in (1) and (2) have the meaning of constant quantities. Some of them are movement parameters. For sufficiently large values of velocities, the friction forces and moments of the wheels are decreasing, and the stick-slip does not occur any more. An important remark is that the moments $M_{tj}(y_j),\, y_j = \dot{\varphi}_j - v/r,$ are continuous everywhere and piecewise analytic as functions of $y_j,\, j = 1{,}2,$ being non-differentiable at $y_j = \pm\dot{\varphi}_p$. Around any point t_0, local solutions do exist, and they are $C^{(1)}$ functions. If at a point, we have

$$\left|y_j(t_0)\right| \ne \dot{\varphi}_p,$$

then there exist analytic local solutions, which are defined on a maximal subinterval. We have the following relation on the movement parameters, which show that the moments $\widetilde{M}_{tj}(y_j):=M_{tj}(\dot{\varphi}_j - v/r),\ j=1,2$ are continuous at $y_j=\pm\dot{\varphi}_p$:

$$\mu r = a/\dot{\varphi}_p + b r \Leftrightarrow \dot{\varphi}_p = a/[r(\mu-b)] \tag{3}$$

Observe that M_{tj} are even functions of $y_j,\ j=1,2;$ slightly modifying $M_{tj}(y_j)$ around $y_j=\dot{\varphi}_p$ with the aid of a parabola, one obtains a $C^{(1)}$ function on the whole interval, preserving the property of attaining the maximum at $\dot{\varphi}_p$. The advantage of using smooth curves is that of possible applying results of differential calculus. The velocity $\dot{\varphi}_p$ corresponds to the adherence limit, which is (almost) constant on the stick-slip interval of time.

The rest of the paper contains the following results. In Section 2, one gives explicit approximate solutions of system (1). Section 3 contains "local solutions as fixed points for local contractions". Some remarks on the norm of derivation-operator, as well as related orthogonal systems are contained in Section 4. As an application of Hermite's functions, an explicit solution for a particular case concerning the movement of a perfect fluid is proposed. Then one applies these results in solving the general case. Section 5 is devoted to subspaces associated to a bounded linear operator, on which the derivation operator equals the given operator. The end of the paper contains some conclusions.

2 Explicit solutions for constrained stick-slip and stability results

In the following theorems, we will call solution of the system (1), functions $\dot{\varphi}_j, \dot{\varphi}_r,\ j\in\{1,2\}$ verifying the system. The initial conditions will be homogeneous.

For this problem, an equilibrium point is a point t_e at which we have:

$$\begin{aligned}&\ddot{\varphi}_j(t_e)=\ddot{\varphi}_r(t_e)=0, M_{t1}(\dot{\varphi}_1)(t_e)=M_{t2}(\dot{\varphi}_2)(t_e),\\ &\dot{\varphi}_r(t_e)=u\dot{\varphi}_j(t_e),\ j=1,2,\ uM_m(\dot{\varphi}_r)(t_e)=2M_{tj}(\dot{\varphi}_j)(t_e),\ j=1,2\end{aligned} \tag{4}$$

At such point, the adherence is optimal, and it is an local extremum point for $\dot{\varphi}_j,\ j=1,2,\ \dot{\varphi}_r$. Thus, if we have a maximum point for $\dot{\varphi}_1$, on a subinterval $[t_e, t_e+\delta]$ the signature of $\ddot{\varphi}_j, \ddot{\varphi}_r,\ j=1,2$ will be constant, for $\ddot{\varphi}_1$ being negative. To the left of t_e, the signature of $\ddot{\varphi}_1$ will be positive. In all cases, at an equilibrium point, on an interval situated on one side of this point, the signature of each of the second derivatives is constant. The first condition (3) is the usual equilibrium point definition. The rest of conditions (3) concern the stick-slip. A useful remark is that the continuous functions $M_{tj}(y_j)$ can be approximated in various ways around the non-smooth point $y=\dot{\varphi}_p$ and on the stick-slip interval $[0,B]$ of variation of $y_j=\dot{\varphi}_j - v/r,\ j=1,2.$ Next, we give a method of approximation. This method preserves the form of the graph of $\widetilde{M}_{tj}(y_j)$, $j=1,2$ outside a small interval. It avoids rapidly decreasing moments of the wheels, realizing a smooth and almost flat behavior around $\dot{\varphi}_p$. This behavior avoids the loss of adherence, and allows the increasing of accelerations $\ddot{\varphi}_j$, $j=1,2$.

Lemma 1 *Let* $\varepsilon > 0$ *sufficiently small. There is* $\delta(\varepsilon) > 0$ *and a two-degree algebraic polynomial* $p_2 = p_2(y) = \alpha_2 y^2 + \alpha_1 y + \alpha_0$ *in* $y = \dot{\varphi} - v/r$, $\alpha_2 < 0$, *defined on* $[\dot{\varphi}_p - \varepsilon, \dot{\varphi}_p + \delta(\varepsilon)]$, *such that:*

$$f(y) = p_2(y),\ y \in [\dot{\varphi}_p - \varepsilon, \dot{\varphi}_p + \delta(\varepsilon)],$$

$$f(y) = \widetilde{M}_{t1}(y),\ y \in R_+ \setminus [\dot{\varphi}_p - \varepsilon, \dot{\varphi}_p + \delta(\varepsilon)],\ f(-y) = f(y),\ y \le 0,$$

is a $C^{(1)}$ *function, approximating uniformly the non-derivable function* $\widetilde{M}_{t1}(y_1)$.

Proof. The derivative of $\widetilde{M}_{tj}(y_j)$ with respect to y_j is constant on $[-\dot{\varphi}_p, \dot{\varphi}_p]$, being equal to $\mu Q r / \dot{\varphi}_p$ on this interval. Outside this interval, its value is

$$-aQ/y^2,\ y \notin [-\dot{\varphi}_p, \dot{\varphi}_p]$$

We will modify this derivative around $y_1 = \dot{\varphi}_p$, making it linear on $[\dot{\varphi}_p - \varepsilon, \dot{\varphi}_p + \delta] = I_{\varepsilon,\delta}$. The derivative must vanish at $\dot{\varphi}_p$ and be continuous on $[0, \infty[$. Pasting the graphs of the affine functions, and integrating, one obtains its primitive on the positive semi- axis. It will be a second-degree algebraic polynomial on the interval $I_{\varepsilon,\delta}$, and a $C^{(1)}$ function on the positive semi-axis, with maximum point $\dot{\varphi}_p$. Then one extends the obtained function to the whole R, such that to obtain an even $C^{(1)}$ function. Finally, one considers the restriction of this function to the interval of the values of y_j, which is bounded for each $j \in \{1,2\}$. The equation of the straight line defined by the points:

$$(\dot{\varphi}_p - \varepsilon, \mu Q r / \dot{\varphi}_p), (\dot{\varphi}_p, 0)$$

and its intersection with the graph of the right-side derivative $f'(y) = -(aQ)/y^2$ are:

$$h = l(y) = \frac{\mu Q r}{\varepsilon}[(y/\dot{\varphi}_p) - 1],\ l(y^*) = -(aQ)/y^{*2} \Rightarrow$$

$$\mu r y^{*^2}(\dot{\varphi}_p - y^*) = \varepsilon a \dot{\varphi}_p \Rightarrow \exists!\, y^* = \dot{\varphi}_p + \delta, \delta > 0$$

Whence, there is a unique $y^* = \dot{\varphi}_p + \delta, \delta = \delta(\varepsilon) \to 0, \varepsilon \to 0,$ verifying the last condition. This follows from qualities of the graph of the three degree polynomial function involved in the last equation in y^*. Integration on $[\dot{\varphi}_p - \varepsilon, y + \delta(\varepsilon)]$ yields:

$$p_2(y) = \frac{\mu Qr}{\varepsilon}\left[-\frac{1}{2\dot{\varphi}_p}\left(y^2-(\dot{\varphi}_p-\varepsilon)^2\right)+\left(y-(\dot{\varphi}_p-\varepsilon)\right)\right] =$$

$$\frac{\mu Qr}{\varepsilon}\left(y-(\dot{\varphi}_p-\varepsilon)\right)\left[1-\frac{1}{\dot{\varphi}_p}\frac{y+\dot{\varphi}_p-\varepsilon}{2}\right] \Rightarrow \max p_2(y) = p_2(\dot{\varphi}_p) = \frac{\mu Qr}{2\dot{\varphi}_p}\varepsilon \qquad (5)$$

$$y \in \left[\dot{\varphi}_p-\varepsilon, \dot{\varphi}_p+\delta(\varepsilon)\right].$$

To obtain our function at any point y of $[0,\infty[$, one integrates the continuous function

$$f'(y) := \frac{\mu Qr}{\dot{\varphi}_p},\ y \in \left[0, \dot{\varphi}_p - \varepsilon\right],\ f'(y) := l(y),\ y \in \left[\dot{\varphi}_p-\varepsilon, \dot{\varphi}_p+\delta\right],$$

$$f'(y) := -\frac{aQ}{y^2},\ y > \dot{\varphi}_p + \delta,$$

on the interval $[0, y]$. We obtain the following formula for the modified moments of the wheels:

$$f_j(y_j) = \frac{\mu Qr}{\dot{\varphi}_p} y_j,\ y_j \in \left[0, \dot{\varphi}_p - \varepsilon\right],$$

$$f_j(y_j) = P_2(y) := \frac{\mu Qr}{\dot{\varphi}_p}(\dot{\varphi}_p-\varepsilon) + p_2(y_j),\ y_j \in \left[\dot{\varphi}_p-\varepsilon, \dot{\varphi}_p+\delta(\varepsilon)\right], \qquad (6)$$

$$f_j(y_j) = Q\,r\left(\frac{a}{r}\cdot\frac{1}{y_j}+b\right),\ y_j \in \left[\dot{\varphi}_p+\delta(\varepsilon), B\right]$$

In (6) p_2 is from (5), and B is an upper bound for the values of $y_j\, j=1,2.$ By its construction, f' is continuous, so that f is a $C^{(1)}$ function verifying the conditions of the lemma. Extension of f' to an odd function on the real axes preserves the continuity, since f' is linear around the origin. Thus, f has an even extension to $R,$ with the properties mentioned in the statement. •

Denote by $(1')$ the system obtained from (1) by replacing the moments of the wheels with the similar $C^{(1)}$ functions $f = f_j$ in variables $y_j = \dot{\varphi}_j - v/r,\ j=1,2.$ The measured data show that $K_m < 0,\ K_m^2 - 4c_r I_r < 0.$

Corollary 1 *The function from Lemma 1 has the following properties*:

$$\max f_{j,\varepsilon} = f_{j,\varepsilon}(\dot{\varphi}_p) = \mu\, Q\, r\left(1-\frac{\varepsilon}{2\dot{\varphi}_p}\right) \uparrow \mu\, Q\, r = \max M_{t\,j},$$

$$\widetilde{M}_{t\,j}(y) = \sup_{\varepsilon\downarrow 0} f_{j\,\varepsilon}(y),\ y \in R,\ j=1,2.$$

Proof. The first assertion is obvious. For the second one, we observe that $f_{j\,\varepsilon}$ is a concave two-degree polynomial. Its graph is tangent to that of $M_{t\,j}$. There are two tangency points, situated on different sides with respect to $\dot{\varphi}_p$. because to the left

we have a linear behavior of M_{tj}, while on the right we have a convex one, the conclusion follows. Moreover, for the right hand size, there is a common tangent for the two graphs, at the corresponding point. •
Denote by $(1')$ the system obtained from (1) by replacing the moments of the wheels with the corresponding $C^{(1)}$ functions $f = f_j$ in variables $y_j = \dot{\varphi}_j - v/r,\ j = 1,2,$ given by lemma 1. The measured data show that $K_m < 0, K_m^2 - 4c_r I_r < 0.$ The aim of the following result is to establish explicit approximate solutions of $(1')$.

Theorem 1 *The general form of the approximate solution* of $(1')$ *in a neighborhood of an equilibrium point* t_e *is:*

$$
\begin{aligned}
&I\ddot{\varphi}_1 + I'\ddot{\varphi}_2 + uI_r\ddot{\varphi}_r \approx 0\\
&\dot{\varphi}_r(t) - \dot{\varphi}_r(t_e) \approx a_1 e^{-p_3(t-t_e)}\cos(\omega_3(t-t_e)) + a_2 e^{-p_3(t-t_e)}\sin(\omega_3(t-t_e)),\\
&p_3 = K_m/2I_r,\ \omega_3 = \frac{\left[4c_r I_r\left(1+\left(u^2 I_r\right)/(I')\right) - K_m^2\right]^{1/2}}{2I_r}\\
&\dot{\varphi}_1(t) = \left[\dot{\varphi}_{1,e}^{-1} e^{(\alpha_1/I)(t-t_e)} + (\alpha_2/\alpha_1)\left(e^{(\alpha_1/I)(t-t_e)} - 1\right)\right]^{-1} +\\
&\beta_0 + \beta_1 e^{-p_3(t-t_e)}\cos(\omega_3(t-t_e)) + \beta_2 e^{-p_3(t-t_e)}\sin(\omega_3(t-t_e)),\\
&\dot{\varphi}_2(t) = \left[\dot{\varphi}_{2,e}^{-1} e^{(\alpha_1/I')(t-t_e)} + (\alpha_2/\alpha_1)\left(e^{(\alpha_1/I')(t-t_e)} - 1\right)\right]^{-1} +\\
&\lambda_0 + \lambda_1 e^{-p_3(t-t_e)}\cos(\omega_3(t-t_e)) + \lambda_2 e^{-p_3(t-t_e)}\sin(\omega_3(t-t_e))
\end{aligned}
\tag{7}
$$

Proof. Addition of the three equations of (1), the last one being multiplied by u, leads to:

$$I\ddot{\varphi}_1 + I'\ddot{\varphi}_2 + uI_r\ddot{\varphi}_r = -\sum_{j=1}^{2} M_{tj}(\dot{\varphi}_j) + uM_m(\dot{\varphi}_r) \tag{8}$$

Because the free terms are $C^{(1)}$ functions of $\dot{\varphi}_j,\ j = 1,2, \dot{\varphi}_r,$ the solutions will be of class $C^{(2)}$, obtained by local integration of composition of $C^{(1)}$ functions of t. Relations (4) and derivation in each equation $(1')$ show that the second order derivatives at t_e vanish too, as the first ones do. Due to Lagrange's Theorem, in a neighborhood of t_e the first of relations (7) holds; these remarks yield:

$$I\ddot{\varphi}_1 \approx I'\ddot{\varphi}_2 \approx I_r u\ddot{\varphi}_r \Rightarrow \varphi_2 \approx (I_r u/I')\varphi_r + b_1(t-t_e) + b_0 \tag{9}$$

Inserting this in the third equation (1) or $(1')$ lead to the following equation in φ_r :

$$I_r\ddot{\varphi}_r + K_m\dot{\varphi}_r + c_r\left[1 + \left(u^2 I_r/I'\right)\right]\varphi_r = d_1(t-t_e) + d_0.$$

The explicit form of the solution $\dot{\varphi}_r$ follows. Using (9) once more, lemma 1, addition of the last two equations of $(1')$, the third one multiplied by u, yield:

$$
\begin{aligned}
&2I_r u\ddot{\varphi}_r - uM_m(\dot{\varphi}_r) - c_1(\varphi_1 - \varphi_2) =\\
&-M_{t2}(\dot{\varphi}_2) = \widetilde{M}(y_2) \approx -\mu Qr \Rightarrow\\
&c_1(\varphi_1 - \varphi_2) \approx 2I_r u\ddot{\varphi}_r - uM_m(\dot{\varphi}_r) + \mu Qr
\end{aligned}
$$

Inserting this expression in the first equation $(1')$ and using (9) once more, lead to:

$$I\dot{y}_1 + \alpha_1 y_1 + \alpha_2 y_1^2 = -c_1(\varphi_1 - \varphi_2) \approx$$
$$-2I_r u\ddot{\varphi}_r + uM_m(\dot{\varphi}_r) - \mu Qr =$$
$$= \gamma_0 + \lambda_1 e^{-p_3(t-t_e)} \cos(\omega_3(t - t_e)) + \gamma_2 e^{-p_3(t-t_e)} \sin(\omega_3(t - t_e))$$

Solving firstly the homogeneous Bernoulli equation, the general form of the solution of non-homogeneous equation in $y_1 = \dot{\varphi}_1 - v/r$ follows. The solution $\dot{\varphi}_2$ follows in the same way. Relations (4) show that in the explicit form of the functions $\dot{\varphi}_j$ the free terms and the coefficient of the functions involving "cos" are the same. •

3 Local solutions as fixed points

This section is devoted to generalizations of the results of the previous section. We consider functions as elements of $L^2(0,T)$, where $(0,T)$ is the interval of time on which the phenomenon is studied. Thus, we have another method for the proof of the local existence and uniqueness of the solutions, with more general free terms. Because of slowly decreasing behavior of $\dot{\varphi}_p$ on this relatively small interval, this function is "almost" constant. In computations, it is used as a constant. Let us consider an interval on which both moments $M_{tj}(\dot{\varphi}_j)$ are linear as functions of $\dot{\varphi}_j$. The other notations and hypothesis are the same as those of Theorem 1.

Theorem 2 *On small intervals defined by strict inequalities* (2), *the solution* $S = \left(\dot{\varphi}_1 - \frac{v}{r}, \dot{\varphi}_2 - \frac{v}{r}, \dot{\varphi}_r - \frac{vu}{r}\right)$ *of the system* (1) *can be determined by the successive approximation method.*

Proof. Let consider an interval on which both moments $M_{tj}(\dot{\varphi}_j)$ are linear. Our system appears below in the integral form, as well as in its "given" form:

$$S^{tr} = \left(A \circ \int \; + B \circ \iint \;\right)\left(S^{tr}\right) + \left(0,0,M_0/I_r\right)^{tr} \int dt = W\left(S^{tr}\right)$$

$$\ddot{\varphi}_1 = \frac{1}{I}\left[-\mu Q r(\dot{\varphi}_1 - v/r) - c_1(\varphi_1 - \varphi_2)\right];$$
$$\ddot{\varphi}_2 = \frac{1}{I'}\left[-\mu Q r(\dot{\varphi}_2 - v/r) + c_1(\varphi_1 - \varphi_2) - c_r u(u\varphi_2 - \varphi_r)\right]; \qquad (10)$$
$$\ddot{\varphi}_r = \frac{1}{I_r}\left[M_0 + K_m(\dot{\varphi}_r - vu/r) + c_r(u\varphi_2 - \varphi_r)\right].$$

In the above first matrix relation, $S^{tr} := (\dot{\varphi}_1 - v/r, \dot{\varphi}_2 - v/r, \dot{\varphi}_r - vu/r)^{tr}$, A is the negative definite diagonal matrix having as entries the coefficients of

$\dot{\varphi}_1 - v/r, \dot{\varphi}_2 - v/r, \dot{\varphi}_r - vu/r,$

while B is the linear operator defined by the matrix applied to $(\varphi_1, \varphi_2, \varphi_r)$ The other two linear operators are the first, respectively the second order integration operators, defined on the subspace of functions with compact support contained in our open interval. Consider the space $\widetilde{H} = H_0^{(2)}(J_\varepsilon)$, the closure of the space of all test functions on J_ε in the Hilbert space $H^{(2)}(J_\varepsilon)$. One approximates any function from this space by functions with the support contained in J_ε, which coincide with the old ones on intervals $(t_0 - \varepsilon_n, t_0 + \varepsilon_n), \varepsilon_n \uparrow \varepsilon$. Due to Poincaré's inequality [11], for the linear part $W_0 \in B(\widetilde{H}^3)$ of the affine operator W defined in (10), we have:

$$\|W_0\| \le \|A\| \cdot 2^{1/2} \varepsilon + \|B\| \cdot 2\varepsilon^2 < 1, \text{ if } \varepsilon > 0 \text{ is sufficiently small.}$$

Here S^{tr}, as well as the other vectors from (10), is considered as an element of $\widetilde{H}^3$. For sufficiently small $\varepsilon > 0$, W is a contraction on $\widetilde{H}^3$. This assertion remains true for any bounded linear operators A, B. It follows that for any sequence defined by: $S_{n+1,\varepsilon}^t = W(S_{n,\varepsilon}^t), n \in N, (S_{n,\varepsilon}^0 \in \widetilde{H}_\varepsilon^3$ arbitrary chosen), we have: $\lim_{n\to\infty} S_{n,\varepsilon}^t = S_\varepsilon^{\ t}$. Moreover, the well known basic evaluation of absolute error for contractions holds. •

Remark 1 In the preceding proof, the presence of a "locally" constant matrix, and also the linearization on small intervals seem not to be essential. To illustrate this remark, we give the statement and the sketch of the proof for a more general result. However, the linearization can lead to finding explicit solutions on small intervals.

Theorem 3 *Let the system of nonlinear equations on a closed small interval* $\bar{J}_\varepsilon = [t_0 - \varepsilon, t_0 + \varepsilon]$:

$$\begin{aligned} & -\lambda_j \ddot{\varphi}_j = u_j(t, \varphi_1, \varphi_2, \varphi_3, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3), \quad \lambda_j \in R, \quad \varphi_j \in C^{(2)}(J_\varepsilon) \\ & \sup(\varphi_j) \subset J_\varepsilon, \ j \in \{1,2,3\}, \end{aligned} \tag{11}$$

, and u_j *are* $C^{(1)}$ *of the set of variables. Then for* $\varepsilon > 0$ *sufficiently small, there exists a solution, which can be determinate by successive approximation method.*

Proof. We write the system (11) in a more convenient way, in a small interval on which the smoothness holds:

$$-\lambda_j(\dot{\varphi}_j(t)) = \int_{t_0}^{t} u_j(x, \varphi_1(x), \dots, \dot{\varphi}_3(x))dx \Leftrightarrow$$

$$\dot{\varphi}_j(t) = -\frac{1}{\lambda_j} \int_{t_0}^{t} u_j(x, \int_{t_0}^{x} \dot{\varphi}_1(s)ds + \varphi_1(t_0), \dots, \dot{\varphi}_3(x))dx =$$

$$y_j(\dot{\varphi}_j)(t), \ t \in \bar{J}_\varepsilon(t_0), \quad j \in \{1,2,3\} \Leftrightarrow \quad \Psi = \Upsilon(\Psi)$$

In the above notation $\Psi := (\dot{\varphi}_j)_{j=1}^3$, while Y is the "double integration" applied to appear only $\dot{\varphi}_j$ in the right hand size. The nonlinear operator Y applies $\overline{X} = (C^{(1)}(\overline{J}_\varepsilon))^3$ into itself. The norm is $\|h\|_{1,\infty} = \|h\|_\infty + \|\dot{h}\|_\infty$. Because the functions $u_j, j = 1,2,3$ are smooth, with bounded first order derivatives on $\overline{J}_\varepsilon(t_0)$, we infer that Y is a contraction for a sufficiently small length of the interval $\overline{J}_\varepsilon, l(\overline{J}_\varepsilon) = 2\varepsilon.$ From the conditions:

$$\varepsilon^2 + \varepsilon < B := \min_{j=1,2,3} \frac{|\lambda_j|}{M}, \textit{ where}$$

$$M = \max_{j,k,l,t} \left\{ \left| \frac{\partial u_j}{\partial \varphi_l} \right|, \left| \frac{\partial u_j}{\partial \dot{\varphi}_k} \right| \right\}, \textit{ where } t \in \overline{J}_\varepsilon, \quad j,k,l \in \{1,2,3\},$$

one deduces:

$$\|Y(\Psi_1) - Y(\Psi_2)\|_{\infty,1} \le \frac{1}{\min_j |\lambda_j|} \max_{j,k,l} \left\{ \left\| \frac{\partial u_j}{\partial \varphi_l} \right\|_\infty, \left\| \frac{\partial u_j}{\partial \dot{\varphi}_k} \right\|_\infty \right\} (\varepsilon + \varepsilon^2) \|\Psi_1 - \Psi_2\|_{\infty,1} =$$

$$= \frac{\varepsilon + \varepsilon^2}{B} \|\Psi_1 - \Psi_2\|_{\infty,1} = q \|\Psi_1 - \Psi_2\|_{\infty,1}, \ 0 < q < 1.$$

For such a small length of the interval, the successive approximation method works for $Y.$ An important case is that of u_j having piecewise continuous, uniformly bounded partial derivatives with respect to $\varphi_j, \dot{\varphi}_j$ on the whole interval $[0,T]$. In this case, M, B from above do not depend on $\varepsilon.$ In all cases, one obtains an approximating sequence of functions $(\Psi_n)_n$ from $\overline{X},$ which converges to the solution $\Psi \in \overline{X}$ in the norm of this space. •

Remark 2 There are different variants for a proof of Theorem 3. If we leave the system as in the statement, approximating the right hand size member by its affine part, then the solution follows as in the proof of Theorem 2.

Remark 3 In theorems no. 2 and respectively 3, we have obtained solutions on non-overlapping intervals, the join of these subintervals being the whole interval. We extend each such local solution by taking zero value outside its small interval of definition. Thus one obtains an orthogonal system in $L^2((0,T)),$ (respectively in $L^2([0,T])$) of local solutions, with non-overlapping supports. A problem, which arises naturally, is to prove the completeness or the non-completeness of this system. In general, such a system is not complete.

4 On the continuity of derivation operation. Solving a partial differential equation

In the proof of Theorem 2 we have used Poincaré's inequality, the idea being to point up the fact that on small intervals, integration operation is a contraction. As it is well known, for several norms, usually the real differential operators are not continuous. In case of continuity, it should be difficult to determine their norms. The following result points up a class of orthogonal systems, which generates real, and complex Hilbert spaces of smooth functions, such that the derivation-operator has norm at most one, or a norm which can be determined. Some examples are also given.

Proposition 1. *Let $J \subset R$ be an open interval, or respectively $A \subset C$ an open subset. Assume that there exists an orthogonal system $\{e_n\}_n \subset L^2(J)$, (respectively in $L^2(A)$) of smooth (respectively complex analytic) functions such that the system formed by the derivatives $\{e'_n\}_n$ is also orthogonal, and $\|e'_n\| \le \|e_n\| \; \forall n \in N$. Then the derivation operator D from $\widetilde{H} := cl(Sp\{e_n\}_n)$ into $H_1 = cl(Sp\{e'_n\}_n)$ is continuous. Moreover, we have: $\|D^m\| \le 1$, $m \in N$.*

Proof. Let $\{e_n\}_{n\in N}$ be the orthogonal system from the statement of the present Proposition. Then for any $f \in Sp\{e_n\}_n$, derivation term by term in a finite sum, leads to:

$$f = \sum_{j\in S(f)} \langle f, e_j\rangle e_j, \quad f' = \sum_{j\in S(f)} \langle f, e_j\rangle e'_j,$$

where the sum is over a finite subset $S(f) \subset N$. These expansions yield:

$$\|f\|^2 - \|f'\|^2 = \sum_{j\in S(f)} \langle f, e_j\rangle^2 \|e_j\|^2 - \sum_{j\in S(f)} \langle f, e_j\rangle^2 \|e'_j\|^2$$

$$= \sum_{j\in S(f)} \langle f, e_j\rangle^2 \left(\|e_j\|^2 - \|e'_j\|^2\right) \ge 0,$$

, because of : $\|e'_j\| \le \|e_j\|$. The conclusion is: $\|f'\| \le \|f\|$ on the dense subspace $Sp(\{e_n\}_n)$ of $\widetilde{H}$, which leads to the same inequality on the whole space $\widetilde{H}$. Thus we reach the conclusion $\|D\| \le 1$. •

Example 1. An example of an orthogonal system in a real Hilbert space H, such that $u'_n = \alpha_n u_n, |\alpha_n| \le 1 \; \forall n$ is the following one:

$$u_n(x) = a_n^x, \quad \{a_n\}_n = Q \cap [1/e, e] \cup \{1/e, e\},$$

$$\langle \sum_{n=0}^{p} \alpha_n u_n, \sum_{j=0}^{q} \beta_j u_j \rangle := \sum_{m=0}^{\min\{p,q\}} \alpha_m \beta_m,$$

$\widetilde{H}$ being the completion of the Euclidean space $(Sp\{u_n\}_n, \langle \;\; \rangle)$. Obviously, we have

$$u'^2_n = u_n^2 (\ln a_n)^2 \le u_n^2, n \in N, \langle u_n, u_j \rangle = \delta_{n,j},$$

$$\|u'_n\|^2 = (\ln a_n)^2 \le 1 = \|u_n\|^2.$$

In this example, a stronger condition is accomplished:

$$|u'_n| \le |u_n| \Rightarrow \|u'_n\|_p \le \|u_n\|_p \; \forall p \in [1, \infty].$$

This is a consequence of the monotony of the norms of the $L^p(J)$ spaces, $J \subset R$ being a bounded interval. Application of the Proposition 1 leads to $\|D\| \le 1$, and because of $D(\exp) = \exp$, we actually have $\|D\| = 1$.
Since $a_n \in [e^{-1}, e], \forall n \in N$, we have:

$$f_n(x) := u_n^2(x) - u_n'^2(x) = a_n^{2x}\left(1 - (\log a_n)^2\right) \ge 0,$$

, and equality holds only for $a_n \in \{e^{-1}, e\}$. Consider the function $f(t) := 1 - (\ln t)^2$, which is vanishing at the ends of the interval and has as unique maximum point $t_m = 1 = e^{-1} \cdot e.$ Moreover, the ratio in which this maximum point divide the interval equals e^{-1}. Obviously, the complex analogue can be discussed.
The next example refers to a not complete orthogonal system in a real Hilbert space, obtained by the aid of disjoint supports.

Example 2 Let $X = C^{(1)}(R) \cap L^2(R), e_n(x) = [(x-n)(n+1-x)]^2 \cdot \chi_{[n,n+1)}(x), n \in Z.$
Obviously, the system $\{e_n\}_n$ is orthogonal in

$L^2(R), \|e_n\|^2 = \dfrac{1}{15 \times 42}, e_n' \in C(R) \cap L^2(R), \|e_n'\|^2 = \dfrac{2}{15 \times 7}\ \forall n \in Z, \{e_n'\}_n$ is also an orthogonal system. By the proof of Proposition 1, the derivation-operator

$D : cl(Sp(\{e_n\}_n)) \to cl(Sp(\{e_n'\})) \subset L^2(R)$ is continuous and we have $\|D\| = \dfrac{\|e_n'\|}{\|e_n\|} = 2 \cdot \sqrt{3}.$

Remark 4 Let $n = 2$; (for arbitrary n, the proof is similar). If $\{e_n\}_n$ is a complete orthogonal system in $L_\rho^2(J)$, then $\{e_n \otimes e_m\}_{(n.m) \in N^2}$ is an orthogonal complete system in $L_\rho^2(J^2)$.

As an application of Remark 4, we propose a solution for a particular case of the modified equation describing the movement of a perfect fluid:

$$\rho_t + div(\rho \cdot \vec{v}) = f(\vec{x}, t) \tag{12}$$

Here $\vec{v} = \vec{v}(\vec{x}, t)$ is the velocity vector field, $\rho = \rho(\vec{x}, t)$ its density, $f(\vec{x}, t)$ being the intensity of the sources. The unknown function is $\vec{v}$.
One considers the particular case of a rotation-free field

$$\vec{v}, \vec{v} \to \vec{v}_0, \|\vec{x}\| \to \infty, 0 < \delta \le \rho(\vec{x})\ \forall \vec{x} \in R^3, \rho_t = 0.$$

The preceding conditions assume that $\vec{v}_0$ is given. Because the intensity of the sources converges to zero when the norm of the position vector converges to infinity, the natural condition is $\vec{v}_0 = \vec{0}$. One also assumes that ρ, f are analytic functions and belong respectively to $L^2(R^3), L^2(R^4)$. Because of the assumptions on f, it is possible to find a solution $u \in L^2(R^4)$ of the following Poisson problem, related to the above one, for $\rho = const. > 0$:

$$\rho \cdot \Delta u = f, \quad u, f \in L^2(R^4) \tag{13}$$

The classical problem requires (by physical reasons) solving the problem in the case of vanishing source f, when $\|\vec{x}\| \to \infty$, namely:

$$\Delta u(\vec{x}, t) = 0 \ \forall \vec{x} \notin \Omega, \ \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0, \ \lim_{\|\vec{x}\|\to\infty} (\nabla u)(\vec{x}, t) = \vec{v}_0 = const.vector$$

Here $\Omega \subset R^3$ is a simply connected large bounded domain. If Ω is a parallelepiped centered at the origin, with faces parallel to the coordinate planes, any affine function in (x, y, z):

$$u(x, y, z, t) = \varphi(t)[ax + by + cz + d]$$

is a trivial solution for the latter problem. Next we sketch a possible way of solving (13), , by using Remark 4 and the continuity of the inverse of the Laplace-operator. This computational method allows finding explicit solutions and related consequences. The condition of a square-integrable solution implies a vanishing at infinity-solution, which stands for the homogeneous boundary condition.

Theorem 4 *If the source-function* $f \in Sp\{h_m \otimes h_n \otimes h_p \otimes h_q\}_{(m,n,p,q)} \subset L^2(R^4)$*, then the problem* (13) *has a unique solution, which belongs to* S*. In particular, it is an analytic function. The solution is:*

$$u = \Delta^{-1}(f/\rho).$$

Its coefficients with respect to the Hilbert basis associated to Hermite's functions are given by (15)*, written for the coefficients of* f/ρ.

Proof. The idea is to reduce the computations related to the Laplace operator to the corresponding operations related to the operator D from below. For this operator, the proper numbers and "eigenvectors" are given in [13]. The sums from below are finite.

$$\begin{aligned} D(u) &= \Delta u - \left(\sum_{j=1}^{3} x_j^2\right) u = \\ &= \sum_{(m,n,p,q)\in F} \lambda_{(m,n,p,q)} [-(2m+1) - (2n+1) - (2p+1)] h_m \otimes h_n \otimes h_p \otimes h_q = \\ &= v = \sum_{(m,n,p,q)\in F} a_{(m,n,p,q)}(v) h_m \otimes h_n \otimes h_p \otimes h_q \Leftrightarrow \\ u &= D^{-1}(v) = -\sum_{(m,n,p,q)\in F} \frac{a_{(m,n,p,q)}}{2(m+n+p)+3} h_m \otimes h_n \otimes h_p \otimes h_q \end{aligned} \tag{14}$$

The last relation shows that D^{-1} is continuous on the dense subspace of $L^2(R^4)$ generated by the elements of its Hilbert basis, so it has a continuous extension given by (14), for infinite sums which define elements of L^2. Going back to the explicit solution of (13), we can proceed in the same way, by determining the coefficients of u:

$$\Delta u = D(u) + \left(\sum_{j=1}^{3} x_j^2\right) u =$$
$$= \sum_{(m,n,p,q)} \left[-2(m+n+p) - 3 + \left(\sum_{j=1}^{3} x_j^2\right)\right] \lambda_{(m,n,p,q)} h_m \otimes h_n \otimes h_p \otimes h_q =$$
$$= \left(\sum_{(m,n,p,q)} \beta_{(m,n,p,q)}(f) h_m \otimes h_n \otimes h_p \otimes h_q \right) \Rightarrow \qquad (15)$$
$$\lambda_{(m,n,p,q)} = \frac{\beta_{(m,n,p,q)}(f)}{-2(m+n+p) - 3 + \alpha_{(m,n,p,q)}}$$

In the above expression, the sums are finite. We have denoted by $\beta_{(m,n,p,q)}(f)$ the Fourier coefficients of f/ρ with respect to the complete orthogonal system related to Hermite's functions, and by $\alpha_{(m,n,p,q)}$ the numbers given by:

$$\alpha_{(m,n,p,q)} = \left[\left(\int_{R^4} \left(\sum_{j=1}^{3} x_j^2 \right) h_m^2 \otimes h_n^2 \otimes h_p^2 \otimes h_q^2 \right) / \left(\|h_m\| \cdot \|h_n\| \cdot \|h_p\| \cdot \|h_q\| \right)^2 \right]$$

It is easy to observe that:

$$\alpha_{(m,n,p,q)} \neq 0 \Leftrightarrow (m,n,p) \in \{0,1\}^3$$

There are a finite number of such numbers, which does not change the asymptotic behavior of $\lambda_{(m,n,p,q)}$ given by (15). It follows that $f/\rho \in L^2(R^4)$ and that it is analytic. We conclude that $u = \Delta^{-1}(f/\rho) \in L^2(R^4)$ and the inverse of the Laplace operator is continuous. Its norm can be determined from the preceding computations. Because Δu is analytic by hypothesis, and the series defining u is absolute convergent in any point (due to Schwarz inequality), the theorem of term-by-term derivation holds. Hence $u \in C^{(\infty)}(R^4)$ and it is analytic in R^4 (by using once more Schwarz inequality, this time for the remainder). •

The next result gives another method, which seems to work for the general equation (12).

Theorem 5 *Under the hypothesis of Theorem 5, there is a unique solution of the equation* (12) *in the class* S, *and it can be determined by:*

$$\vec{v} = \frac{1}{\rho} div^{-1}(f - \rho_t) = \frac{1}{\rho}(\nabla \circ \Delta^{-1})(f - \rho_t)$$

The operator div^{-1} *has a continuous extension to the space* $L^2(R^4)$, *and all the components* $v_j,\ j = 1,2,3$ *of the solution* $\vec{v}$ *are analytic functions on* R^4.

Proof. As in the proof of Theorem 4, we will work with finite linear combinations of the base related to Hermite's functions. On the vector subspace S generated in this way, both operators Δ, ∇ are injective, the Laplace operator being also onto, by the proof of Theorem 4. If $\nabla s = 0, s \in S$, then s must be a constant. This constant is zero, because of vanishing at infinity of all functions in S. Due to the maximum modulus principle for harmonic functions, the Laplace operator is also injective. These remarks lead to:

$$\Delta = div \circ \nabla \Rightarrow div = \Delta \circ \nabla^{-1} \ on \ R(\nabla) \Rightarrow \exists \ div^{-1} = \nabla \circ \Delta^{-1} \ on \ S.$$

Due to the behavior on Hermite's functions, we have:

$$\Delta^{-1}\left(h_m \otimes h_n \otimes h_p \otimes h_q\right) = -\frac{h_m \otimes h_n \otimes h_p \otimes h_q}{2(m+n+p)+3}$$

for all (m,n,p,q), except a finite subset (see the proof of Theorem 4). For each component $v_j,\ j=1,2,3$ of the field $\vec{v}$, we have:

$$\langle \Delta h_{j,k}, h_{j,k} \rangle = -\left\|\nabla h_{j,k}\right\|^2 \Rightarrow \left\|\nabla h_{j,k}\right\| = (2k+1)^{1/2}\left\|h_{j,k}\right\|,\ k \geq 2.$$

These relations yield:

$$\left\|(\nabla \circ \Delta^{-1})\left(h_m \otimes h_n \otimes h_p \otimes h_q\right)\right\| \leq \frac{\left(2(m+n+p)+3\right)^{1/2}}{2(m+n+p)+3}\left\|h_m \otimes h_n \otimes h_p \otimes h_q\right\| =$$

$$\frac{1}{\left(2(m+n+p)+3\right)^{1/2}}\left\|h_m \otimes h_n \otimes h_p \otimes h_q\right\|,\ m,n,p,q \geq 2.$$

It follows that our operator multiplies the corresponding Fourier coefficients respectively by the numbers $\left(1/(2(m+n+p)+3)\right)^{1/2}$, hence their absolute values are diminished. Obviously, it has an extension to the space $L^2\left(R^4\right)$, preserving the norm. From the proof of Theorem 4, we infer that Δ^{-1} applies analytic functions from S into analytic functions from the same class. Since the power series-functions can be derived term by term, we infer that each of the components $\partial/\partial x_j,\ j=1,2,3$ of ∇, applies S into S. It follows that on the subspace S, the following conclusion holds:

$$s \in S \Rightarrow \exists \ div^{-1}(s) = \left(\nabla \circ \Delta^{-1}\right)(s) \in S.$$

In particular, $div^{-1}(s)$ is analytic. Going back to the equation (12), since $f - \rho_t \in S$, we have $\rho\vec{v} = div^{-1}\left(f - \rho_t\right) \in S^3$ and $\vec{v} = \frac{1}{\rho} div^{-1}\left(f - \rho_t\right) \in S^3$.

Now the conclusion follows. •

5 "Continuity subspaces for complex differentiation" associated to a linear bounded operator

In this Section, we mention that the derivation operator can be continuous on some special subspaces of a Hilbert space, associated to an arbitrary linear bounded operator. Some of the following results are valid for continuous operators on different spaces, which contain the space of all entire functions on the complex plane (for examples of such spaces, see [17]). We work in the Hilbert space

$$H = L^2(\Gamma_1), \Gamma_1 = \{|z| = 1\},$$

, endowed with the scalar product defined by:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g}(e^{i\theta}) d\theta.$$

For any two elements $f, g \in H^2 \subset H$ ([17]), we obviously have:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} c_n \overline{d}_n,\ f(z) = \sum_{n=0}^{\infty} c_n z^n, g(z) = \sum_{n=0}^{\infty} d_n z^n,\ z \in C,$$

Theorem 6 Let

$$T \in B(H^2), S = S_D(T) = Sp\{(\exp((z-\lambda)T)(h_0)); \lambda \in C\}, h_0(z) = 1\ \forall z \in C .$$

For all $s \in S$, *we have* $D(s) = T(s)$. *In particular, derivation-operator* D_z *is continuous from* S *into* $T(S)$. *Consequently, it has a unique linear continuous extension* $\widetilde{D} \in B(\overline{S}, H^2)$, *preserving the norm.*

Proof. Let $s \in S$,

$$s(z) = \sum_{n\in F} c_n \exp((z-\lambda_n)T)(h_0)$$

$$T(s)(z) = \sum_{n\in F} c_n T\left[\exp((z-\lambda_n)T)(h_0)\right] = \sum_{n\in F} c_n \left[\exp((z-\lambda_n)T)(T(h_0))\right]$$

Comparing with the behavior of the derivation operator on S, yields:

$$D_z(s)(z) = \sum_{n\in F} c_n T\left[\exp((z-\lambda_n)T)(h_0)\right] = T(s)(z),\ z \in C.$$

Consequently, if T is continuous on S, so is the derivation operator $D_z : \|D_z\|_S = \|T\|_S$. Hence the assertions of the statement follow. •

Proposition 2 *Let the system* (1) *be such that the moments* $M_{tj}(\dot{\varphi}_j), j = 1,2$ *are analytic functions of* $\dot{\varphi}_j$. *Assume that there exists an analytic solution and an equilibrium point such that* (3) *hold. Then the components* $\dot{\varphi}_j(t), j = 1,2, \dot{\varphi}_r$ *are constant functions.*

Proof. If the functions in the right size of the equations (1) are analytic, then the solutions $\dot{\varphi}_j, j = 1,2, \dot{\varphi}_r$ might be also analytic functions of t. We have already observed that the solution verify the relation:

$$I\ddot{\varphi}_1 + I'\ddot{\varphi}_2 + uI_r\ddot{\varphi}_r = -\sum_{j=1}^{2} M_{tj}(\dot{\varphi}_j) + M_m(\dot{\varphi}_r).$$

Assume that there exist analytic solutions and an equilibrium-point t_e, with the qualities (4). Then derivation of the last relation from above, of the equations (1), and application of relations (3) yield:

$$\left(I\ddot{\varphi}_1 + I'\ddot{\varphi}_2 + uI_r\ddot{\varphi}_r\right)(t_e) = 0;$$
$$\left[I\ddot{\varphi}_1 + c_1(\dot{\varphi}_1 - \dot{\varphi}_2)\right](t_e) = I\ddot{\varphi}_1(t_e) = 0 \Rightarrow \ddot{\varphi}_1(t_e) = 0;$$
$$\left[I_r\ddot{\varphi}_r + c_r(\dot{\varphi}_r - u\dot{\varphi}_2)\right](t_e) = I_r\ddot{\varphi}_r(t_e) = 0 \Rightarrow \ddot{\varphi}_r(t_e) \Rightarrow$$
$$\ddot{\varphi}_2(t_e) = 0$$

Successive derivation and application of (4) lead to the conclusion that all the derivatives of all orders greater or equal to two at t_e, of the functions $\varphi_j,\ j = 1,2,\ \varphi_r$, are zero. Now the conclusion follows. •

Remark 5 Let H be a Hilbert space and $T = A + iB \in B(H),\ A, B$ selfadjoint operators. The solution of the problem:

$$Y' = TY,\ Y(0) = u_0,\ Y : C \to H$$

can be written as:

$$Y(z) = \prod_{\alpha \in A}\left[\left(e^{\lambda_\alpha z}P_\alpha + (I - P_\alpha)\right)\right] \cdot i\prod_{\beta \in B}\left[\left(e^{\mu_\beta z}Q_\beta + (I - Q_\beta)\right)\right](u_0)$$

where $(P_\alpha)_\alpha, (Q_\beta)_\beta$ are projectors associated to the decomposition of unity of the operator A, respectively B.

As an application of the continuity of the function defining the moments:

$$\widetilde{M}_{tj}(y_j) = M_{tj}(\dot{\varphi}_j - v/r),\ j = 1,2,$$

we prove the following consequence. The aim is to make another connection to different fields.

Proposition 3 *Let $g = g(y)$ be the function:*

$$g(y) := \frac{\mu Q r}{\dot{\varphi}_p} y,\ |y| \le \dot{\varphi}_p,\ g(y) := \frac{aQ}{y} + bQr,\ |y| > \dot{\varphi}_p$$

Assume that $\dot{\varphi}_p$ satisfies (3). There is a unique nonincreasing solution

$$f : I = (b\dot{\varphi}_p/\mu, \infty) \to I,\ \textit{of the equation}\ g = g \circ f,$$

with the following properties:

$$f \circ f = id_J,\ f(\dot{\varphi}_p) = \dot{\varphi}_p,\ f\ \textit{is continuous},$$

$$\textit{analytic on}\ J \setminus \{\dot{\varphi}_p\};\ f_l(y) = \frac{a}{(\mu r/\dot{\varphi}_p)y - br},\ y \le \dot{\varphi}_p,$$

$$f_r(y) = \frac{\dot{\varphi}_p}{\mu r}\left(\frac{a}{y} + br\right),\ y \ge \dot{\varphi}_p,\ y \in J,\ f_l\left(\frac{b\dot{\varphi}_p}{\mu}\right) = \infty,\ f_r(\infty -) = \frac{b\dot{\varphi}_p}{\mu}.$$

See [14], [15] for related results.

6 Conclusions

The first part of this work gives a method of finding "local" approximate explicit solutions for a two order nonlinear system of differential equations, motivated by a practical phenomenon. We prove some generalizations, by using local linear approximation, local contractions and Poincaré's inequality. Sufficient conditions for the possibility of determining the norm of the derivation-operator are established. Related examples are given. One uses orthogonal basis of analytic functions in solving the movement equation of a perfect fluid. Concerning related aspects in complex differential equations and linear bounded operators, for an arbitrary bounded operator acting on $L^2_C(\{|z|=1\})$, one gives a constructive method for finding an associated infinite dimensional subspace, on which the derivation operator equals the given operator. A last statement concerns the “virtual” analytic trivial solution around an equilibrium point.

Some of the methods used in this work should be applicable in solving problems of related fields: functional equations, elements of operator theory, of complex analysis, optimization.

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