A STUDY ON CONFORMAL SEMI-INVARIANT RIEMANNIAN MAPS TO COSYMPLECTIC MANIFOLDS

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In [3]*, Akyol and Şahin introduced the concept of conformal semi-invariant Riemannian maps to almost Hermitian manifolds. In this article, we expand this concept to almost contact metric manifolds as a generalization of totally real submanifolds and semi-invariant Riemannian maps. Herewith, we present conformal semi-invariant Riemannian maps from Riemannian manifolds to cosymplectic manifolds. To ensure the existence of this concept, we prepare a illustrative example, investigate the geometry of the leaves of* $D_1, D_2, \bar{D}_1$ *and* $\bar{D}_2$*. We find necessary and sufficient conditions for conformal semi-invariant Riemannian maps to be totally geodesic. We also investigate the harmonicity of such maps.*

Keywords: Semi-invariant Riemannian map, Conformal semi-invariant Riemannian map, cosymplectic manifold.

MSC2000: 53C15, 53C43.

1. Introduction

Fischer introduced Riemannian map between Riemannian manifolds as a generalization of an isometric immersion and Riemannian submersion that satisfies the well known generalized eikonal equation $\|\vartheta_*\|^2 = rank\vartheta$, which is a bridge between geometric optics and physical optics [5]. Let $\vartheta : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a smooth map between Riemannian manifolds such that $0 < rank\vartheta < \min\{\dim(N_1), \dim(N_2)\}$. We state the kernel space of ϑ_* by $V_q = \ker\vartheta_{*_q}$ at $q \in N_1$ and consider the orthogonal complementary space $H_q = (\ker\vartheta_{*_q})^\perp$ to $\ker\vartheta_{*_q}$ in T_qN_1. Then the tangent space T_qN_1 of N_1 at q has the decomposition $T_qN_1 = (\ker\vartheta_{*_q}) \oplus (\ker\vartheta_{*_q})^\perp = V_q \oplus H_q$. We state the range of ϑ_* by $range\vartheta_*$ at $q \in N_1$ and consider the orthogonal complementary space $(range\vartheta_{*_q})^\perp$ to $range\vartheta_{*_q}$ in the tangent space $T_{\vartheta(q)}N_2$ of N_2 at $\vartheta(q) \in N_2$. Since $rank\vartheta < \min\{\dim(N_1), \dim(N_2)\}$, we have $(\ker\vartheta_{*_q})^\perp \neq \{0\}$. Therefore the tangent space $T_{\vartheta(q)}N_2$ of N_2 at $\vartheta(q) \in N_2$ has the decomposition $T_{\vartheta(q)}N_2 = (range\vartheta_{*_q}) \oplus (range\vartheta_{*_q})^\perp$. Then ϑ is called Riemannian map at $q \in N_1$ if the horizontal restriction $\vartheta^h_{*_q} : (\ker\vartheta_{*_q})^\perp \to (range\vartheta_{*_q})$ is a linear isometry between the spaces $((\ker\vartheta_{*_q})^\perp, g_{N_{1_q}}|_{(\ker\vartheta_{*_q})^\perp})$ and $(range\vartheta_{*_q}, g_{N_2\vartheta(q)}|_{range\vartheta_{*_q}})$.

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In other words, ϑ satisfies

$$g_{N_2}(\vartheta_* A_1, \vartheta_* A_2) = g_{N_1}(A_1, A_2), \tag{1}$$

for all A_1, A_2 vector field tangent to $\Gamma(\ker \vartheta_{*_q})^\perp$.

Different features of Riemannian maps have been investigated extensively by many authors such as [1, 7, 9, 10, 14, 18, 19, 11, 6, 20]. Detailed development in the theory of Riemannian map can be found in [15].

Conformal Riemannian maps as a generalization of Riemannian maps and the harmonicity of such maps have been introduced in [16, 17]. Conformal anti-invariant Riemannian maps have been studied in [2, 12]. In this article, we expand this concept to almost contact metric manifolds as a generalization of semi-invariant Riemannian maps and totally real submanifolds.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 includes conformal semi-invariant Riemannian maps from Riemannian manifolds to cosymplectic manifolds and provides this notion by non-trivial example. Then, we get a decomposition theorem by using the existence of conformal semi-invariant Riemannian maps. Moreover, conformal semi-invariant Riemannian maps allow us to obtain new conditions for a map to be harmonic. We also investigated the total geodesicity of conformal semi-invariant maps.

2. **Preliminaries**

Let N be an odd-dimensional smooth manifold. Then, N has an almost contact structure [15] if there exist a tensor field P of type$-(1,1)$, a vector field ξ, and 1-form η on N such that

$$P^2 E_1 = -E_1 + \eta(E_1)\xi, P\xi = 0, \eta \circ P = 0, \eta(\xi) = 1. \tag{2}$$

If there exists a Riemannian metric g_N on an almost contact manifold N satisfying:

$$\begin{aligned} g_N(PE_1, PE_2) &= g_N(E_1, E_2) - \eta(E_1)\eta(E_2), \qquad (3) \\ g_N(E_1, PE_2) &= -g_N(PE_1, E_2), \end{aligned}$$

$$\eta(E_1) = g_N(E_1, \xi), \tag{4}$$

where E_1, E_2 are any vector fields on N, then N is called an almost contact metric manifold [4] with an almost contact structure (P, ξ, η, g_N) and is symbolized by (N, P, ξ, η, g_N).

A manifold N with the structure (P, ξ, η, g_N) is said to be cosymplectic [15] if

$$(\nabla^N_{E_1} P)E_2 = 0, \tag{5}$$

for any vector fields E_1, E_2 on N, where ∇ stands for the Riemannian connection of the metric g_N on N. For a cosymplectic manifold, we get

$$\nabla^N_{E_1}\xi = 0, \tag{6}$$

for any vector field E_1 on N.

ϑ_* can be considered as a part of bundle $\hom(TN_1, \vartheta^{-1}TN_2) \to N_1$, where $\vartheta^{-1}TN_2$ is the pullback bundle. The bundle has a connection $\triangledown$ induced from the pullback connection $\overset{N_2}{\triangledown^{\vartheta}}$ and the Levi-Civita connection $\triangledown^{N_1}$. Then the second fundamental form $(\triangledown\vartheta_*)(A_1, A_2)$ of ϑ is given by [8]

$$(\triangledown\vartheta_*)(A_1, A_2) = \overset{N_2}{\triangledown}{}^{\vartheta}_{A_1}\vartheta_*A_2 - \vartheta_*(\triangledown^{N_1}_{A_1}A_2), \tag{7}$$

for all $A_1, A_2 \in \Gamma(TN_1)$, where $\overset{N_2}{\triangledown}{}^{\vartheta}_{A_1}\vartheta_*A_2 \circ \vartheta = \overset{N_2}{\triangledown}{}^{\vartheta}_{\vartheta_*A_1}\vartheta_*A_2$. It is known that $(\triangledown\vartheta_*)(A_1, A_2)$ is symmetric and $(\triangledown\vartheta_*)(A_1, A_2)$ has no component in $range\vartheta_*$, for all $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ [15]. It means that, we get

$$(\triangledown\vartheta_*)(A_1, A_2) \in \Gamma(range\vartheta_*)^\perp.$$

The tension field of ϑ is defined to be the trace of the second fundamental form of ϑ, i.e. $\tau(\vartheta) = trace(\triangledown\vartheta_*) = \sum_{i=1}^{m} (\triangledown\vartheta_*)(e_i, e_i)$, where $m = \dim(N_1)$ and $\{e_1, e_2, ..., e_m\}$ is the orthonormal frame on N_1. Moreover, a map $\vartheta : (N_1, g_{N_1}) \to (N_2, g_{N2})$ is harmonic if and only if the tension field of ϑ vanishes at each point $q \in N_1$.

For any section B_1 of $(range\vartheta_*)^\perp$ and vector field A_1 on N_1, we get $\triangledown^{\vartheta\perp}_{A_1}B_1$, which is the orthogonal projection of $\triangledown^{N_2}_{A_1}B_1$ on $(range\vartheta_*)^\perp$, where $\triangledown^{\vartheta\perp}$ is linear connection on $(range\vartheta_*)^\perp$ such that $\triangledown^{\vartheta\perp}g_{N_2} = 0$. For a Riemannian map ϑ we describe S_{B_1} as ([15], p. 188)

$$\triangledown^{N_2}_{\vartheta_*A_1}B_1 = -S_{B_1}\vartheta_*A_1 + \triangledown^{\vartheta\perp}_{A_1}B_1, \tag{8}$$

where $S_{B_1}\vartheta_*A_1$ is the tangential component of $\triangledown^{N_2}_{\vartheta_*A_1}B_1$ and $\triangledown^{N_2}$is Levi-Civita connection on N_2. Therefore, we have

$$\triangledown^{N_2}_{\vartheta_*A_1}B_1(q) \in T_{\vartheta(q)}N_2, S_{B_1}\vartheta_*A_1 \in \vartheta_{*q}(T_qN_1)$$

and

$$\triangledown^{\vartheta\perp}_{A_1}B_1 \in (\vartheta_{*q}(T_qN_1))^\perp$$

at $q \in N_1$. We know that $S_{B_1}\vartheta_*A_1$ is bilinear in B_1, and ϑA_1 at q depends only on B_{1q} and $\vartheta_{*q}A_{1q}$. From here, using (7) and (8) we have

$$g_{N_2}(S_{B_1}\vartheta_*A_1, \vartheta_*A_2) = g_{N_2}(B_1, (\triangledown\vartheta_*)(A_1, A_2)), \tag{9}$$

where S_{B_1} is self adjoint operator for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$.

For all $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ we define

$$\triangledown^{N_2}_{B_1}B_2 = R(\triangledown^{N_2}_{B_1}B_2) + \triangledown^{\vartheta\perp}_{B_1}B_2,$$

where $R(\triangledown^{N_2}_{B_1}B_2)$) and $\triangledown^{\vartheta\perp}_{B_1}B_2$ denote $range\vartheta_*$ and $(range\vartheta_*)^\perp$ part of $\triangledown^{N_2}_{B_1}B_2$, respectively. Therefore $(range\vartheta_*)^\perp$ is totally geodesic if and only if

$$\triangledown^{N_2}_{B_1}B_2 = \triangledown^{\vartheta\perp}_{B_1}B_2. \tag{10}$$

3. **Conformal semi-invariant Riemannian maps to cosymplectic manifolds**

Definition 3.1. [17] *Let* $\vartheta : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ *be a conformal Riemannian map (cRm). Then,* ϑ *is a horizontally homothetic map if* $H(grad\lambda) = 0$.

Definition 3.2. [16] *Let* $\vartheta : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ *be a smooth map between Riemannian manifolds. Then,* ϑ *is a cRm at* $q \in N_1$ *if* $0 < rank\vartheta_{*q} \leq \min\{\dim(N_1), \dim(N_2)\}$ *and* ϑ_{*q} *maps the horizontal space* $H(q) = (\ker\vartheta_{*q})^{\perp}$ *conformally into* $range\vartheta_{*q}$*, it means that there exists a number* $\lambda^2(q) \neq 0$ *such that*

$$g_{N_2}(\vartheta_{*q}A_1, \vartheta_{*q}A_2) = \lambda^2(q) g_{N_1}(A_1, A_2),$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^{\perp}$*. Moreover, if* ϑ *is cRm at any* $q \in N_1$*, then* ϑ *is called cRm.*

Lastly, the second fundamental form of ϑ is given by [16]

$$\begin{aligned} (\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} &= A_1(\ln\lambda)\vartheta_*A_2 + A_2(\ln\lambda)\vartheta_*A_1 \\ &\quad - g_{N_1}(A_1, A_2)\vartheta_*(grad\ln\lambda). \end{aligned} \tag{11}$$

Therefore, if we state the $(range\vartheta_*)^{\perp}$ component of $(\nabla\vartheta_*)(A_1, A_2)$ by $(\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^{\perp}}$, then we can write

$$\begin{aligned} (\nabla\vartheta_*)(A_1, A_2) &= (\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} \\ &\quad + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^{\perp}}, \end{aligned} \tag{12}$$

for $A_1, A_2 \in \Gamma(\ker\vartheta_*)^{\perp}$. Therefore we get

$$\begin{aligned} (\nabla\vartheta_*)(A_1, A_2) &= A_1(\ln\lambda)\vartheta_*A_2 + A_2(\ln\lambda)\vartheta_*A_1 \\ &\quad - g_{N_1}(A_1, A_2)\vartheta_*(grad\ln\lambda) \\ &\quad + (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^{\perp}}. \end{aligned} \tag{13}$$

Definition 3.3. *Let* ϑ *be a cRm from a Riemannian manifold* (N_1, g_{N_1}) *to an almost contact metric manifold* $(N_2, P, \xi, \eta, g_{N_2})$*. Then* ϑ *is a conformal semi-invariant Riemannian map (csiRm) at* $q \in N_1$ *if there is a subbundle* $D_1 \subseteq (range\vartheta_*)$ *such that*

$$range\vartheta_{*q} = D_1 \oplus D_2, P(D_1) = D_1, P(D_2) \subseteq (range\vartheta_{*q})^{\perp},$$

where D_2 *is orthogonal complementary to* D_1 *in* $range\vartheta_*$*. If* ϑ *is a csiRm for any* $q \in N_1$*, then* ϑ *is called a csiRm.*

For $\vartheta_*A_1 \in \Gamma(range\vartheta_*)$, then we write

$$P\vartheta_*A_1 = \phi\vartheta_*A_1 + \omega\vartheta_*A_1, \tag{14}$$

where $\phi\vartheta_*A_1 \in \Gamma(D_1)$ and $\omega\vartheta_*A_1 \in \Gamma(PD_2)$. Also, for $\vartheta_*A_1 \in \Gamma(D_1)$ and $\vartheta_*A_2 \in \Gamma(D_2)$, we have $g_{N_2}(\vartheta_*A_1, \vartheta_*A_2) = 0$. Thus we have two orthogonal distributions $\bar{D}_1$ and $\bar{D}_2$ such that

$$(\ker\vartheta_{*_q})^{\perp} = \bar{D}_1 \oplus \bar{D}_2.$$

On the other hand, for $B_1 \in \Gamma((range\vartheta_*)^{\perp})$, then we have

$$PB_1 = \beta_1 B_1 + \alpha_1 B_1, \tag{15}$$

where $\beta_1 B_1 \in \Gamma(D_1)$ and $\alpha_1 B_1 \in \Gamma(\eta)$. Here η is the complementary orthogonal distribution to $\omega(D_2)$ in $(range\vartheta_*)^{\perp}$. It is easy to see that η is invariant with respect to P.

Example 3.1. *Let N_1 be an Euclidean space given by*

$$N_1 = \left\{(u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5 : u_1 \neq 0, u_2 \neq 0, u_5 \neq 0\right\}.$$

We describe the Riemannian metric g_{N_1} on N_1 given by

$$g_{N_1} = du_1^2 + du_2^2 + du_3^2 + du_4^2 + du_5^2.$$

Let $N_2 = \{(v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5\}$ be a Euclidean space with metric g_{N_2} on N_2 given by

$$g_{N_2} = e^{2u_1}dv_1^2 + e^{2u_1}dv_2^2 + e^{2u_1}dv_3^2 + dv_4^2 + dv_5^2.$$

An almost contact structure (P, ξ, η) on (N_2, g_{N_2}) can be choosen as

$$\begin{aligned} P(\frac{\partial}{\partial v_1}) &= \frac{\partial}{\partial v_2}, P(\frac{\partial}{\partial v_2}) = -\frac{\partial}{\partial v_1}, \\ P(\frac{\partial}{\partial v_3}) &= \frac{\partial}{\partial v_4}, P(\frac{\partial}{\partial v_4}) = -\frac{\partial}{\partial v_3}, \\ \eta &= dv_5, \xi = \frac{\partial}{\partial v_5}, P(\xi) = 0. \end{aligned}$$

Then a basis of T_qN_1 is

$$\left\{e_i = e^{u_1}\frac{\partial}{\partial u_i} \text{ for } 1 \leq i \leq 5\right\},$$

and a P-basis on $T_{\vartheta(q)}N_2$ is

$$\left\{e_j^* = \frac{\partial}{\partial v_j} \text{ for } 1 \leq j \leq 4, e_4^* = e^{u_1}\frac{\partial}{\partial v_4}, \xi = e_5^* = \frac{\partial}{\partial v_5}\right\},$$

for all $q \in N_1$. Now, we define a map $\vartheta : (N_1, g_{N_1}) \to (N_2, g_{N_2}, P)$ by

$$\vartheta(u_1, u_2, u_3, u_4, u_5) = (u_1, u_2, u_5, 0, 0).$$

Then, we have

$$ker\vartheta_* = Span\{U_1 = e_3, U_2 = e_4\},$$

$$(ker\vartheta_*)^{\perp} = Span\{A_1 = e_1, A_2 = e_2, A_3 = e_5\}.$$

Hence it is easy to see that

$$\vartheta_* A_1 = e^{u_1}e_1^*, \vartheta_* A_2 = e^{u_1}e_2^*, \vartheta_* A_3 = e^{u_1}e_3^*$$

and

$$g_{N_2}(\vartheta_*(A_{1i}), \vartheta_*(A_{1j})) = e^{2u_1}g_{N_1}(A_{1i}, A_{1j})$$

for $i, j = 1, 2, 3$. *Thus* ϑ *is a cRm with* $\lambda = e^{2u_1}$ *and we get*

$$\begin{aligned} range\vartheta_* &= Span\left\{e^{u_1}e_1^*, e^{u_1}e_2^*, e^{u_1}e_3^*\right\}, \\ (range\vartheta_*)^\perp &= Span\left\{e_4^*, \xi\right\}, \end{aligned}$$

$$D_1 = Span\left\{e^{u_1}e_1^*, e^{u_1}e_2^*\right\}, D_2 = Span\left\{e^{u_1}e_3^*\right\}.$$

Moreover it is easy to see that

$$P\vartheta_*A_1 = e^{u_1}e_2^*, P\vartheta_*A_2 = -e^{u_1}e_1^*, P\vartheta_*A_3 = e^{u_1}e_4^*.$$

Thus ϑ *is a csiRm.*

Remark 3.1. *Throughout this article* $\xi \in (range\vartheta_*)^\perp$ *will be taken as the Reeb vector field.*

We obtain the following theorem for the geometry of the leaves of D_1.

Theorem 3.1. *Let* ϑ *be a csiRm from a Riemannian manifold* (N_1, g_{N_1}) *to a cosymplectic manifold* $(N_2, P, \xi, \eta, g_{N_2})$. *Then* D_1 *describes a totally geodesic foliation on* N_2 *if and only if*

i. $g_{N_2}(\beta_1 B_1 (\ln\lambda)\, \vartheta_*A_1 + \vartheta_*(\nabla^{N_1}_{A_1}A_3), P\vartheta_*A_2) = g_{N_2}(S_{\alpha_1 B_1}\vartheta_*A_1, P\vartheta_*A_2) + \eta(\nabla^{N_2}_{A_1}\vartheta_*A_2)\eta(B_1)$,

ii. $\phi S_{P\vartheta_*B_2}\vartheta_*A_1$ *has no components in* $\Gamma(D_1)$,

for any $A_1, A_2, A_3, B_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that*

$$\vartheta_*A_1, \vartheta_*A_2 \in \Gamma(D_1), \vartheta_*B_2 \in \Gamma(D_2)$$

and $B_1 \in \Gamma(range\vartheta_*)^\perp$ *such that* $\vartheta_*A_3 = \beta_1 B_1$.

Proof. For $\vartheta_*A_1, \vartheta_*A_2 \in \Gamma(D_1), B_1 \in \Gamma(range\vartheta_*)^\perp$ and $\vartheta_*B_2 \in \Gamma(D_2)$, since ϑ is a cRm, using (2), (3) and then from (4), (5), (6), (7) and (15), we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_*A_2, B_1) \\ = & -g_{N_2}((\nabla^{N_2}\vartheta_*)(A_1, A_3) + \vartheta_*(\nabla^{N_1}_{A_1}A_3), P\vartheta_*A_2) \\ & + g_{N_2}(S_{\alpha_1 B_1}\vartheta_*A_1, P\vartheta_*A_2) - g_{N_2}(\nabla^{\vartheta\perp}_{A_1}\vartheta_*A_1, P\vartheta_*A_2) \\ & + \eta(\nabla^{N_2}_{A_1}\vartheta_*A_2)\eta(B_1), \end{aligned}$$

where $\beta_1 B_1 = \vartheta_*A_3 \in \Gamma(D_2)$ for $A_3 \in \Gamma(ker\vartheta_*)^\perp$. From (11) and using (12) in the above equation and since $grad \ln\lambda \in (range\vartheta_*)^\perp$, using (3) and $\vartheta_*A_3 = \beta_1 B_1$ we get

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_*A_2, B_1) \\ = & -g_{N_2}(\beta_1 B_1 (\ln\lambda)\, \vartheta_*A_1 + \vartheta_*(\nabla^{N_1}_{A_1}A_3), P\vartheta_*A_2) \\ & + g_{N_2}(S_{\alpha_1 B_1}\vartheta_*A_1, P\vartheta_*A_2) + \eta(\nabla^{N_2}_{A_1}\vartheta_*A_2)\eta(B_1). \end{aligned}$$

This implies the proof of i.

On the other hand, by using (3) and from (4), (5), (6), (14) and (8) we get

$$g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_*A_2, \vartheta_*B_2) = g_{N_2}(-\phi S_{P\vartheta_*B_2}\vartheta_*A_1, \vartheta_*A_2).$$

This implies the proof of ii. ∎

We obtain the following theorem for the geometry of the leaves of D_2.

Theorem 3.2. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$. Then D_2 describes a totally geodesic foliation on N_2 if and only if*

i. $\eta(\nabla^{N_2}_{B_3}\vartheta_* B_4)\eta(B_2) = g_{N_2}((\nabla^{N_2}\vartheta_*)(B_3, A_4)^{(range\vartheta_*)^\perp} - \nabla^{\varphi\perp}_{B_3}\alpha_1 B_2, P\vartheta_* B_4)$,

ii. $\beta_1(\nabla^{N_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}$ *has no components in* $\Gamma(D_2)$

for any $A_3, A_4, B_3, B_4 \in \Gamma(ker\vartheta_*)^\perp$ *such that*

$$\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$$

and $\vartheta_* A_4 = \beta_1 B_2$.

Proof. For $\vartheta_* A_3, \vartheta_* B_3, \vartheta_* B_4 \in \Gamma(D_2), B_2 \in \Gamma(range\vartheta_*)^\perp$, using (3), (5), (14), (15) and since ϑ is a cRm, then from (7), (8) and $\vartheta_* A_4 = \beta_1 B_2$ we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{B_3}\vartheta_* B_4, B_2) \\ = \ & -g_{N_2}((\nabla^{N_2}\vartheta_*)(B_3, A_4) + \vartheta_*(\nabla^{N_1}_{B_3} A_4) \\ & -S_{\alpha_1 B_2}\vartheta_* B_3 + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, P\vartheta_* B_4) \\ & +\eta(\nabla^{N_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on N_2, using (12) we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{B_3}\vartheta_* B_4, B_2) \\ = \ & g_{N_2}((\nabla^{N_2}\vartheta_*)(B_3, A_4)^{(range\vartheta_*)^\perp} \\ & + \nabla^{\vartheta\perp}_{B_3}\alpha_1 B_2, P\vartheta_* B_4) + \eta(\nabla^{N_2}_{B_3}\vartheta_* B_4)\eta(B_2). \end{aligned}$$

This implies the proof of i.

On the other hand, by the virtue of (3), (8), (12) and (15) we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{B_3}\vartheta_* A_3, \vartheta_* B_3) \\ = \ & g_{N_2}(\beta_1(\nabla^{N_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}, \vartheta_* B_3). \end{aligned}$$

Since D_2 defines a totally geodesic foliation on N_2 then we can say that $\beta_1(\nabla^{N_2}\vartheta_*)(B_3, A_3)^{(range\vartheta_*)^\perp}$ has no components in $\Gamma(D_2)$. This completes the proof of ii. □

Theorem 3.3. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$. If $(range\vartheta_*)$ defines a totally geodesic foliation on N_2 and ϑ is a horizontally homothetic cRm then we have*

$$\begin{aligned} & g_{N_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) \\ & -g_{N_2}(\vartheta_*(\nabla^{N_1}_{A_1} A_3), \phi\vartheta_* A_2) \\ = \ & g_{N_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \beta_1 B_1) - g_{N_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ & -\eta(\nabla^{N_2}_{A_1}\vartheta_* A_2)\eta(B_1) \end{aligned} \tag{16}$$

for any $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that*

$$\vartheta_* A_1, \vartheta_* A_2 \in \Gamma(range\vartheta_*), B_1 \in \Gamma(range\vartheta_*)^\perp$$

and $\vartheta_* A_3 = \beta_1 B_1$.

Proof. For $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$, using (3), (5) and from (15), (14) and $\vartheta_* A_3 = \beta_1 B_1$we get

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_* A_2, B_1) \\ = \; & -g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_* A_3, \phi\vartheta_* A_2) + g_{N_2}(\nabla^{N_2}_{A_1}\omega\vartheta_* A_2, \vartheta_* A_3) \\ & -g_{N_2}(\nabla^{N_2}_{A_1}\alpha_1 B_1, \phi\vartheta_* A_2) + g_{N_2}(\nabla^{N_2}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) \\ & +\eta(\nabla^{N_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned}$$

Since ϑ is a cRm, using $\vartheta_* A_3 = \beta_1 B_1$, (7), (13), (8) and if we take $A_1(\ln\lambda) = g_{N_1}(A_1, Hgrad\ln\lambda)$ and $A_3(\ln\lambda) = g_{N_1}(A_3, Hgrad\ln\lambda)$, then we obtain

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{A_1}\vartheta_* A_2, B_1) \\ = \; & -g_{N_1}(A_1, Hgrad\ln\lambda)g_{N_2}(\vartheta_* A_3, \phi\vartheta_* A_2) \\ & -g_{N_1}(A_3, Hgrad\ln\lambda)g_{N_2}(\vartheta_* A_1, \phi\vartheta_* A_2) \\ & -g_{N_1}(A_1, A_3)g_{N_2}(\vartheta_*(grad\ln\lambda), \phi\vartheta_* A_2) \\ & -g_{N_2}(\vartheta_*(\nabla^{N_1}_{A_1}A_3), \phi\vartheta_* A_2) \\ & -g_{N_2}(S_{\omega\vartheta_* A_2}\vartheta_* A_1, \beta_1 B_1) + g_{N_2}(S_{\alpha_1 B_1}\vartheta_* A_1, \phi\vartheta_* A_2) \\ & +g_{N_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_2, \alpha_1 B_1) + \eta(\nabla^{N_2}_{A_1}\vartheta_* A_2)\eta(B_1). \end{aligned} \tag{17}$$

Since $(range\vartheta_*)$ describes a totally geodesic foliation on N_2 and ϑ is a horizontally homothetic cRm, then from (17) we obtain (16). □

Theorem 3.4. *Let* ϑ *be a csiRm from a Riemannian manifold* (N_1, g_{N_1}) *to a cosymplectic manifold* $(N_2, P, \xi, \eta, g_{N_2})$*. Then* $(range\vartheta_*)^\perp$ *defines a totally geodesic foliation on* N_2 *if and only if*

$$\begin{aligned} & g_{N_2}(\alpha_1 B_1, (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) \\ = \; & g_{N_2}(B_2, [B_1, \vartheta_* A_1] + \nabla^{\vartheta\perp}_{\vartheta_* A_1}P\beta_1 B_1 \\ & +\alpha_1 \nabla^{\vartheta\perp}_{\vartheta_* A_1} P\alpha_1 B_1), \end{aligned}$$

for any $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ *and* $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$ *such that* $\vartheta_* A_2 = \beta_1 B_2$.

Proof. For any $B_1, B_2 \in \Gamma(range\vartheta_*)^\perp$ and $A_1, A_2 \in \Gamma(ker\vartheta_*)^\perp$, using (3), (5) and since N_2 is a cosymplectic manifold,

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{B_1}B_2, \vartheta_* A_1) \\ = \; & -g_{N_2}(B_2, [B_1, \vartheta_* A_1]) - g_{N_2}(PB_2, \nabla^{N_2}_{\vartheta_* A_1}PB_1) \\ & +\eta(\nabla^{N_2}_{B_1}B_2)\underbrace{\eta(\vartheta_* A_1)}_{0}. \end{aligned}$$

Then using (7), (8), (14), (15) and from (12), (10) and since $(range\vartheta_*)^\perp$ defines a totally geodesic foliation we have

$$\begin{aligned} & g_{N_2}(\alpha_1 B_1, (\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) \\ = \;& g_{N_2}(B_2, [B_1, \vartheta_* A_1] + \nabla^{\vartheta\perp}_{\vartheta_* A_1} P\beta_1 B_1 \\ & +\alpha_1 \nabla^{\vartheta\perp}_{\vartheta_* A_1} P\alpha_1 B_1). \end{aligned}$$

This completes the proof. □

Remark 3.2. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$. From the second fundamental form, one can easly see that $ker\vartheta_*$ and $(ker\vartheta_*)^\perp$ define a totally geodesic foliation on N_1.*

From the above fact we can state following theorem.

Theorem 3.5. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$. Then ϑ is totally geodesic foliation if and only if*

$$\begin{aligned} & \phi((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{N_1}_{A_1} A_2) - S_{\omega\vartheta_* A_3}\vartheta_* A_1) \\ = \;& -\beta_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_3) - \vartheta_*(\nabla^{N_1}_{A_1} A_3), \end{aligned} \tag{18}$$

$$\begin{aligned} & \omega((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*} - \vartheta_*(\nabla^{N_1}_{A_1} A_2) - S_{\omega\vartheta_* A_3}\vartheta_* A_1) \\ = \;& -\alpha_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_3) + \eta(\nabla^{N_2}_{A_1}\vartheta_* A_3)\xi, \end{aligned} \tag{19}$$

for any $A_1, A_2, A_3 \in \Gamma(ker\vartheta_*)^\perp$ such that $\vartheta_* A_2 = \phi\vartheta_* A_3$.

Proof. For $A_1, A_3 \in \Gamma(ker\vartheta_*)^\perp$, using (2), (7), (14) and from (8) and (12) we have

$$\begin{aligned} & (\nabla^{N_2}\vartheta_*)(A_1, A_3) \\ = \;& -P((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*}) \\ & -P((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp}) \\ & -P(\vartheta_*(\nabla^{N_1}_{A_1} A_2)) + P(S_{\omega\vartheta_* A_3}\vartheta_* A_1) \\ & -P(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_3) \\ & -\vartheta_*(\nabla^{N_1}_{A_1} A_3) + \eta(\nabla^{N_2}_{A_1}\vartheta_* A_3)\xi. \end{aligned}$$

Since ϑ is a cRm, from (14), (15) and taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$\begin{aligned} & \phi((\nabla\vartheta_*)(A_1, A_3)^{range\vartheta_*} \\ = \;& -\phi((\nabla\vartheta_*)(A_1, A_2)^{range\vartheta_*}) + \vartheta_*(\nabla^{N_1}_{A_1} A_2) \\ & -S_{\omega\vartheta_* A_3}\vartheta_* A_1) - \beta_1((\nabla\vartheta_*)(A_1, A_2)^{(range\vartheta_*)^\perp} \\ & + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_3) - \vartheta_*(\nabla^{N_1}_{A_1} A_3) \end{aligned}$$

$$\begin{aligned}&(\bigtriangledown\vartheta_*)(A_1,A_3)^{(range\vartheta_*)^\perp}\\ =\;& -\omega((\bigtriangledown\vartheta_*)(A_1,A_2)^{range\vartheta_*}+\vartheta_*(\bigtriangledown^{N_1}_{A_1}A_2)\\ &-S_{\omega\vartheta_*A_3}\vartheta_*A_1)-\alpha_1((\bigtriangledown\vartheta_*)(A_1,A_2)^{(range\vartheta_*)^\perp}\\ &+\bigtriangledown^{\vartheta\perp}_{A_1}\omega\vartheta_*A_3)+\eta(\bigtriangledown^{N_2}_{A_1}\vartheta_*A_3)\xi.\end{aligned}$$

Thus $(\bigtriangledown\vartheta_*)(A_1,A_3)=0$ if and only if (18) and (19) are satisfied. This completes the proof. □

Proposition 3.1. *Let ϑ be a csiRm from a Riemannian manifold (N_1,g_{N_1}) to a cosymplectic manifold (N_2,P,ξ,η,g_{N_2}) such that $\dim(range\vartheta_*)>1$. Then the following statements are true.*

i. $\bar{D}_1$ defines a totally geodesic foliation if and only if $(\bigtriangledown\vartheta_*)(A_1,U_1)$ has no component in D_1 such that

$$S_{P\vartheta_*A_1'}(.)g_{N_2}(\vartheta_*A_1,P\vartheta_*A_2)=\eta(\bigtriangledown^{N_1}_{A_1}A_2)\eta(A_1')$$

for $A_1,A_2\in\Gamma(\bar{D}_1),U_1\in\Gamma(ker\vartheta_*)$ and $A_1'\in\Gamma(\bar{D}_2)$.

ii. $\bar{D}_2$ defines a totally geodesic foliation if and only if $(\bigtriangledown\vartheta_*)(A_2,U_1)$ has no component in D_2 such that

$$S_{P\vartheta_*A_4}(.)g_{N_2}(\vartheta_*A_3,P\vartheta_*A_3')=\eta(\bigtriangledown^{N_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(A_3')$$

for $A_2,A_3,A_4\in\Gamma(\bar{D}_2),U_1\in\Gamma(ker\vartheta_*)$ and $A_3'\in\Gamma(\bar{D}_2)$.

Proof. We know that $\bar{D}_1$ defines totally geodesic foliation if and only if

$$g_{N_1}(\bigtriangledown^{N_1}_{A_1}A_2,U_1)=0$$

and

$$g_{N_1}(\bigtriangledown^{N_1}_{A_1}A_2,A_1')=0$$

for $A_1,A_2\in\Gamma(\bar{D}_1),U_1\in\Gamma(ker\vartheta_*)$ and $A_1'\in\Gamma(\bar{D}_2)$. Now, since ϑ is Riemannian map, using (1),(7) and (8) we have

$$g_{N_1}(\bigtriangledown^{N_1}_{A_1}A_2,U_1)=-g_{N_2}((\bigtriangledown\vartheta_*)(A_1,U_1),\vartheta_*A_2),$$

and similarly

$$g_{N_1}(\bigtriangledown^{N_1}_{A_1}A_2,A_1')=-g_{N_2}(\bigtriangledown^{N_2}_{\vartheta_*A_1}\vartheta_*A_1',\vartheta_*A_2).$$

Since N_2 is cosymplectic manifold, using (3) and then (8), we have

$$\begin{aligned}&g_{N_1}(\bigtriangledown^{N_1}_{A_1}A_2,A_1')\\ =\;& -S_{P\vartheta_*A_1'}(.)g_{N_2}(\vartheta_*A_1,P\vartheta_*A_2)+\eta(\bigtriangledown^{N_1}_{A_1}A_2)\eta(A_1').\end{aligned}$$

This completes the proof of i.

On the other hand, we know that $\bar{D}_2$ defines a totally geodesic foliation if and only if

$$g_{N_1}(\bigtriangledown^{N_1}_{A_3}A_4,U_1)=0$$

and

$$g_{N_1}(\bigtriangledown^{N_1}_{A_3}A_4,A_3')=0$$

for $A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_1)$. Now, since ϑ is Riemannian map, using (1) and (7) we have

$$g_{N_1}(\nabla^{N_1}_{A_3}A_4, U_1) = -g_{N_2}((\nabla\vartheta_*)(A_3, U_1), \vartheta_*A_4),$$

and similarly

$$g_{N_1}(\nabla^{N_1}_{A_3}A_4, A_3^{'}) = -g_{N_2}(\nabla^{N_2}_{\vartheta_*A_3}\vartheta_*A_4, \vartheta_*A_3^{'}).$$

Since N_2 is cosymplectic manifold, using (3),(5) and then (8), we have

$$\begin{aligned} & g_{N_1}(\nabla^{N_1}_{A_3}A_4, A_3^{'}) \\ = \; & -S_{P\vartheta_*A_4}(.)g_{N_2}(\vartheta_*A_3, P\vartheta_*A_3^{'}) \\ & +\eta(\nabla^{N_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(\vartheta_*A_3^{'}). \end{aligned}$$

This completes the proof of ii. □

Definition 3.4. [13] *Let (N_1, g_{N_1}) be a Riemannian manifold and assume that the canonical foliations K_1 and K_2 such that $K_1 \cap K_2 = \{0\}$ everywhere. Then (N_1, g_{N_1}) is a locally product manifold if and only if K_1 and K_2 are totally geodesic foliations.*

Theorem 3.6. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$ such that $\dim(range\vartheta_*) > 1$. Then $(ker\vartheta_*)^{\perp}$ is a locally product manifold of $\bar{D}_1$ and $\bar{D}_2$ if and only if*

i. $(\nabla\vartheta_*)(A_1, U_1)$ has no component in D_1 such that

$$S_{P\vartheta_*A_1^{'}}(.)g_{N_2}(\vartheta_*A_1, P\vartheta_*A_2) = \eta(\nabla^{N_1}_{A_1}A_2)\eta(A_1^{'})$$

for $A_1, A_2 \in \Gamma(\bar{D}_1), U_1 \in \Gamma(ker\vartheta_*)$ and $A_1^{'} \in \Gamma(\bar{D}_2)$,

ii. $(\nabla\vartheta_*)(A_3, U_1)$ has no component in D_2 such that

$$S_{P\vartheta_*A_4}(.)g_{N_2}(\vartheta_*A_3, P\vartheta_*A_3^{'}) = \eta(\nabla^{N_1}_{\vartheta_*A_3}\vartheta_*A_4)\eta(A_3^{'})$$

for $A_2, A_3, A_4 \in \Gamma(\bar{D}_2), U_1 \in \Gamma(ker\vartheta_*)$ and $A_3^{'} \in \Gamma(\bar{D}_2)$.

Proof. The proof is clear by Proposition (3.1) and Definition (3.4). □

Theorem 3.7. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$ such that $\dim(range\vartheta_*) > 1$. Then the base manifold is locally product manifold $N_{2_{range\vartheta_*}} \times N_{2_{(range\vartheta_*)^{\perp}}}$ if and only if*

$$\begin{aligned} 0 = \; & g_{N_2}(\vartheta_*(\nabla^{N_2}_{A_1}\phi\vartheta_*A_1), \beta_1B_1) + g_{N_2}(\vartheta_*(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_*A_1, \beta_2B_1) \\ & +g_{N_2}(S_{\beta_2B_1}\vartheta_*A_1, \phi\vartheta_*A_1) + \eta(\nabla^{N_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1) \end{aligned}$$

for $A_1 \in \Gamma(\bar{D}_1)$ and $B_1 \in \Gamma(range\vartheta_*)^{\perp}$.

Proof. Since N_2 is cosymplectic manifold, using (3) we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{\vartheta_*A_1}\vartheta_*A_1, B_1) \\ = \; & g_{N_2}(\nabla^{N_2}_{\vartheta_*A_1}P\vartheta_*A_1, PB_1) + \eta(\nabla^{N_2}_{\vartheta_*A_1}\vartheta_*A_1)\eta(B_1), \end{aligned}$$

for $\vartheta_* A_1 \in \Gamma(range\vartheta_*)$ and $B_1 \in \Gamma(range\vartheta_*)^\perp$. Using (14), (15), (8) and then (7) we have

$$\begin{aligned} & g_{N_2}(\nabla^{N_2}_{\vartheta_* A_1}\vartheta_* A_1, B_1) \\ = \; & g_{N_2}((\nabla\vartheta_*)(A_1, {}^*\vartheta_*(\phi\vartheta_* A_1)), \beta_1 B_1) \\ & + g_{N_2}(S_{\beta_2 B_1}\vartheta_* A_1, \phi\vartheta_* A_1) + g_{N_2}(\nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1, \beta_2 B_1) \\ & + \eta(\nabla^{N_2}_{\vartheta_* A_1}\vartheta_* A_1)\eta(B_1). \end{aligned}$$

From Definition (3.4), the proof is completed. □

Now, we will examine the harmonicity of csiRm from a Riemannian manifold (N_1, g_{N_1}) to cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$ in the following theorem.

Theorem 3.8. *Let ϑ be a csiRm from a Riemannian manifold (N_1, g_{N_1}) to a cosymplectic manifold $(N_2, P, \xi, \eta, g_{N_2})$. Then ϑ is harmonic if and only if the following conditions are satisfied*

i. The fibres are minimal,

ii.

$$\begin{aligned} 0 = \; & trace\phi S_{\omega\vartheta_* A_1} A_1 - \beta_1 \nabla^{\vartheta\perp}_{A_1} \omega\vartheta_* A_1 \\ & - \vartheta_*(\nabla^{N_1}_{A_1} A_1) - (\nabla^{N_2} P\phi\vartheta_* A_1)^{range\vartheta_*}, \end{aligned}$$

iii.

$$\begin{aligned} 0 = \; & trace\omega S_{\omega\vartheta_* A_1} A_1 - \alpha_1 \nabla^{\vartheta\perp}_{A_1} \omega\vartheta_* A_1 \\ & - (\nabla^{N_2}_{A_1} P\phi\vartheta_* A_1)^{(range\vartheta_*)^\perp} + \eta((\nabla\vartheta_*)(A_1, A_1)^{(range\vartheta_*)^\perp})\xi, \end{aligned}$$

for $A_1 \in (ker\vartheta_*)^\perp$.

Proof. For $U_1 \in ker\vartheta_*$, since $\vartheta_* U_1 = 0$, using (7) we get

$$(\nabla\vartheta_*)(U_1, U_1) = -\vartheta_*(\nabla^{N_1}_{U_1} U_1), \tag{20}$$

For $A_1 \in (ker\vartheta_*)^\perp$ using (3), (7), (15), (12) and (8) we have

$$\begin{aligned} & (\nabla\vartheta_*)(A_1, A_1) \\ = \; & -\nabla^{N_2}_{A_1} P\phi\vartheta_* A_1 - P(-S_{\omega\vartheta_* A_1}\vartheta_* A_1 + \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1) \\ & - \vartheta_*(\nabla^{N_1}_{A_1} A_1) + \eta(\nabla^{N_2}_{A_1}\vartheta_* A_1)\xi. \end{aligned}$$

Since ϑ is a cRm, from (14), (15) and taking $range\vartheta_*$ and $(range\vartheta_*)^\perp$ components we have

$$\begin{aligned} & (\nabla\vartheta_*)(A_1, A_1)^{range\vartheta} \\ = \; & \phi S_{\omega\vartheta_* A_1}\vartheta_* A_1 - \beta_1 \nabla^{\vartheta\perp}_{A_1} \omega\vartheta_* A_1 \\ & - \vartheta_*(\nabla^{N_1}_{A_1} A_1) - (\nabla^{N_2}_{A_1} P\phi\vartheta_* A_1)^{range\vartheta} \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
& (\nabla\vartheta_*)(A_1, A_1)^{(range\vartheta_*)^\perp} \\
= \;& \omega S_{\omega\vartheta_*\vartheta_* A_1}\vartheta_* A_1 - \alpha_1 \nabla^{\vartheta\perp}_{A_1}\omega\vartheta_* A_1 \\
& -(\nabla^{N_2}_{A_1} P\phi\vartheta_* A_1)^{(range\vartheta_*)^\perp} \\
& +\eta((\nabla\vartheta_*)(A_1, A_1)^{(range\vartheta_*)^\perp})\xi.
\end{aligned} \tag{22}
$$

Thus the proof is completed from (20), (21) and (22). □

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