U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 2, 2015 ISSN 1223-7027

POLYNOMIAL APROXIMATION ON UNBOUNDED SETS AND THE MULTIDIMENSIONAL MOMENT PROBLEM

Octav OLTEANU[1]

We solve a two dimensional moment problem on a space of absolutely integrable functions in a strip. To this end, we approximate nonnegative continuous compactly supported functions by sums of tensor products of positive polynomials on the corresponding intervals. Thus, one characterizes the solutions in terms of "computable" quadratic mappings. Next, we prove an application of an extension theorem for linear operators defined on a subspace that is "distanced" with respect to a bounded convex subset. Finally, one considers an application of the abstract Markov moment problem to a space of analytic functions. A common characteristic of all these results is the Hahan-Banach principle and its generalizations.

Keywords: approximation; extension of linear operators; constraints; unbounded sets; the moment problem

MSC 2010: 41A10, 47A57, 47A50

1. Introduction

Using polynomial decomposition or approximation in existence, uniqueness and construction of the solution of classical moment problems is a well - known and natural technique [1] - [22]. For the study of the uniqueness and the construction of the solutions, one uses L^2 - approximation [20], [17]. The approximation in L^1 norm is sometimes sufficient for the characterization of the existence of the solution. This is one of the ideas of the present work. In the real multidimensional moment problem, one of the main difficulties is the fact that the positive polynomials are not writable in terms of sums of squares. We solve this difficulty by means of appropriate polynomial L^1 approximation. For various results concerning decomposition of polynomials of several variables see [1], [3], [6] - [9]. For approximation results applied to the complex moment problem, similar to our results in the real case, see [11] and the reference there. Uniqueness of the solution is considered in [20]-[22], and in many other works of the References and of outside. Probabilistic approach of the uniqueness problem appears in [22]. The background is partially contained in [1], [23], [24]. For earlier basic related results, see the Introduction of [17] and the references therein.

[1] Prof., Matehmatics Department, University POLITEHNCA of Bucharest, Romania, email: olteanuoctav@yahoo.ie

The paper is organized as follows. In Section 2, we prove an application of polynomial approximation results to a Markov moment problem in an unbounded strip. Section 3 is devoted to an application of an earlier result concerning extension of linear operators defined on a subspace distanced with respect to a convex set. Section 4 contains an application of the solution of an abstract moment problem to a space of analytic functions. Section 5 concludes the paper.

2. Approximation and the Markov moment problem

The following lemma appeared firstly in [15]. A complete proof has been published in [16].

Theorem 1. *Let* $A \subset R^n$ *be an arbitrary closed subset and* ν *a positive regular determinate Borel measure on* A, *with finite moments of all orders. Then for any nonnegative continuous vanishing at infinity function* $\psi \in (C_0(A))_+$, *there exists a sequence* $(p_m)_m$ *of polynomials on* A, $p_m \geq \psi$, $p_m \to \psi$ *in* $L^1_\nu(A)$. *We have*

$$\lim \int_A p_m d\nu = \int_A \psi \, d\nu,$$

the cone P_+ *of positive polynomials is dense in* $\left(L^1_\nu(A)\right)_+$ *and* P *is dense in* $L^1_\nu(A)$.

Recall that a determinate measure is, by definition, uniquely determinate by its moments [22]. Here the novelty is that approximation holds by "dominating" polynomials. Let $\nu = \nu_1 \times \nu_2$ be the product of two measures on the strip $S = R \times [0,1]$, ν_1 verifying the conditions from Lemma 1 for $n = 1$, $A = R$, and ν_2 being a positive Borel regular measure on $[0,1]$. Let Y be an order complete Banach lattice with solid norm:

$$|y_1| \leq |y_2| \Rightarrow \|y_1\| \leq \|y_2\|.$$

Theorem 2. *Let*

$$X = L^1_\nu(S), \varphi_{j,k}(t_1, t_2) = t_1^j t_2^k, (j,k) \in N^2, (t_1, t_2) \in S.$$

Let $\left(y_{j,k}\right)_{(j,k)\in \mathrm{N}^2}$ *be a sequence in* Y *and* $G \in B_+(X,Y)$ *a linear positive bounded operator. The following statements are equivalent:*

(a) *there exists a unique operator* F *satisfying the conditions*

$$F \in B(X,Y), F\left(\varphi_{j,k}\right) = y_{j,k}, \forall (j,k) \in \mathrm{N}^2,$$
$$0 \leq F(\psi) \leq G(\psi), \forall \psi \in X_+, \|F\| \leq \|G\|;$$

(b) *for any finite subsets* $(m,n,p)\in N^3$ *and any* $\{\alpha_j\}_{j=0}^{m}$, *we have:*

$$0\leq \sum_{i,j=0}^{m}\alpha_i\alpha_j\left(\sum_{k=0}^{p}(-1)^k C_p^k y_{(i+j,n+k)}\right)\leq \sum_{i,j=0}^{m}\alpha_i\alpha_j\left(\sum_{k=0}^{p}(-1)^k C_p^k G\left(\varphi_{(i+j,n+k)}\right)\right)$$

Proof. Let ψ be a continuous nonnegative compactly supported function, and K its support contained in S. One chooses a rectangle R_2 containing

$$K_1\times[0,1],\ K_1:=pr_1(K).$$

One approximates the extension of ψ to R_2 vanishing outside its support by means of Luzin's theorem. Next, we approximate this continuous function on the rectangle by the corresponding Bernstein polynomials in two variables. Each term of such Bernstein polynomial is the tensor product of two positive polynomials on

$$K_j,\ j=1,2,\ K_2=[0,1].$$

Extend the polynomial in the first variable t_1 such that it vanishes outside $pr_1(R_2)$. Then we use Luzin's theorem, followed by Theorem 1, applied to $n=1,\ A=R.$ Hence one obtains a positive approximating polynomial on the whole real axes in the first variable, which is a sum of two squares. On the other hand, the polynomial in the second variable t_2 is a linear combination with positive coefficients of special polynomials $t_2^n(1-t_2)^p,\ (n,p)\in N^2$ [6]. Using these conclusions, one obtains approximation of ψ in $L_\nu^1(S)$ by sums of tensor products

$$p_1\otimes p_2,\ p_1(t_1)=q^2(t_1)\ \forall t_1\in R,\ p_2(t)=t_2^n(1-t_2)^p.$$

From the preceding arguments, we infer that the assertion (b) says that

$$0\leq F_0(\tilde{p}_1\otimes\tilde{p}_2)\leq G(\tilde{p}_1\otimes\tilde{p}_2),\ \tilde{p}_j(t_j)>0,\ j=1,2,\ (t_1,t_2)\in S,$$

where F_0 is defined on the subspace of polynomials, such that the moment conditions are accomplished. Application of Theorem from Section 5.1.2 [23] p. 160, leads to the existence of a positive linear extension $F\in L_+(X_1,Y)$ of F_0, where $X_1\subset X$ is the subspace of all functions from X having their modulus dominated by a polynomial. This subspace contains the subspace of continuous compactly supported functions. Hence $h\circ F$ has a representing positive measure for all linear positive functional h on Y. Using these conclusions, one obtains approximation of ψ in $L_\nu^1(S)$ by sums of tensor products

$$\sum_{j=0}^{k(m)} p_{m,1.j} \otimes p_{m,2,j} \to \psi,\ m \to \infty,$$

$$p_{m,1.j}(t_1) = q_{m,1,j}^2(t_1),\ t_1 \in R,\ p_{m,p,j}(t_2) = t_2^{n\,j}(1-t_2)^{p\,j},\ t_2 \in [0,1].$$

Now, using the density of sums of tensor products of positive polynomials in X_+, we prove that

$$0 \le F(\psi) \le G(\psi),\ \psi \in C_c(S),\ \psi \ge 0.$$

To this end, we proceed as follows. Applying Fatou's lemma, one obtains:

$$0 \le h(F(\psi)) \le \liminf_m (h \circ F)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j}\right) \le$$

$$\lim_m (h \circ F_2)\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j}\right) = \tag{1}$$

$$h(F_2(\psi)),\ \ \psi \in (C_c(S))_+,\ h \in Y_+^*.$$

Assume that

$$F_2(\psi) - F(\psi) \notin Y_+$$

Using a separation theorem, it should exist a positive linear continuous functional $h \in Y_+^*$ such that

$$h(F_2(\psi) - F(\psi)) < 0,$$

that is $h(F_2(\psi)) < h(F(\psi))$. This relation contradicts (1). The conclusion is that we must have

$$F(\psi) \le F_2(\psi),\ \psi \in (C_c(S))_+.$$

Then for arbitrary $g \in C_c(S)$ one writes

$$|F(g)| \le F_2(g^+) + F_2(g^-) = F_2(|g|) \Rightarrow \|F(g)\| \le \|F_2\| \cdot \|g\|_1.$$

The conclusion is that the operator F is positive and continuous, of norm dominated by $\|F_2\|$, on a dense subspace of $L_\nu^1(S)$. It has a unique linear extension preserving these properties. This concludes the proof. □

3. Distanced subspace with respect to a convex subset and the moment problem

The following extension result for linear operators has a nice geometric meaning and leads to interesting results in the moment problem.

If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V. See also [17] and the references there.

Theorem 3. *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following qualities:*

(a) *there exists a (convex) neighborhood V of the origin such that*

$$(S+V)\bigcap A = \Phi$$

(*A and S are distanced*);

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j(s) = f_j(s),\ s \in S \ \text{and}\ F_j(a) \geq \tilde{y},\ a \in A,\ j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V \cap S) \subset [-u_0, u_0], \quad (S+V)\bigcap A = \Phi,$$

$$0 < \alpha \in R \ s.t.\ p_V(a) \leq \alpha,\ \forall a \in A,\ \alpha_1 > 0 \ s.t.\ \tilde{y} \leq \alpha_1 u_0,$$

then the following relations hold

$$F_j(x) \leq (1+\alpha+\alpha_1)p_V(x)\cdot u_0, \quad x \in X,\ j \in J.$$

We denote by X the space of all continuous functions in the polydisc $\overline{D} = \prod_{j=1}^{n}\{|z_j| \leq 1\}$, which can be writen as a power series with real coefficients, centered at $(0,...,0)$ in the open polydisc D. Let

$$\varphi_j(z_1,...,z_n) = z_1^{j_1} \cdots z_n^{j_n},\ j = (j_k)_{k=1}^n,\ |j| = \sum_{k=1}^{n} j_k \geq 1.$$

On the other hand, consider a complex Hilbert space H, $U_0 \in \mathrm{A}(H)$ a selfadjoint operator acting on H. Denote

$Y_1 = \{U \in \mathrm{A}(H); UU_0 = U_0U\},\ Y = \{U \in Y_1; UV = VU\ \forall V \in Y_1\},$

$Y_+ = \{U \in Y; \langle U(h),h\rangle \geq 0\ \forall h \in H\}$

Here $A(H)$ stands for the real vector space of all selfadjoint operators acting on H. Obviously, Y is a commutative algebra of selfadjoint operators. Moreover, Y is an order complete vector lattice [23], [10], and the operatorial norm is solid on Y:

$$|U| \le |V| \Rightarrow \|U\| \le \|V\|, U, V \in Y.$$

Theorem 4. *Let* $(B_j)_{j \in \mathrm{N}^n}, \sum_{k=1}^{n} j_k \ge 1$ *be a sequence in* Y, $0 < \varepsilon < 1$, *such that*

$$\|B_j\| \le M \cdot \varepsilon^{j_1 + \cdots + j_n}, \quad \forall j = (j_1, \ldots, j_n) \in \mathrm{N}^n, |j| \ge 1.$$

Let $\{\psi_k\}_{k \in \mathrm{N}^n}$ *be a sequence in* X, *such that* $\psi_k(0, \ldots, 0) = 1, \|\psi_k\| \le 1, \forall k \in \mathrm{N}^n$. *Let* $\widetilde{B} \in Y_+$. *Then there is a linear operator applying* X *into* Y *such that:*

$$F(\varphi_j) = B_j, \, j \in \mathrm{N}^n, \sum_{k=1}^{n} j_k \ge 1, \, F(\psi_k) \ge \widetilde{B},$$

$$F(\varphi) \le \left(2 + \|\widetilde{B}\| \cdot \left(M^{-1}(1-\varepsilon)^n\right)\right) \cdot \|\varphi\|_\infty \cdot u_0, \quad u_0 := \cdot\left(M(1-\varepsilon)^{-n}\right) \cdot I.$$

Proof. Due to the behavior at $(0, \ldots, 0)$ of the functions $\varphi_j, |j| := \sum_{k=1}^{n} j_k \ge 1$ and $\psi_k, k \in \mathrm{N}^n$, we have

$$\|s - a\|_\infty \ge |s(0) - a(0)| \ge 1, \forall s \in S := Span\{\varphi_j; |j| \ge 1\},$$
$$\forall a \in A := conv\left\{\psi_k; k \in \mathrm{N}^n\right\} \Rightarrow (S + B(0,1)) \cap A = \Phi.$$

Thus using the hypothesis on the norms of the functions $\psi_k, k \in \mathrm{N}^n$, (where the unit ball $B(0,1)$ stands for V, and $\|\cdot\|$ stands for p_V), the above relations hold. Now let $s = \sum_{j \in J_0} \lambda_j \varphi_j \in S \cap B(0,1)$ and define the linear operator F_0 on the subspace S, such that the moment conditions $F_0(\varphi_j) = B_j, |j| \ge 1$ be accomplished. Cauchy's inequalities yield

$$|\lambda_j| \le \|s\|_\infty \le 1, \, j \in J_0 \Rightarrow f(s) = \sum_{j \in J_0} \lambda_j B_j \le \sum_{j \in J_0} |\lambda_j| \cdot |B_j| \le$$

$$\left(\sum_{j \in J_0} \|B_j\|\right) \cdot I \le M \cdot \left(\sum_{j_1 \in N} \varepsilon^{j_1}\right) \cdots \left(\sum_{j_n \in N} \varepsilon^{j_n}\right) \cdot I = M \cdot (1-\varepsilon)^{-n} \cdot I = u_0.$$

On the other hand, we have:

$$\widetilde{B} \le \left\|\widetilde{B}\right\| \cdot I = \left\|\widetilde{B}\right\| \cdot \left(M^{-1}(1-\varepsilon)^n\right) \cdot u_0 .$$

Application of theorem 3 leads to the conclusion. □

4. An application to a space of analytic functions

We recall the following result [19] on the abstract Markov moment problem, as an extension with two constraints theorem for linear operators.

Theorem 5. *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $F \in L(X,Y)$ such that*

$$F_1(x) \le F(x) \le F_2(x) \, \forall x \in X_+ , F(x_j) = y_j \, \forall j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

From Theorem 5 we deduce the following result. Let Y be an commutative real Banach algebra, which is also an order complete Banach lattice, with solid norm. Let

$$a_k, b_k \in Y_+, \|a_k\| < 1, \|b_k\| < 1, k = 1,\ldots,n.$$

Let $(y_j)_{j \in N^n}$ be a sequence in Y_+. Consider the space X of all continuous functions in the unit closed polydisc, which can be represented by sums of absolutely convergent power series with real coefficients in the open polydisc. The order relation on X is given by the coefficients of the power series. Namely,

$$X_+ = \left\{ \sum_{j \in N^n} c_j z^j ; c_j \ge 0 \; \forall j \in N^n \right\}.$$

Let

$$\varphi_j(z_1,...,z_n) = z_1^{j_1} \cdots z_n^{j_n},\ j = (j_1,..., j_n) \in N^n, |z_k| \le 1,\ k = 1,...,n.$$

Theorem 6. (see also [18]) *With the above notations, the following statements are equivalent:*

(a) *there exists* $F \in B(X,Y)$ *such that*

$$F(\varphi_j) = y_j,\ j \in \mathrm{N}^n,\ 0 \le F(\psi) \le \psi(a_1,...,a_n) + \varepsilon \cdot \psi(b_1,...,b_n),$$
$$\psi \in X_+, \|F\| \le 1 + \varepsilon;$$

(b) *we have:* $0 \le y_j \le a_1^{j_1} \cdots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \cdots b_n^{j_n},\ j = (j_1,..., j_n) \in \mathrm{N}^n$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious, because of the relations

$$\varphi_j \in X_+ \Rightarrow y_j = F(\varphi_j) \in [0, \varphi_j(a_1,...,a_n) + \varepsilon \cdot \varphi_j(b_1,...,b_n)] =$$
$$\left[0, a_1^{j_1} \cdots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \cdots b_n^{j_n}\right],\quad j \in \mathrm{N}^n.$$

Conversely, assume that (b) holds. We verify the implication in (b), Theorem 5. Namely, we have:

$$\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1 = \sum_{m \in \mathrm{N}^n} \alpha_m \varphi_m - \sum_{m \in \mathrm{N}^n} \beta_m \varphi_m,\ \ \alpha_m, \beta_m \ge 0,\ m \in \mathrm{N}^n \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j y_j \le \sum_{j \in J_0^+} \lambda_j y_j \le \sum_{j \in \mathrm{N}^n} \alpha_j y_j \le \sum_{j \in \mathrm{N}^n} \alpha_j \left(a_1^{j_1} \cdots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \cdots b_n^{j_n}\right) =$$
$$= \psi_2(a_1,...,a_n) + \varepsilon \cdot \psi_2(b_1,...,b_n) = F_2(\psi_2) - F_1(\psi_1),$$
$$F_1 := 0,\ J_0^+ = \{j \in J_0;\ \lambda_j \ge 0\}.$$

A direct application of Theorem 5 leads to the existence of a linear operator $F \in L(X,Y)$, such that

$$0 \le F(\psi) \le \psi(a_1,...,a_n) + \varepsilon \cdot \psi(b_1,...,b_n),\ \forall \psi \in X_+.$$

For an arbitrary $\varphi \in X$, one obtains:

$$|F(\varphi)| \le F(\varphi^+) + F(\varphi^-) \le |\varphi|(a_1,...,a_n) + \varepsilon \cdot |\varphi|(b_1,...,b_n) \Rightarrow$$
$$\|F(\varphi)\| \le (1 + \varepsilon) \cdot \|\varphi\|_\infty,\ \forall \varphi \in X \Rightarrow \|F\| \le 1 + \varepsilon.$$

This concludes the proof. □

5. Conclusions

In the first part of this work, we apply polynomial approximation results on unbounded subsets to the real multidimensional Markov moment problem on a strip. One approximates nonnegative compactly supported continuous functions, having their support contained in the strip, by sums of tensor products of positive polynomials in each separate variable, on the corresponding intervals. For such polynomials, the analytic expression is well known. Thus, one characterizes the existence of the solution in terms of quadratic mappings. Secondly, one proves applications of a theorem of extension of linear operators with two constraints to the Markov moment problem.

REFERENCES

[1] *N. I. Akhiezer*, The Classical Moment Problem and some related Questions in Analysis*,* Oliver and Boyd, Edinburgh-London, 1965.

[2] *G. Choquet*, Le problème des moments, Séminaire d'Initiation à l'Analise, Institut H. Poincaré, Paris, 1962.

[3] *C. Berg, J. P. R. Christensen and C. U. Jensen*, A remark on the multidimensional moment problem, Mathematische Annalen, 243, (1979), 163-169.

[4] *C. Berg and P. H. Maserick*, Polynomially positive definite sequences, Mathematische Annalen, 259, (1982), 487-495.

[5] *C. Berg, J. P. R. Christensen and P. Ressel*, Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions, Springer, 1984.

[6] *G. Cassier*, Problèmes des moments sur un compact de R^n et décomposition des polyn□mes à plusieurs variables, Journal of Functional. Analysis*,* 58, (1984), 254-266.

[7] *K. Schmüdgen*, The $K-$ moment problem for compact semi-algebraic sets, Mathematische Annalen, 289, (1991), 203-206.

[8] *M. Putinar*, Positive polynomials on compact semi-algebraic sets, Indiana University Mathematical. Journal, **42**, 3 (1993), 969-984.

[9] *F. H. Vasilescu*, Spectral measures and moment problems; in Spectral Analysis and its Applications, pp. 173-215, Ed. Theta, Bucharest, 2003.

[10] *L. Lemnete-Ninulescu*, Using the solution of an abstract moment problem to solve some classical complex moment problems, Romanian Journal of Pure and Applied Mathematics, **51,** (2006), 703-711.

[11] *L. Lemnete-Ninulescu and A. Zlătescu*, Some new aspects of the L moment problem, Romanian Journal of Pure and Applied Mathematics, **55**, 3, (2010), 197-204.

[12] *J. M. Mihăilă, O. Olteanu and C. Udrişte*, Markov-type and operator-valued multidimensional moment problems, with some applications, Romanian Journal of Pure and Applied Mathematics, **52,** (2007), 405-428.

[13] *J. M. Mihăilă, O. Olteanu and C. Udrişte*, Markov-type moment problems for arbitrary compact and for some non-compact Borel subsets of R^n, Romanian Journal of Pure and Applied Mathematics, **52**, (2007), 655-664.

[14] *J. M. Mihăilă, O. Olteanu and C. Udrişte*, La construction de quelque fonction par des moments donnes*,* Balkan Journal of Geometry and Its Applications, **13**, *1*(2008), 77-86.

[15] *A. Olteanu and O. Olteanu*, Some unexpected problems of the Moment Problem, Proc. of the Sixth Congress of Romanian Mathematicians, Academiei, Vol I, Scientific contributions, pp. 347-355, Bucharest, 2009.

[16] *O. Olteanu*, Approximation and Markov moment problem on concrete spaces, Rendiconti del Circolo Matematico di Palermo, (2014) 63: 161-172 DOI 10.1007/s12215-014-0149-7

[17] *O. Olteanu*: New results on Markov moment problem, International Journal of Analysis, Vol. 2013, Article ID 901318, (2013), 17 pages.

[18] *O. Olteanu*, Polynomial approximation on unbounded subsets and the Markov moment problem, International Journal of Analysis and Applications, **3,** 2 (2013), 68-80.

[19] *O. Olteanu,* Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une generalisation d'un théorème de Mazur-Orlicz, C. R. Acad. Sci Paris **313***,* Série I (1991), 739-742.

[20] *C. Berg and M. Thill*, Rotation invariant moment problems, Acta Mathematica, **11,** (1991), 207-227.

[21] *B. Fuglede*, The multidimensional moment problem, Expositiones Mathematicae I, (1983), 47-64.

[22] *C. Kleiber and J. Stoyanov*, Multivariate distributions and the moment problem, Journal of Multivariate Analysis, **113,** 1 (2013), 7-18.

[23] *R. Cristescu*, Ordered Vector Spaces and Linear Operators, Ed. Academiei, Bucharest and Abacus Press, Tunbridge Wells, Kent, 1976.

[24] *I. Colojoară*, Elemente de Teorie Spectrală (Elements of Spectral Theory), Ed. Academiei, Bucharest, 1968, (in Romanian).