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THE PROPERTIES OF A POLARIZATION INDEX FOR BOUNDED EXPONENTIAL DISTRIBUTIONS

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În literatura de specialitate sunt definiţi mai mulţi coeficienţi destinaţi stabilirii nivelului de polarizare a veniturilor unei populaţii P *. Indicatorul* $\Delta_*(f)$ *ce a fost propus în [14], [15] calculează diferenţa dintre "polii" a două grupări din* P *, grupări delimitate în raport cu densitatea de repartiţie f(x) a veniturilor indivizilor populaţiei* P *. In [15] s-a determinat expresia coeficientului* $\Delta_*(f)$ *pentru densităţi de repartiţie f(x) de tip exponenţial* Exp(θ, a, b) *cu suport* $[a , b]$ *marginit. Remarcăm faptul ca valoarea indicatorului* $\Delta_*(f)$ *depinde numai de un singur parametru, anume* $\lambda = \theta(b-a)$ *.*

Prezentul articol stabileşte în cazul repartiţiei Exp(θ, a, b) *limitele de variaţie ale coeficientului* $\Delta_*(f)$ *evidenţiindu-se totodata şi alte proprietăţi ale indicatorului de polarizare analizat.*

In the literature are presented different coefficients to measure the polarization level of the income for the individuals from a given population P*. The polarization index* $\Delta_*(f)$ *proposed in [14], [15] computes the difference between the poles of two disjoint groups which are determined by the probability density function f(x) of the income values of* P*. The paper [15] gives the expression of the coefficient* $\Delta_*(f)$ *for exponential* Exp(θ,a,b) *distributions having the support* $[a , b]$*. In this case the index* $\Delta_*(f)$ *depends on a single parameter, that is* $\lambda = \theta(b-a)$ *.*

Considering an Exp(θ, a, b) *distribution, in the present work are suggested bounds for the coefficient* $\Delta_*(f)$ *and are also emphasized different other properties of this polarization index.*

Key words : *measuring the inequality, polarization index, exponential distribution with bounded support.*

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Introduction

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A lot of social studies analyses the concentration and also the polarization phenomena. The Gini index is the most known coefficient for measuring the concentration aspect. The literature mentions many other polarization indices based on the income distribution of the individuals from a given population P (see especially [2]-[9], [16]-[18]).

We proposed in [14]-[15] a polarization coefficient Δ(f) which takes into consideration the probability density function (p.d.f.) f(x), $a \le x \le b$, of the income variable X.

More precisely, the suggested index Δ(f) measures the difference between the "poles" of two disjointed sets $I_1 = \{x \mid a \le x \le \mu\}$ and $I_2 = \{x \mid \mu < x \le b\}$, the separation threshold between these sets being just the mean μ of the random variable (r.v.) X ,

$$\mu = Mean(X) = \int_a^b x f(x)\, dx \tag{1}$$

The polarization measure Δ(f) is defined by the expression ([14], [15])

$$\Delta(f) = \frac{4p(1-p)(\mu_2 - \mu_1)}{b-a} \tag{2}$$

where

$$p = Pr(X \le \mu) = \int_a^{\mu} f(x)\, dx$$

$$\mu_1 = Mean(X \mid I_1) = \int_a^{\mu} \frac{x f(x)}{p}\, dx \tag{3}$$

$$\mu_2 = Mean(X \mid I_2) = \int_{\mu}^{b} \frac{x f(x)}{1-p}\, dx$$

We intend to emphasize some properties of the polarization index Δ(f) where $f(x;\theta,a,b)$ is the p.d.f. of an Exp(θ , a , b) exponential distribution with a finite support [a , b] . More exactly

$$f(x;\theta,a,b) = \frac{\theta e^{\theta x}}{e^{\theta b} - e^{\theta a}} \quad , \quad a \le x \le b \quad , \quad \theta \in R\, , \; \theta \ne 0 \tag{4}$$

In the subsequent we'll use the following notations

$$g(y;\lambda)=\frac{\lambda e^{\lambda y}}{e^{\lambda}-1} \quad , \quad 0\le y\le 1$$

$$G(y;\lambda)=\int_0^y g(t;\lambda)\,dt=\frac{e^{\lambda y}-1}{e^{\lambda}-1} \quad , \quad 0\le y\le 1$$

$$G_1(y;\lambda)=\int_0^y t\,g(t;\lambda)\,dt=\frac{ye^{\lambda y}}{e^{\lambda}-1}-\frac{e^{\lambda y}-1}{\lambda(e^{\lambda}-1)} \quad , \quad 0\le y\le 1$$

$$\nu=G_1(1;\lambda)=\frac{e^{\lambda}}{e^{\lambda}-1}-\frac{1}{\lambda} \tag{5}$$

Proposition 1 ([15]). If $f(x;\theta,a,b)$ is an exponential type p.d.f. (formula (4)), $\lambda=(b-a)\theta$ and ν has the expression (5) then the polarization index Δ(f) has the expression

$$\Delta_*(\lambda)=\Delta(f)=4\left(\nu G(\nu;\lambda)-G_1(\nu;\lambda)\right) \quad , \quad \lambda\neq 0 \tag{6}$$

The properties of $\Delta_*(\lambda)$

Having in mind the last relation (5) we define for any $\lambda\neq 0$ the function $\gamma(\lambda)$,

$$\gamma(\lambda)=\lambda\nu=\frac{\lambda e^{\lambda}}{e^{\lambda}-1}-1 \tag{7}$$

Proposition 2. The polarization index $\Delta_*(\lambda)$ has the form

$$\Delta(f)=\Delta_*(\lambda)=4\frac{(e^{\lambda}-1)e^{\gamma(\lambda)}-\lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} \tag{8}$$

Proof. A straightforward calculus gives

$$\Delta(f)=4\left[\nu G(\nu;\lambda)-G_1(\nu;\lambda)\right]=$$

$$=4\left[\left(\frac{e^{\lambda}}{e^{\lambda}-1}-\frac{1}{\lambda}\right)\frac{e^{\lambda\nu}}{e^{\lambda}-1}-\frac{\nu e^{\lambda\nu}}{e^{\lambda}-1}+\frac{e^{\lambda\nu}-1}{\lambda(e^{\lambda}-1)}\right]=$$

$$= 4\left[\frac{e^{\lambda}}{e^{\lambda}-1}\,\frac{e^{\gamma(\lambda)}-1}{e^{\lambda}-1} - \left(\frac{e^{\lambda}}{e^{\lambda}-1} - \frac{1}{\lambda}\right)\frac{e^{\gamma(\lambda)}}{e^{\lambda}-1}\right] =$$

$$= 4\left[\frac{e^{\gamma(\lambda)}}{\lambda(e^{\lambda}-1)} - \frac{e^{\lambda}}{(e^{\lambda}-1)^2}\right] =$$

$$= 4\,\frac{(e^{\lambda}-1)e^{\gamma(\lambda)} - \lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} = \Delta_*(\lambda)$$

Proposition 3. $\gamma(-\lambda) = \gamma(\lambda) - \lambda \;,\; \forall\; \lambda \in \mathbf{R},\; \lambda \neq 0\,.$ (9)

Proof. $\gamma(-\lambda) + \lambda = -\dfrac{\lambda e^{-\lambda}}{e^{-\lambda}-1} - 1 + \lambda = \dfrac{\lambda}{e^{\lambda}-1} - 1 + \lambda = \dfrac{\lambda e^{\lambda}}{e^{\lambda}-1} - 1 = \gamma(\lambda)$

Proposition 4 (the symmetry property). We have

$\Delta_*(-\lambda) = \Delta_*(\lambda) \;,\; \forall\; \lambda \in \mathbf{R},\; \lambda \neq 0$ (10)

Proof. Applying the formulas (8) and (9) we get

$$\Delta_*(-\lambda) = -4\frac{(e^{-\lambda}-1)e^{\gamma(-\lambda)} + \lambda e^{-\lambda}}{\lambda(e^{-\lambda}-1)^2} = -4\frac{(e^{-\lambda}-1)e^{\gamma(\lambda)-\lambda} + \lambda e^{-\lambda}}{\lambda(e^{-\lambda}-1)^2} =$$

$$= -\frac{4e^{\lambda}\left[(1-e^{\lambda})e^{\gamma(\lambda)-\lambda} + \lambda\right]}{\lambda(1-e^{\lambda})^2} = 4\,\frac{(e^{\lambda}-1)e^{\gamma(\lambda)} - \lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} = \Delta_*(\lambda)$$

Remark 1. Taking into consideration the symmetry property of the index $\Delta_*(\lambda)$, in the following we'll study the behavior of the function $\Delta_*(\lambda)$ only for $\lambda > 0$.

Proposition 5. For any $\lambda > 0$ we have

$\gamma(\lambda) < \lambda$ (11)

Proof. It is sufficient to show that the function $h(\lambda) = \lambda - \gamma(\lambda)$ is strictly positive when $\lambda > 0$. Since $h(\lambda) = \lambda - \gamma(\lambda) = \lambda - \dfrac{\lambda e^{\lambda}}{e^{\lambda}-1} + 1 = \dfrac{e^{\lambda}-\lambda-1}{e^{\lambda}-1}$ and $e^{\lambda}-1 > 0$ it remains to prove the inequality $h_1(\lambda) = e^{\lambda} - \lambda - 1 > 0$ for any $\lambda > 0$.

Indeed, the function $h_1(\lambda)$ is strictly increasing on the interval $(0,\infty)$ because its derivative $h_1^{(1)}(\lambda) = e^{\lambda} - 1$ is strictly positive when $\lambda > 0$. So $h_1(\lambda) > h_1(0) = 0$ for any $\lambda > 0$.

Proposition 6. The behavior of the index $\Delta_*(\lambda)$ for large values of λ is given by

$$\lim_{\lambda\to\infty} \Delta_*(\lambda) = 0 \tag{12}$$

Proof. Using the inequality (11) we obtain

$$0 \le \lim_{\lambda\to\infty} \frac{e^{\gamma(\lambda)}}{\lambda(e^{\lambda}-1)} \le \lim_{\lambda\to\infty} \frac{e^{\lambda}}{\lambda(e^{\lambda}-1)} = \left(\lim_{\lambda\to\infty} \frac{1}{\lambda}\right)\left(\lim_{\lambda\to\infty} \frac{e^{\lambda}}{e^{\lambda}-1}\right) = \lim_{\lambda\to\infty} \frac{1}{\lambda} = 0$$

Therefore

$$\lim_{\lambda\to\infty} \Delta_*(\lambda) = \lim_{\lambda\to\infty} 4\frac{(e^{\lambda}-1)e^{\gamma(\lambda)} - \lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} =$$

$$= 4 \lim_{\lambda\to\infty} \frac{e^{\gamma(\lambda)}}{\lambda(e^{\lambda}-1)} - 4 \lim_{\lambda\to\infty} \frac{e^{\lambda}}{(e^{\lambda}-1)^2} = 0$$

Proposition 7. For any $\lambda > 0$ the function $\gamma(\lambda)$ satisfies the inequality

$$max\{\lambda/2\,;\lambda-1\} < \gamma(\lambda) \tag{13}$$

Proof. To prove the inequality $\gamma(\lambda) = \dfrac{\lambda e^{\lambda}}{e^{\lambda}-1} - 1 > \dfrac{\lambda}{2}$ it is sufficient to show that the function $h(\lambda) = \lambda e^{\lambda} - 2e^{\lambda} + \lambda + 2$ is strictly positive on the whole interval $(0,\infty)$.

Indeed $h^{(1)}(\lambda) = \lambda e^{\lambda} - e^{\lambda} + 1 > 0$ for any $\lambda > 0$ because $h^{(2)}(\lambda) = \lambda e^{\lambda} > 0$ when $\lambda > 0$ and therefore $h^{(1)}(\lambda) > h^{(1)}(0) = 0$. Since the function $h(\lambda)$ is a strict increasing one it results that $h(\lambda) > h(0) = 0$ for $\lambda > 0$. More, we have also the inequality $\gamma(\lambda) > \lambda - 1$ with $\lambda > 0$ since this relation is equivalent with $\dfrac{\lambda e^{\lambda}}{e^{\lambda}-1} > \lambda$. But the last inequality is true for any $\lambda > 0$.

Proposition 8. The function $\gamma(\lambda)$ increases strictly on the domain $(0,\infty)$ and in addition $\gamma(0) = 0$, $\gamma(\infty) = \infty$.

Proof. We'll show that the first derivative $\gamma^{(1)}(\lambda)$ of the function $\gamma(\lambda)$ is strictly positive when $\lambda > 0$.

So, the inequality $h(\lambda) = e^{\lambda} - \lambda - 1 > h(0) = 0$ is true for an arbitrary $\lambda > 0$ since the function $h(\lambda)$ is strictly increasing ($h^{(1)}(\lambda) = e^{\lambda} - 1 > 0$ if $\lambda > 0$). Therefore

$$\gamma^{(1)}(\lambda) = \frac{e^{\lambda}(e^{\lambda} - \lambda - 1)}{(e^{\lambda} - 1)^2} = \frac{e^{\lambda} h(\lambda)}{(e^{\lambda} - 1)^2} > 0$$

More, using the inequalities (11) and (13) we deduce

$$0 \le \lim_{\lambda \downarrow 0} \gamma(\lambda) \le \lim_{\lambda \downarrow 0} \lambda = 0$$

$$\lim_{\lambda \to \infty} \gamma(\lambda) \ge \lim_{\lambda \to 0} \frac{\lambda}{2} = \infty$$

Proposition 9. We have the following limit

$$\lim_{\lambda \downarrow 0} \frac{\gamma(\lambda)\gamma^{(1)}(\lambda)}{\lambda} = \frac{1}{4} \tag{14}$$

Proof. Making the transform $t = e^{\lambda}$ and applying l'Hospital rule we obtain successively

$$\lim_{\lambda \downarrow 0} \frac{\gamma(\lambda)\gamma^{(1)}(\lambda)}{\lambda} = \lim_{t \downarrow 1} \frac{\big(t\, ln(t) - t + 1\big)\, t\big(t - 1 - ln(t)\big)}{(t-1)^3\, ln(t)} =$$

$$= \left(\lim_{t \downarrow 1} \frac{t\, ln(t) - t + 1}{(t-1)\, ln(t)} \right) \left(\lim_{t \downarrow 1} \frac{t - 1 - ln(t)}{(t-1)^2} \right) =$$

$$= \left(\lim_{t \downarrow 1} \frac{ln(t)}{ln(t) + 1 - \frac{1}{t}} \right) \left(\lim_{t \downarrow 1} \frac{1 - \frac{1}{t}}{2(t-1)} \right) = \frac{1}{2} \left(\lim_{t \downarrow 1} \frac{\frac{1}{t}}{\frac{1}{t} + \frac{1}{t^2}} \right) = \frac{1}{4}$$

Proposition 10. The following expression could also be used to compute the polarization index $\Delta_*(\lambda)$,

$$\Delta_*(\lambda) = 4 \frac{(e^{\gamma(\lambda)} - \gamma(\lambda) - 1)(\gamma(\lambda) - \lambda + 1)}{\lambda^2} \tag{15}$$

Proof. Since $e^{\lambda} = \dfrac{\gamma(\lambda)+1}{\gamma(\lambda)-\lambda+1}$ we get $e^{\lambda}-1 = \dfrac{\lambda}{\gamma(\lambda)-\lambda+1}$ and hence

$$\Delta_*(\lambda) = 4\frac{(e^{\lambda}-1)e^{\gamma(\lambda)}-\lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} = 4\frac{(e^{\gamma(\lambda)}-\gamma(\lambda)-1)(\gamma(\lambda)-\lambda+1)}{\lambda^2}$$

Proposition 11. The behavior of the polarization coefficient $\Delta_*(\lambda)$ for very small positive λ values is given by

$$\lim_{\lambda\downarrow 0} \Delta_*(\lambda) = \frac{1}{2} \tag{16}$$

Proof. Using the results of Propositions 8-10 and applying l'Hospital rule we deduce

$$\lim_{\lambda\downarrow 0} \Delta_*(\lambda) = 4\left(\lim_{\lambda\downarrow 0} \frac{e^{\gamma(\lambda)}-\gamma(\lambda)-1}{\lambda^2}\right)\left(\lim_{\lambda\downarrow 0}(\gamma(\lambda)-\lambda+1)\right) =$$

$$= 4\left(\lim_{\lambda\downarrow 0} \frac{e^{\gamma(\lambda)}-\gamma(\lambda)-1}{\lambda^2}\right) = 4\left(\lim_{\lambda\downarrow 0} \frac{(e^{\gamma(\lambda)}-1)\gamma^{(1)}(\lambda)}{2\lambda}\right) =$$

$$= 2\left(\lim_{\lambda\downarrow 0} \frac{e^{\gamma(\lambda)}-1}{\gamma(\lambda)}\right)\left(\lim_{\lambda\downarrow 0} \frac{\gamma(\lambda)\gamma^{(1)}(\lambda)}{\lambda}\right) = 2\left(\lim_{\gamma\downarrow 0} \frac{e^{\gamma}-1}{\gamma}\right)\left(\frac{1}{4}\right) = \frac{1}{2}$$

Proposition 12. The polarization index $\Delta_*(\lambda)$ could be computed by the formula

$$\Delta_*(\lambda) = 4\frac{e^{\gamma(\lambda)}-\gamma(\lambda)-1}{\lambda(e^{\lambda}-1)} \tag{17}$$

Proof. Indeed, using the equality $\lambda e^{\lambda} = (\gamma(\lambda)+1)(e^{\lambda}-1)$ in the expression (8) we get

$$\Delta_*(\lambda) = 4\frac{(e^{\lambda}-1)e^{\gamma(\lambda)}-\lambda e^{\lambda}}{\lambda(e^{\lambda}-1)^2} = 4\frac{e^{\gamma(\lambda)}-\gamma(\lambda)-1}{\lambda(e^{\lambda}-1)}$$

In the subsequent we'll propose a variation interval $[\Delta_1(\lambda), \Delta_2(\lambda)]$ for $\Delta_*(\lambda)$ values. So

Proposition 13. The polarization index $\Delta_*(\lambda)$ is raised by the decreasing function $\Delta_2(\lambda)$,

$$\Delta_*(\lambda) < \Delta_2(\lambda) = 4\frac{e^{\lambda} - \lambda - 1}{\lambda(e^{\lambda} - 1)} \quad , \quad \lambda > 0 \tag{18}$$

where

$$\lim_{\lambda \downarrow 0} \Delta_2(\lambda) = 2 \qquad \lim_{\lambda \to \infty} \Delta_2(\lambda) = 0 \tag{19}$$

Proof. The function $h_0(\lambda) = e^{\lambda} - \lambda - 1$ increases strictly when $\lambda > 0$ ($h_0^{(1)}(\lambda) = e^{\lambda} - 1 > 0$, $\forall \ \lambda > 0$). Using formulas (11) and (17) we obtain

$$\Delta_*(\lambda) = 4\frac{e^{\gamma(\lambda)} - \gamma(\lambda) - 1}{\lambda(e^{\lambda} - 1)} < 4\frac{e^{\lambda} - \lambda - 1}{\lambda(e^{\lambda} - 1)} = \Delta_2(\lambda)$$

More, applying successively l'Hospital rule it results

$$\lim_{\lambda \downarrow 0} \Delta_2(\lambda) = 4 \lim_{\lambda \downarrow 0} \frac{e^{\lambda} - \lambda - 1}{\lambda(e^{\lambda} - 1)} = 4 \lim_{\lambda \downarrow 0} \frac{e^{\lambda} - 1}{\lambda e^{\lambda} + e^{\lambda} - 1)} =$$

$$= 4 \lim_{\lambda \downarrow 0} \frac{e^{\lambda}}{\lambda e^{\lambda} + 2e^{\lambda}} = \frac{4}{2} = 2$$

$$\lim_{\lambda \to \infty} \Delta_2(\lambda) = 4 \lim_{\lambda \to \infty} \frac{e^{\lambda} - \lambda - 1}{\lambda(e^{\lambda} - 1)} = 4 \lim_{\lambda \to \infty} \frac{e^{\lambda} - 1}{\lambda e^{\lambda} + e^{\lambda} - 1} =$$

$$= 4 \lim_{\lambda \to \infty} \frac{e^{\lambda}}{\lambda e^{\lambda} + 2e^{\lambda}} = 4 \lim_{\lambda \to \infty} \frac{1}{\lambda + 2} = 0$$

We'll show that the expression $\Delta_2(\lambda)$ is a decreasing function by proving that its derivative $\Delta_2^{(1)}(\lambda)$ takes only strict negative values when $\lambda > 0$. So

$$\Delta_2^{(1)}(\lambda) = 4\frac{h_1(\lambda)}{\lambda^2(e^{\lambda} - 1)^2}$$

where

$$h_1(\lambda) = (e^{\lambda} - 1)(\lambda e^{\lambda} - \lambda) - (e^{\lambda} - \lambda - 1)(\lambda e^{\lambda} + e^{\lambda} - 1) = \lambda^2 e^{\lambda} - (e^{\lambda} - 1)^2$$

We have

$$h_1^{(1)}(\lambda) = e^{\lambda}(\lambda^2 + 2\lambda + 2 - 2e^{\lambda}) = e^{\lambda} h_2(\lambda)$$

The function $h_2(\lambda)$ decreases for any $\lambda > 0$ because $h_2^{(1)}(\lambda) = 2\lambda + 2 - 2e^{\lambda} < 0$

($h_2^{(2)}(\lambda) = 2 - 2e^{\lambda} < 0$, $\forall\ \lambda > 0$ and $h_2^{(1)}(0) = 0$).

Therefore for an arbitrary $\lambda > 0$ we have $h_1^{(1)}(\lambda) < 0$ and $h_1(0) = 0$. Concluding $h_1(\lambda) < 0$, $\forall\ \lambda > 0$, that is $\Delta_2^{(1)} < 0$ for any $\lambda > 0$ and hence the function $\Delta_2(\lambda)$ decreases when $\lambda > 0$.

Proposition 14. The polarization index $\Delta_*(\lambda)$ is diminished by the decreasing function $\Delta_1(\lambda)$, that is

$$\Delta_*(\lambda) > \Delta_1(\lambda) = 4\frac{e^{\lambda/2} - \lambda/2 - 1}{\lambda(e^{\lambda} - 1)} \quad , \quad \lambda > 0 \tag{20}$$

where

$$\lim_{\lambda \downarrow 0} \Delta_1(\lambda) = \frac{1}{2} \qquad\qquad \lim_{\lambda \to \infty} \Delta_1(\lambda) = 0 \tag{21}$$

Proof. Since the function $h_0(\lambda) = e^{\lambda} - \lambda - 1$ increases strictly on the domain $(0, \infty)$, taking into consideration the formula (13) and (17) we deduce

$$\Delta_*(\lambda) = 4\frac{e^{\gamma(\lambda)} - \gamma(\lambda) - 1}{\lambda(e^{\lambda} - 1)} > 4\frac{e^{\lambda/2} - \lambda/2 - 1}{\lambda(e^{\lambda} - 1)} = \Delta_1(\lambda)$$

But

$$\Delta_1(\lambda) = 4\frac{e^{\lambda/2} - \lambda/2 - 1}{\lambda(e^{\lambda} - 1)} = 4\frac{e^{\lambda/2} - \lambda/2 - 1}{(\lambda/2)(e^{\lambda/2} - 1)}\ \frac{1}{2(e^{\lambda/2} + 1)} = \Delta_2(\lambda/2)\, h_1(\lambda)$$

where the functions $\Delta_2(\lambda)$ and $h_1(\lambda) = \dfrac{1}{2(e^{\lambda/2} + 1)}$, $\lambda > 0$, are decreasing and positive.

Concluding, the function $\Delta_1(\lambda)$ decreases.

In addition, applying more times l'Hospital rule we obtain

$$\lim_{\lambda \downarrow 0} \Delta_1(\lambda) = 4 \lim_{\lambda \downarrow 0} \frac{e^{\lambda/2} - \lambda/2 - 1}{\lambda(e^{\lambda} - 1)} = 2 \lim_{\lambda \downarrow 0} \frac{e^{\lambda/2} - 1}{\lambda e^{\lambda} + e^{\lambda} - 1)} = \lim_{\lambda \downarrow 0} \frac{e^{\lambda/2}}{\lambda e^{\lambda} + 2e^{\lambda}} = \frac{1}{2}$$

$$\lim_{\lambda \to \infty} \Delta_1(\lambda) = 4 \lim_{\lambda \to \infty} \frac{e^{\lambda/2} - \lambda/2 - 1}{\lambda(e^{\lambda} - 1)} = 2 \lim_{\lambda \to \infty} \frac{e^{\lambda/2} - 1}{\lambda e^{\lambda} + e^{\lambda} - 1} =$$

$$= \lim_{\lambda \to \infty} \frac{e^{\lambda/2}}{\lambda e^{\lambda} + 2e^{\lambda}} = \lim_{\lambda \to \infty} \frac{1}{(\lambda + 2)e^{\lambda/2}} = 0$$

Remark 2. As a particular case of the Exp(θ, a, b) distribution is obtained when $\theta \to 0$, that is $\lambda \to 0$. In this situation we'll work with an uniform U([a , b]) distribution which have the p.d.f. $g(x;a,b) = \dfrac{1}{b-a}$, $a \le x \le b$. More exactly

Proposition 15. If $X_\theta \sim$ Exp(θ, a, b) and $Y = \lim\limits_{\theta \downarrow 0} X_\theta$ then Y ~ U([a , b])

Proof. We'll show that the limit of the p.d.f. $f(x;\theta,a,b)$ when $\theta \to 0$ is just the p.d.f. $g(x;a,b) = \dfrac{1}{b-a}$ which characterizes the uniform U([a , b]) distribution. Indeed, for an arbitrary $a \le x \le b$, applying the l'Hospital rule we get

$$\lim_{\theta \downarrow 0} f(x;\theta,a,b) = \lim_{\theta \downarrow 0} \frac{e^{\theta x}}{e^{\theta b} - e^{\theta a}} = \left(\lim_{\theta \downarrow 0} \frac{\theta(b-a)}{e^{\theta(b-a)} - 1} \right) \left(\lim_{\theta \downarrow 0} \frac{e^{\theta x}}{b-a} \right) =$$

$$= \left(\lim_{\theta \downarrow 0} \frac{b-a}{(b-a)e^{\theta(b-a)}} \right) \left(\frac{1}{b-a} \right) = \frac{1}{b-a} = g(x;a,b)$$

According with Proposition 11 we have

Proposition 16. $\Delta(g) = \dfrac{1}{2}$ (22)

Proof. Applying the formula (3) for the p.d.f. $g(x;a,b) = \dfrac{1}{b-a}$, $a \le x \le b$, we deduce

$$\mu = \frac{a+b}{2} \qquad p = \frac{1}{2} \qquad \mu_1 = \frac{3a+b}{4} \qquad \mu_2 = \frac{a+3b}{4}$$

and therefore the formulae (2) gives

$$\Delta(g) = \frac{4p(1-p)(\mu_2 - \mu_1)}{b-a} = \frac{4(1/2)(1/2)}{b-a} \left(\frac{a+3b}{4} - \frac{3a+b}{4} \right) = \frac{1}{2}$$

Conclusions

The present work emphasizes some properties for the polarization index $\Delta_*(\lambda)$, $\lambda = \theta(b-a)$, associated to a bounded exponential distribution Exp(θ, a, b). We mention here the symmetry property (Proposition 4) or the behavior of $\Delta_*(\lambda)$ for very small and also for very large positive values of λ (Propositions 6, 11). There are also proposed more formula to compute the index $\Delta_*(\lambda)$ (compare the expressions (6), (8), (15), (17)).

Finally we suggested two decreasing functions $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ which are variation bounds of the polarization coefficient $\Delta_*(\lambda)$, that is $\Delta_1(\lambda) < \Delta_*(\lambda) < \Delta_2(\lambda)$ (Propositions 13, 14).

Some of these results can be easily extended for an uniform U([a , b]) distribution which is regarded as a limit, when $\theta \to 0$, of the Exp(θ, a, b) distribution (see Propositions 15, 16).

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