THE CONCENTRATION GINI COEFFICIENT VERSUS A POLARIZATION INDEX

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Coeficientul Gini γ este frecvent utilizat pentru a masura nivelul "saraciei" dintr-o populatie P. A fost definit un nou indicator Δ cu scopul evaluarii intensitatii fenomenului de polarizare din P . Prezentul studiu scoate în evidenta deosebirile dintre indicatorii γ si Δ aducând noi argumente în sprijinul folosirii coeficientului Δ pentru determinarea gradului de polarizare din populatia P .

The Gini concentration coefficient γ is frequently used to measure the poverty level from a given population P . A new indicator Δ was defined to evaluate the intensity of the polarization phenomenon in P . The present study emphasizes the differences between the indices γ and Δ , giving new arguments to apply the coefficient Δ for establishing the degree of the income polarization for P individuals.

Keywords : Gini coefficient, polarization index, Lorenz order, antithetic variables.

MSC2000 : primary 62P25 ; secondary 62P20, 91B14, 91D99 .

1. Introduction

1.1. The Gini concentration coefficient.

Let X be a random variable having the probability density function $f(x)$, $a \le x \le b$, and $F(x)$ as cumulative distribution function. We'll suppose $a = 0$ and a finite threshold b (the maximal "income") if we intend to use X to model the income distribution of the individuals from the population P .

Theoretically we can also accept the alternative $b = \infty$.

Obviously $F(a) = 0$, $F(b) = 1$ and $0 \le F(x) \le 1$ for any $a \le x \le b$.

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Definition 1. ([2]; [3], p.179-180). The Lorenz concentration curve $L_X(u)$ attaches for every probability $0 \le u \le 1$ the relative mean of the whole x values which satisfy the restriction $F(x) \le u$. More exactly

$$L_X(u) = \left(\int_a^{F^{-1}(u)} x\, f(x)\, dx \right) / \left(\int_a^b x\, f(x)\, dx \right) = \frac{1}{Mean(X)} \left(\int_a^{F^{-1}(u)} x\, f(x)\, dx \right) \qquad (1)$$

Remark 1. Making the variable transform $t = F(x)$, since the derivative $F^{(1)}(x)$ of the cumulative distribution function $F(x)$ is just the probability density function $f(x)$, from (1) we get

$$L_X(u) = \frac{1}{Mean(X)} \left(\int_0^u F^{-1}(t)\, dt \right) \quad , \quad 0 \le u \le 1 \qquad (2)$$

Based on the Lorenz $L_Z(u)$ curve associated to an arbitrary random variable Z we'll define a partial order relation $\le_L$ between any two variables X and Y . So

Definition 2. For any random variables X and Y we consider ([11])

$X \le_L Y$ if and only if $L_X(u) \ge L_Y(u)$, $\forall \ 0 \le u \le 1$

Definition 3. The Gini index $\gamma(X)$ attached to the random variable X measures the difference between an ideal "egalitarian" situation (the curve $s = u$) and the Lorenz curve $s = L_X(u)$, $0 \le u \le 1$, that is ([2], [3], [11], [12]) ,

$$\gamma(X) = 2\int_0^1 (u - L_X(u))\, du = 1 - 2\int_0^1 L_X(u)\, du =$$

$$= 1 - \frac{2}{Mean(X)} \int_0^1 \left(\int_0^u F^{-1}(t)\, dt \right) du \qquad (3)$$

Proposition 1. For any random variables X and Y if $X \le_L Y$ then $\gamma(X) \le \gamma(Y)$.

Proof. If $X \le_L Y$ then $L_X(u) \ge L_Y(u)$ for every $0 \le u \le 1$. Therefore

$$\gamma(X) = 1 - 2\int_0^1 L_X(u)\, du \le 1 - 2\int_0^1 L_Y(u)\, du = \gamma(Y)$$

Remark 2. In the literature are proposed many others Gini index extensions $\gamma_0(X)$ to evaluate the poverty degree in the population P ([6]- [8], [11], [12]). Generalizing the result of Proposition 1, we'll impose that all the poverty indicators γ_0 must satisfy the "Lorenz principle", that is the order relation $X \le_L Y$ implies always the inequality $\gamma_0(X) \le \gamma_0(Y)$ (referring to the Lorenz order the index γ_0 is an increasing function).

1.2. A polarization indicator.

In this section we'll assume a bounded support $[a, b]$ for the random variable X .

The index $\Delta(X)$ proposed in [9], [10] computes the polarization degree of the variable X. More precisely, the domain $[a, b]$ is partitioned in two disjoint subdomains I_1 and I_2 , the separation threshold of these sets being just the mean μ of the variable X ,

$$\mu = Mean(X) = \int_a^b x\, f(x)\, dx \tag{4}$$

The coefficient $\Delta(X)$ will measure the difference between the "poles" μ_1 and μ_2 of the sets I_1 and I_2 taking in consideration the "weights" p and $1-p$ of the groups $I_1 = \{x \mid a \le x \le \mu\}$, respectively $I_2 = \{x \mid \mu < x \le b\}$,

$$p = Pr(X \le \mu) = \int_a^{\mu} f(x)\, dx$$

$$\mu_1 = Mean(X \mid I_1) = \int_a^{\mu} \frac{x\, f(x)}{p}\, dx \tag{5}$$

$$\mu_2 = Mean(X \mid I_2) = \int_{\mu}^{b} \frac{x\, f(x)}{1-p}\, dx$$

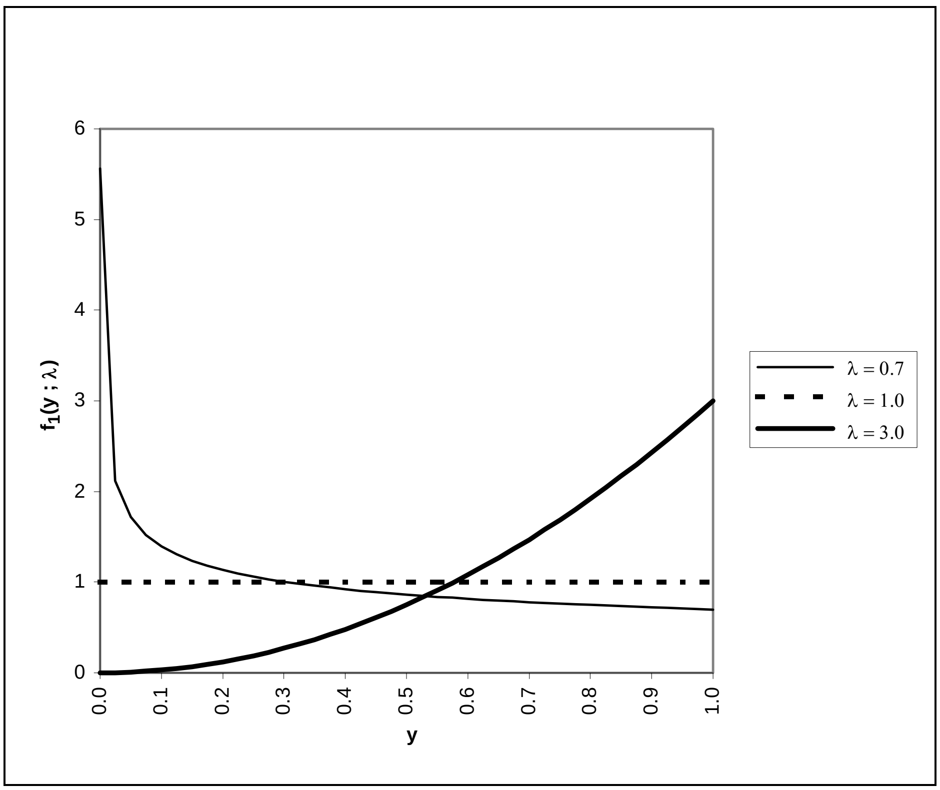
Definition 4. The polarization index $\Delta(X)$ has the expression ([9], [10])

$$\Delta(X) = \frac{4p(1-p)(\mu_2 - \mu_1)}{b-a} \tag{6}$$

2. Comparing the indices $\gamma(X)$ and $\Delta(X)$

2.1. The distribution Pow1(λ) .

Definition 5. The random variable Y has a type 1 power distribution with λ parameter, $\lambda > 0$, Y ~ Pow1(λ) , if its probability density function is given by the expression $f_1(y;\lambda) = \lambda y^{\lambda-1}$, where $0 < y \le 1$.



Graphic 1. The probability density function $f_1(y;\lambda)$.

The Graphic 1 presents for different $\lambda > 0$ the fluctuations of the probability density function $f_1(y;\lambda)$, $0 < y \le 1$.

Proposition 2. If $Y \sim \text{Pow1}(\lambda)$, $\lambda > 0$, then

$$\gamma(Y) = \frac{1}{2\lambda + 1} \tag{7}$$

Proof. Since for $Y \sim \text{Pow1}(\lambda)$, $0 \le u \le 1$, $0 < y \le 1$, $\lambda > 0$ we have

$$Mean(Y) = \int_0^1 y\, f_1(y;\lambda)\, dy = \int_0^1 \lambda y^{\lambda}\, dy = \frac{\lambda}{\lambda+1}$$

$$F_1(y;\lambda) = \int_0^y f_1(t;\lambda)\, dt = y^{\lambda} \qquad F_1^{-1}(u;\lambda) = u^{1/\lambda}$$

$$L_Y(u) = \frac{1}{Mean(Y)}\left(\int_0^u F_1^{-1}(t;\lambda)\, dt\right) = \frac{\lambda+1}{\lambda}\left(\int_0^u t^{1/\lambda}\, dt\right) = u^{1/\lambda+1}$$

it results

$$\gamma(Y) = 1 - 2\int_0^1 L(u)\, du = 1 - 2\int_0^1 u^{1/\lambda+1}\, du = 1 - \frac{2}{1/\lambda + 2} = \frac{1}{2\lambda+1}$$

Proposition 3. If $Y \sim \text{Pow1}(\lambda)$, $\lambda > 0$, then we have

$$\Delta(Y) = \frac{4\lambda^{\lambda+1}}{(\lambda+1)^{\lambda+2}} \tag{8}$$

Proof. We get successively

$$\mu = Med(Y) = \int_0^1 y\, f_1(y;\lambda)\, dy = \int_0^1 \lambda y^{\lambda}\, dy = \frac{\lambda}{\lambda+1}$$

$$p = Pr(Y \le \mu) = \int_0^{\mu} f_1(y;\lambda)\, dy = \int_0^{\lambda/(\lambda+1)} \lambda y^{\lambda-1}\, dy = \left(\frac{\lambda}{\lambda+1}\right)^{\lambda}$$

$$\mu_1 = Med(\, Y \mid Y \le \mu) = \int_0^{\lambda/(\lambda+1)} \frac{\lambda y^{\lambda}}{p}\, dy =$$

$$= \frac{\lambda}{\lambda+1}\left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1} / \left(\frac{\lambda}{\lambda+1}\right)^{\lambda} = \frac{\lambda^2}{(\lambda+1)^2}$$

$$\mu_2 = Med(Y \mid Y > \mu) = \int_{\lambda/(\lambda+1)}^{1} \frac{\lambda y^{\lambda}}{1-p}\, dy =$$

$$= \frac{\lambda}{\lambda+1}\left(1-\left(\frac{\lambda}{\lambda+1}\right)^{\lambda+1}\right) / \left(1-\left(\frac{\lambda}{\lambda+1}\right)^{\lambda}\right) =$$

$$= \frac{\lambda}{(\lambda+1)^2}\left((\lambda+1)^{\lambda+1}-\lambda^{\lambda+1}\right) / \left((\lambda+1)^{\lambda}-\lambda^{\lambda}\right)$$

So

$$\Delta(Y) = \frac{4p(1-p)(\mu_2-\mu_1)}{b-a} =$$

$$= 4\frac{\lambda^{\lambda+1}}{(\lambda+1)^{\lambda+2}}\left(1-\left(\frac{\lambda}{\lambda+1}\right)^{\lambda}\right)\left(\frac{(\lambda+1)^{\lambda+1}-\lambda^{\lambda+1}}{(\lambda+1)^{\lambda}-\lambda^{\lambda}}-\lambda\right) = \frac{4\lambda^{\lambda+1}}{(\lambda+1)^{\lambda+2}}$$

Remark 3. The behavior of the indices $\gamma(Y)$, $\Delta(Y)$ is very different when $Y \sim \text{Pow1}(\lambda)$, $\lambda > 0$ (see Graphic 2).

2.2. The Lorenz order.

Proposition 4. If $Y_k \sim \text{Pow1}(\lambda_k)$, $\lambda_k > 0$, $k \in \{1, 2\}$ then $Y_1 <_L Y_2$ if and only if $\lambda_1 > \lambda_2$.

Proof. If $\lambda_1 > \lambda_2$ then for any $0 \le u \le 1$ we have

$$L_{Y_1}(u) = u^{1/\lambda_1+1} \ge u^{1/\lambda_2+1} = L_{Y_2}(u)$$

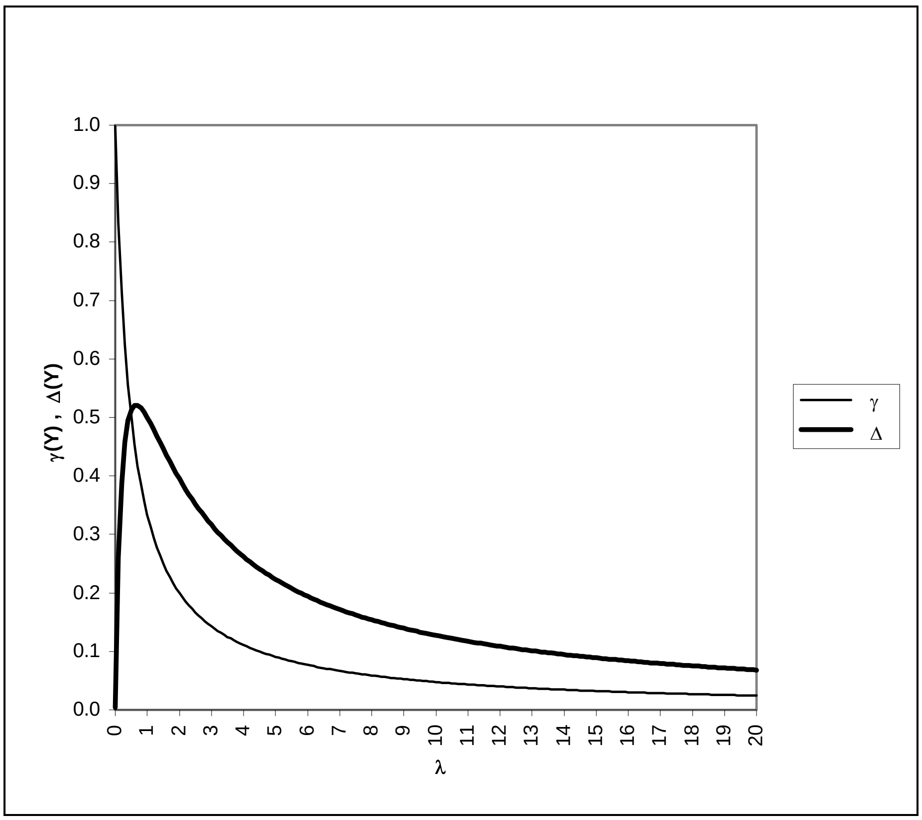
this inequality being strict when $0 < u < 1$. So, from Definition 2 we get $Y_1 <_L Y_2$.

Reciprocally, if $Y_1 <_L Y_2$ then $L_{Y_1}(u) \ge L_{Y_2}(u)$ for every $0 \le u \le 1$ and more it exists $0 \le u_0 \le 1$ so that $L_{Y_1}(u_0) > L_{Y_2}(u_0)$ that is $u_0^{1/\lambda_1+1} > u_0^{1/\lambda_2+1}$. Obviously $0 < u_0 < 1$ ($u_0 \ne 0$, $u_0 \ne 1$). Applying the logarithm to the previous inequality we obtain $(1/\lambda_1+1)\,ln(u_0) > (1/\lambda_2+1)\,ln(u_0)$. Since $ln(u_0) < 0$ it results $1/\lambda_1+1 < 1/\lambda_2+1$ that is $\lambda_1 > \lambda_2$.

Proposition 5. The polarization index $\Delta(Y)$ don't usually satisfy the Lorenz criteria.

Proof. It is sufficient to determine three random variables Y_k which validate the inequalities $Y_1 <_L Y_2 <_L Y_3$ and more without to verify one of the following relations $\Delta(Y_1) < \Delta(Y_2) < \Delta(Y_3)$ or $\Delta(Y_1) > \Delta(Y_2) > \Delta(Y_3)$.

Indeed, applying Proposition 4 for $Y_1 \sim$ Pow1(1.3) , $Y_2 \sim$ Pow1(0.7) , $Y_3 \sim$ Pow1(0.1) since $1.3 > 0.7 > 0.1$ we deduce $Y_1 <_L Y_2 <_L Y_3$. But after a direct computation (formula (8)) we obtain $\Delta(Y_1) = 0.468$, $\Delta(Y_2) = 0.521$, $\Delta(Y_3) = 0.260$ and therefore $\Delta(Y_1) < \Delta(Y_2) > \Delta(Y_3)$.

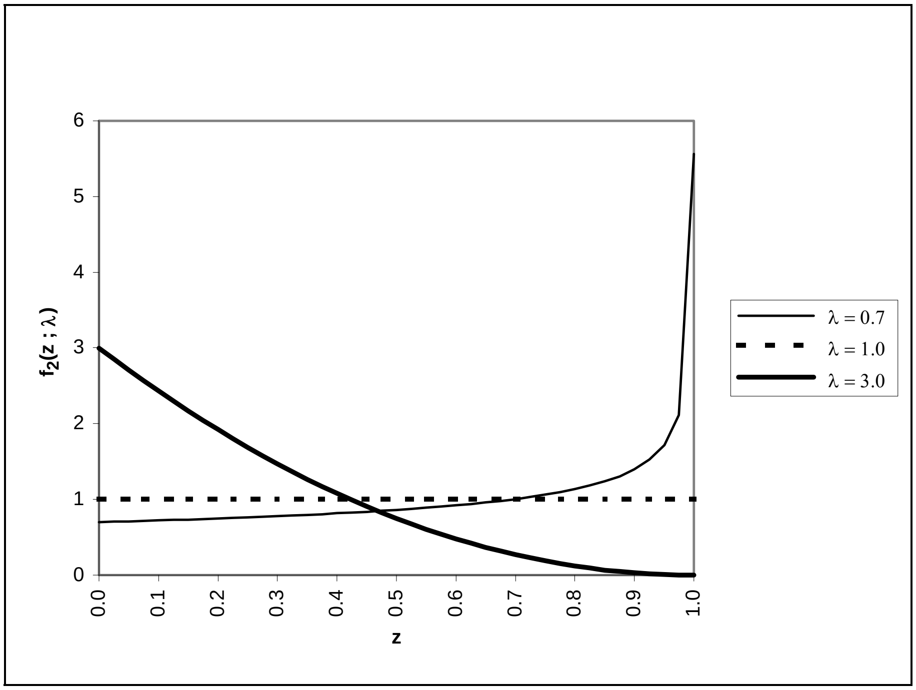


Graphic 2. The variation of the indices $\gamma(Y)$, $\Delta(Y)$ when $Y \sim$ Pow1(λ).

2.3. The distribution Pow2(λ) .

Definition 6. The random variable Z is type 2 power distributed with the parameter λ , $\lambda > 0$, $Z \sim$ Pow2(λ) , if its probability density function has the form $f_2(z\,;\lambda) = \lambda(1 - z)^{\lambda - 1}$, $0 \le z < 1$.

Graphic 3 presents the variation of the probability density function $f_2(z;\lambda) = \lambda(1-z)^{\lambda-1}$ for different values of the parameter λ .



Graphic 3. The probability density function $f_2(z\,;\lambda)$.

Proposition 6. If $Z \sim \mathrm{Pow2}(\lambda)$, $\lambda > 0$, then we have

$$\gamma(Z) = \frac{\lambda}{2\lambda+1} \tag{9}$$

Proof. Considering $Z \sim \mathrm{Pow2}(\lambda)$, $0 \le u \le 1$, $0 \le z < 1$, $\lambda > 0$ we get

$$Mean(Z) = \int_0^1 z f_2(z;\lambda)\,dz = \int_0^1 \lambda z(1-z)^{\lambda-1}\,dz =$$

$$= \int_0^1 \lambda(1-t)t^{\lambda-1}\,dt = 1 - \frac{\lambda}{\lambda+1} = \frac{1}{\lambda+1}$$

$$F_2(z;\lambda)=\int_0^z f_2(t;\lambda)\,dt=1-(1-z)^{\lambda} \qquad F_1^{-1}(u;\lambda)=1-(1-u)^{1/\lambda}$$

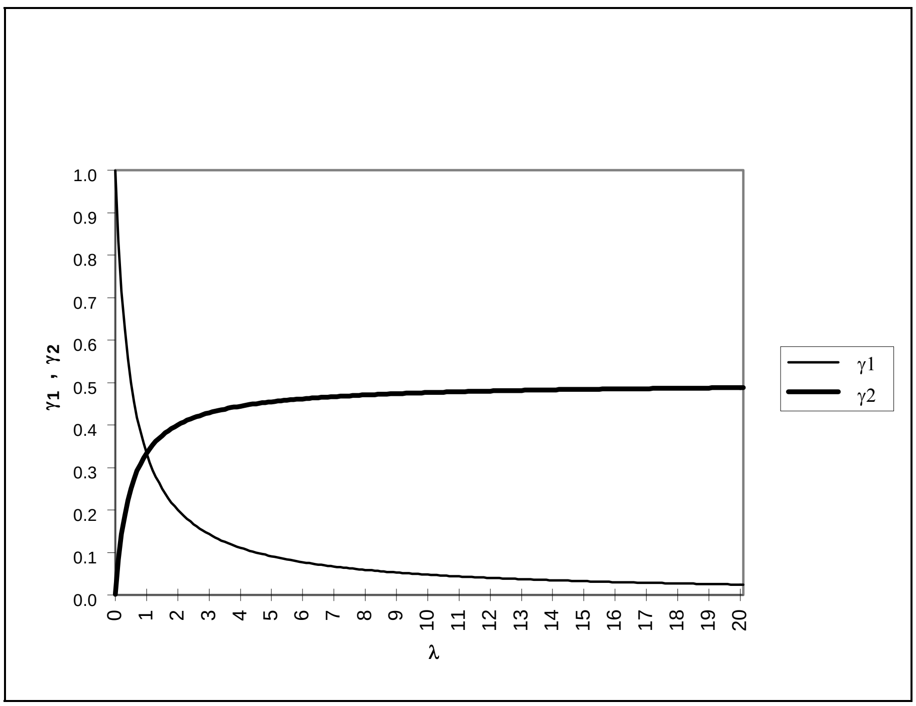
$$L_Z(u)=\frac{1}{Mean(Z)}\left(\int_0^u F_2^{-1}(t;\lambda)\,dt\right)=(\lambda+1)\left(\int_0^u\left(1-(1-t)^{1/\lambda}\right)dt\right)=$$

$$=(\lambda+1)u+\lambda(1-u)^{1/\lambda+1}-\lambda$$

We finally obtain

$$\gamma(Z)=1-2\int_0^1 L(u)\,du=1-2\int_0^1\left((\lambda+1)u+\lambda(1-u)^{1/\lambda+1}-\lambda\right)du=$$

$$=1-2\left(\frac{\lambda+1}{2}+\frac{\lambda}{1/\lambda+2}-\lambda\right)=\frac{\lambda}{2\lambda+1}$$



Graphic 4. The variation of the Gini indices $\gamma_1=\gamma(Y)$, $\gamma_2=\gamma(Z)$ when
Y ~ Pow1(λ) , Z ~ Pow2(λ)

Remark 4. Graphic 4 compares the fluctuations of the Gini coefficients $\gamma_1(\lambda)=\gamma(Y)$ and $\gamma_2(\lambda)=\gamma(Z)$ considering the random variables Y ~ Pow1(λ) , respectively Z ~ Pow2(λ) (see formulas (7), (9)). Although the probability density functions $f_1(x;\lambda)$ and $f_2(x;\lambda)$, $0<x<1$, $\lambda>0$, are symmetric functions referring to $x=0.5$ point (see the Graphics 1 and 3) however their Gini coefficients $\gamma_1(\lambda)$, $\gamma_2(\lambda)$ are very distinct (Propositions 2, 6 ; Graphic 4).

2.4. Antithetic variables.

Definition 7. We'll denote by X̲ the "antithetic" variable attached to the random variable X , the probability density function $g(w)$ of the variable X̲ verifying the equality

$$g(w)=f(a+b-w) \quad , \quad \forall \ a\le w\le b \tag{10}$$

where $f(x)$ is just the probability density function of X .

Remark 5. Since $g(w)=f(a+b-w)\ge 0$ for any $a\le w\le b$ and more

$$\int_a^b g(w)\,dw=\int_a^b f(a+b-w)\,dw=-\int_b^a f(x)\,dx=\int_a^b f(x)\,dx=1$$

we conclude that the application $g(w)$ is really a probability density function.

Remark 6. If X is a "symmetrical" random variable (its probability density function is symmetric) then $\underline{X}=X$ (a symmetrical random variable is identical with its antithetic).

Indeed, from the symmetry relation we get $f((a+b)/2-t)=f((a+b)/2+t)$ for any t with $|t|\le(b-a)/2$. Considering an arbitrary $a\le x\le b$ and taking $t=x-(a+b)/2$ we deduce $|t|\le(b-a)/2$. Therefore we have

$$\begin{aligned} f(x)&=f((a+b)/2+t)=f((a+b)/2-t)=\\ &=f((a+b)/2-x+(a+b)/2)=f(a+b-x)\end{aligned}$$

Proposition 7. For any random variable X with a finite support $[a,b]$ it result $\Delta(X)=\Delta(\underline{X})$.

Proof. We keep the previous notations and in addition we'll consider $g(w)$ as the probability density function of the random variable X̲ , $\nu=Mean(\underline{X})$,

$q = Pr(\underline{X} \le \nu)$, $J_1 = \{ w \mid a \le w \le \nu \}$, $J_2 = \{ w \mid \nu < w \le b \}$, $\nu_1 = Mean(\underline{X} \mid J_1)$, $\nu_2 = Mean(\underline{X} \mid J_2)$. Hence

$$\nu = Mean(\underline{X}) = \int_a^b w g(w)\, dw = \int_a^b w f(a+b-w)\, dw = -\int_b^a (a+b-x) f(x)\, dx =$$

$$= \int_a^b (a+b-x) f(x)\, dx = (a+b)\int_a^b f(x)\, dx - \int_a^b x f(x)\, dx = a+b-\mu$$

$$q = Pr(\underline{X} \le \nu) = \int_a^\nu g(w)\, dw = \int_a^{a+b-\mu} f(a+b-w)\, dw = -\int_b^\mu f(x)\, dx =$$

$$= \int_\mu^b f(x)\, dx = \int_a^b f(x)\, dx - \int_a^\mu f(x)\, dx = 1-p$$

$$\nu_1 = Mean(\underline{X} \mid J_1) = \int_a^\nu \frac{w g(w)}{q}\, dw = \int_a^{a+b-\mu} \frac{w f(a+b-w)}{1-p}\, dw =$$

$$= -\int_b^\mu \frac{(a+b-x) f(x)}{1-p}\, dx = \int_\mu^b \frac{(a+b-x) f(x)}{1-p}\, dx =$$

$$= \frac{a+b}{1-p}\int_\mu^b f(x)\, dx - \int_\mu^b \frac{x f(x)}{1-p}\, dx = a+b-\mu_2$$

$$\nu_2 = Mean(\underline{X} \mid J_2) = \int_\nu^b \frac{w g(w)}{1-q}\, dy = \int_{a+b-\mu}^b \frac{w f(a+b-w)}{p}\, dy =$$

$$= -\int_\mu^a \frac{(a+b-x) f(x)}{p}\, dx = \int_a^\mu \frac{(a+b-x) f(x)}{p}\, dx =$$

$$= \frac{a+b}{p}\int_a^\mu f(x)\, dx - \int_a^\mu \frac{x f(x)}{p}\, dx = a+b-\mu_1$$

So

$$\Delta(\underline{X}) = \frac{4q(1-q)(\nu_2 - \nu_1)}{b-a} = \frac{4(1-p)p(\mu_2 - \mu_1)}{b-a} = \Delta(X)$$

Proposition 8. For any $\lambda > 0$, Y ~ Pow1(λ) , Z ~ Pow2(λ) we have $\Delta(Y) = \Delta(Z)$.

Proof. The random variables Y and Z are antithetics since their probability density functions verifies the equality $f_2(x;\lambda) = f_2(1-x;\lambda)$ for every $0 \le x < 1$ (Definition 7). Applying Proposition 7 we obtain $\Delta(Y) = \Delta(Z)$.

Table 1.

The values $\gamma(Y)$, $\gamma(Z)$ and $\Delta(Y) = \Delta(Z)$ when Y ~ Pow1(λ) , Z ~ Pow2(λ).

λ	0.001	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
γ(Y)	0.998	0.833	0.714	0.625	0.556	0.500	0.455	0.417	0.385	0.357
γ(Z)	0.001	0.083	0.143	0.188	0.222	0.250	0.273	0.292	0.308	0.321
Δ(Y)	0.004	0.260	0.388	0.457	0.495	0.513	0.520	0.521	0.516	0.509
λ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
γ(Y)	0.333	0.313	0.294	0.278	0.263	0.250	0.238	0.227	0.217	0.208
γ(Z)	0.333	0.344	0.353	0.361	0.368	0.375	0.381	0.386	0.391	0.396
Δ(Y)	0.500	0.490	0.479	0.468	0.457	0.446	0.435	0.425	0.415	0.405
λ	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
γ(Y)	0.200	0.192	0.185	0.179	0.172	0.167	0.161	0.156	0.152	0.147
γ(Z)	0.400	0.404	0.407	0.411	0.414	0.417	0.419	0.422	0.424	0.426
Δ(Y)	0.395	0.386	0.377	0.368	0.360	0.352	0.344	0.337	0.330	0.323
λ	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
γ(Y)	0.143	0.139	0.135	0.132	0.128	0.125	0.122	0.119	0.116	0.114
γ(Z)	0.429	0.431	0.432	0.434	0.436	0.438	0.439	0.440	0.442	0.443
Δ(Y)	0.316	0.310	0.304	0.298	0.292	0.287	0.282	0.276	0.272	0.267
λ	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
γ(Y)	0.111	0.109	0.106	0.104	0.102	0.100	0.098	0.096	0.094	0.093
γ(Z)	0.444	0.446	0.447	0.448	0.449	0.450	0.451	0.452	0.453	0.454
Δ(Y)	0.262	0.258	0.253	0.249	0.245	0.241	0.237	0.234	0.230	0.227

Remark 7. Index $\Delta(X)$ remains invariant to an "antithetic" transform (Proposition 8). But this property isn't true when $\Delta(X)$ is replaced by the Gini coefficient $\gamma(X)$ (see Propositions 2 and 6 ; Graphic 4 ; Table 1).

3. Conclusions

The Gini concentration coefficient $\gamma(X)$ is based on the Lorenz order (Definitions 2, 3 ; Proposition 1). Contrary, the polarization index $\Delta(X)$ is build considering a different principle, that is dividing the initial population P in two antithetic groups (Definition 4). Proposition 5 proofs that the Lorenz order don't necessary imply the monotony of the function $\Delta(X)$, but the application $\gamma(X)$ is an increasing function (Proposition 1).

The Gini coefficient $\gamma(X)$ don't treat unitary the poor and rich individuals from the population P (these aspects are confirmed by the Propositions 2 and 6 ; Graphic 4 ; Table 1). We remark here that the polarization index $\Delta(X)$ remains invariant if we apply an "antithetic" transform, that is when are permuted the roles of poor and rich people (Proposition 7).

The particular values and also the qualitative aspects regarding the form and the monotony of the functions $\gamma(X)$ and $\Delta(X)$ too are sometimes very different (see the Graphics 2 and 4 ; Table 1 ; Propositions 2, 6, 8), since the both indicators measure distinct aspects of the social reality. So the Gini coefficient $\gamma(X)$ is oriented for monitoring the poverty phenomenon and in reverse the index $\Delta(X)$ must determine the polarization level from the population P .

Taking in consideration all the previous mentions we conclude that is inadequate to use the Gini coefficient when we intend to emphasize a strength social polarization phenomenon.

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