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C^*-ALGEBRA VALUED EXTENDED b-METRIC SPACES AND FIXED POINT RESULTS WITH AN APPLICATION

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In this paper, we introduce the notion of C^*-algebra valued extended b-metric spaces and utilize the same to prove an analogue of Banach Contraction Principle. We adopt an example to exhibit the utility of our main result. Finally, we apply our result to examine the existence and uniqueness of solution for a system of Fredholm integral equations.

Keywords: C^*-algebra; C^*-algebra valued extended b-metric space; fixed point.

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1. Introduction

Fixed point theory continues to be a fascinating subject of research despite having a history of more than hundred years. The attraction of fixed point theory lies in its application which belongs to numerous domain. The first fundamental result on fixed point for contractive-type mappings is essentially the well known result namely Banach contraction principle by Banach [9] in 1922, which turns out to be very effective tool in guaranteeing the existence and uniqueness of solution of various types of diverse problems arising in several domains within and beyond mathematics. The classical Banach contraction principle has been extended and generalized in number of different directions (see [8, 22, 3, 11, 10, 4, 5, 6, 7, 19]). To enhance the domain of applicability, I.A. Bakhtin [8] and S. Czerwik [11] introduced the concept of b-metric space as a note improvement of metric spaces and proved fixed point results as an analogue of Banach contraction principle. Indeed, many researchers are dealing with the fixed point theory for singlevalued and multivalued mappings in b-metric spaces and by now there exists a considerable literature in such spaces (see [15, 12, 24, 27, 1, 16, 26]). On the other hand, Kamran et al. [17] introduced a new type of generalized b-metric space and termed it as extended b-metric space. Thereafter, several researchers proved

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some existence and uniqueness results on fixed point in extended b-metric spaces (see [25, 23, 13, 18]).

In 2014, Ma et al. [20] established the notion of C^*-algebra valued metric spaces (in short C^*-avMS) by replacing the range set $\mathbb{R}$ with a unital C^*-algebra which is more general class than class of metric spaces and utilize the same to prove some fixed point results in such spaces. One year later, again Ma et al. [21] introduce the notion of C^*-algebra valued b-metric spaces as a generalization of C^*-avMS and proved some fixed point results also used their results as an applications for an integral type operator.

Inspired by foregoing observations, we enlarge the class of C^*-avbMS by introducing the class of C^*-avEbMS and utilize the same to prove fixed point result. We also furnish some examples which demonstrate the utility of our main result. Moreover, we our main result to examine the existence and uniqueness of solution for a system of integral type operator.

2. **Preliminaries**

In this section, we collect notions, definitions and auxiliary results which are needed in our subsequent discussions.

Throughout the paper, we denote $\mathcal{A}$ by an unital (*i.e.*, unity element I) C^*-algebra with linear involution $*$ such that for all $\rho, \varsigma \in \mathcal{A}$, $(\rho\varsigma)^* = \varsigma^*\rho^*$ and $\rho^{**} = \rho$. Let $\mathcal{A}$ be an unital C^*-algebra with unity element I, then we denote $\mathcal{A}^I = \{a \in \mathcal{A}; ab = ba, a \succcurlyeq I \text{ and } \forall\ b \in \mathcal{A}\}$. A positive element $\rho \in \mathcal{A}$ is denoted by $0_{\mathcal{A}} \preccurlyeq \rho$, if $\rho = \rho^*$ and $\sigma(\rho) = \{\lambda \in \mathbb{R} : \lambda I - \rho \text{ is non-invertible }\} \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is a zero element in $\mathcal{A}$. The partial ordering on $\mathcal{A}$ can be defined as follows: $\rho \preccurlyeq \varsigma$ if and only if $0_{\mathcal{A}} \preccurlyeq \varsigma - \rho$. The pair $(\mathcal{A}, *)$ is said to be an unital $*$-algebra, if it contains the unity element I. A unital $*$-algebra $(\mathcal{A}, *)$ is called a Banach $*$-algebra, if it satisfies $\|\rho^*\| = \|\rho\|$ along with a complete sub-multiplicative norm. A Banach C^*-algebra satisfying $\|\rho^*\rho\| = \|\rho\|^2$, for all $\rho \in \mathcal{A}$ is called a C^*-algebra.

The following definition is introduced by Ma et al. [20]:

Definition 2.1. *Let $A \neq \emptyset$. The mapping $d : A \times A \to \mathcal{A}$ is called a C^*-av metric on A, if it satisfies the following (for all $\varsigma, \sigma, \rho \in A$):*
(1) $d(\varsigma, \sigma) \succcurlyeq 0_{\mathcal{A}}$ *and* $d(\varsigma, \sigma) = 0_{\mathcal{A}}$ *iff* $\varsigma = \sigma$;
(2) $d(\varsigma, \sigma) = d(\sigma, \varsigma)$;
(3) $d(\varsigma, \sigma) \preccurlyeq d(\varsigma, \rho) + d(\rho, \sigma)$.
The triplet $(A, \mathcal{A}, d)$ is called a C^*-avMS.

I.A. Bakhtin [8] and S. Czerwik [11] introduced the notion of b-metric spaces.

Definition 2.2. *Let $A \neq \emptyset$. The mapping $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric with coefficient $b \geq 1$, if σ satisfies the following (for all $\varsigma, \sigma, \rho \in A$):*
(1) $d(\varsigma, \sigma) = 0$ *if and only if* $\varsigma = \sigma$;

(2) $d(\varsigma, \sigma) = d(\sigma, \varsigma)$;
(3) $d(\varsigma, \sigma) \leq b[d(\varsigma, \rho) + d(\rho, \sigma)]$.
Then the pair (A, d) *is said to be a b-metric space.*

In 2015, again Ma et al. [21] introduced the notion of C^*-av b-metric space as follows:

Definition 2.3. *Let* $A \neq \emptyset$ *and* $s \in \mathcal{A}$ *such that* $s \succcurlyeq I$. *The mapping* $d : A \times A \to \mathcal{A}$ *is called a* C^**-av b-metric on* A*, if it satisfies the following (for all* $\varsigma, \sigma, \rho \in A$*):*
(1) $d(\varsigma, \sigma) \succcurlyeq 0_{\mathcal{A}}$ *and* $d(\varsigma, \sigma) = 0_{\mathcal{A}}$ *iff* $\varsigma = \sigma$;
(2) $d(\varsigma, \sigma) = d(\sigma, \varsigma)$;
(3) $d(\varsigma, \sigma) \preccurlyeq s[d(\varsigma, \rho) + d(\rho, \sigma)]$.
The triplet $(A, \mathcal{A}, d)$ *is called a* C^**-avbMS.*

In 2017, T. Kamran et al. [17] introduced the following definition of extended b-metric spaces.

Definition 2.4. *Let* $A \neq \emptyset$ *and* $\xi : X \times X \to [1, \infty)$. *The mapping* $d : X \times X \to \mathbb{R}_+$ *is said to be an extended b-metric, if* σ *satisfies the following (for all* $\varsigma, \sigma, \rho \in A$*):*
(1) $d(\varsigma, \sigma) = 0$ *if and only if* $\varsigma = \sigma$;
(2) $d(\varsigma, \sigma) = d(\sigma, \varsigma)$;
(3) $d(\varsigma, \sigma) \leq \xi(\varsigma, \sigma)[d(\varsigma, \rho) + d(\rho, \sigma)]$.
Then the pair (A, d) *is said to be an extended b-metric space.*

Remark 2.1. *Clearly, if* $s = I$ *then a* C^**-avbMS reduced to a* C^**-avMS.*

3. **Results**

In this section, we introduce yet another type of generalized C^*-avMS, which we refer as C^*-avEbMS. We also establish a fixed point theorem besides deducing natural corollaries. Now, we define C^*-algebra valued extended b-metric space (in short C^*-avEbMS) as follows:

Definition 3.1. *Let* $A \neq \emptyset$ *and* $\xi : A \times A \to \mathcal{A}^I$. *The mapping* $d_\xi : A \times A \to \mathcal{A}$ *is called a* C^**-av extended b-metric on* A*, if it satisfies the following (for all* $\varsigma, \sigma, \rho \in A$*):*
(1) $d_\xi(\varsigma, \sigma) \succcurlyeq 0_{\mathcal{A}}$ *and* $d_\xi(\varsigma, \sigma) = 0_{\mathcal{A}}$ *iff* $\varsigma = \sigma$;
(2) $d_\xi(\varsigma, \sigma) = d_\xi(\sigma, \varsigma)$;
(3) $d_\xi(\varsigma, \sigma) \preccurlyeq \xi(\varsigma, \sigma)[d_\xi(\varsigma, \rho) + d_\xi(\rho, \sigma)]$.
The triplet $(A, \mathcal{A}, d_\xi)$ *is called a* C^**-avEbMS.*

Remark 3.1. *Observe that, if* $\xi(\varsigma, \sigma) = s \succcurlyeq I$, *then* $(A, \mathcal{A}, d_\xi)$ *reduces to a* C^**-avbMS (see* [21]*).*

$$\begin{array}{ccccc} \text{Metric space} & \longrightarrow & \text{b-metric space} & \longrightarrow & \text{Extended b-metric space} \\ \downarrow & & \downarrow & & \downarrow \\ C^*\text{-avMS} & \longrightarrow & C^*\text{-avbMS} & \longrightarrow & C^*\text{-avEbMS} \end{array}$$

Example 3.1. *Let $A = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$, the class of bounded and linear operators on a Hilbert space $\mathbb{C}^2$. Define a mapping $\xi : A \times A \to \mathcal{A}$ by (for all $\varsigma, \sigma \in \chi$):*

$$\xi(\varsigma,\sigma) = \begin{cases} \begin{bmatrix} |\varsigma-\sigma|^{p-1} & 0 \\ 0 & |\varsigma-\sigma|^{p-1} \end{bmatrix} & \text{if } \varsigma \neq \sigma \\ I_{2\times 2} \quad \text{if } \varsigma = \sigma \end{cases}$$

where $I_{2\times 2}$ is a square identity matrix in $\mathcal{A}$ and $k > 0$ is a constant.

Define $d_\xi : A \times A \to \mathcal{A}$ by (for all $\varsigma, \sigma \in A$):

$$d_\xi(\varsigma,\sigma) = \begin{bmatrix} |\varsigma-\sigma|^{p} & 0 \\ 0 & |\varsigma-\sigma|^{p} \end{bmatrix}$$

Then $(A, \mathcal{A}, d_\xi)$ be a C^*-avEbMS.

Proof. By routine calculation one can verify, conditions $(i) - (ii)$ of Definition 3.1. Now, we give the following inequality (for all $\alpha, \beta \in A$):

$$\begin{bmatrix} (\alpha+\beta)^p & 0 \\ 0 & (\alpha+\beta)^p \end{bmatrix} \leq \begin{bmatrix} (\alpha+\beta)^{p-1} & 0 \\ 0 & (\alpha+\beta)^{p-1} \end{bmatrix} \begin{bmatrix} \alpha^p+\beta^p & 0 \\ 0 & \alpha^p+\beta^p \end{bmatrix}.$$

Above inequality is trivial for $\alpha = \beta = 0$. For $|\alpha| \geq 1$ or $|\beta| \geq 1$, we obtain

$$\begin{aligned} \begin{bmatrix} (\alpha+\beta)^p & 0 \\ 0 & k(\alpha+\beta)^p \end{bmatrix} &= \frac{\begin{bmatrix} (\alpha+\beta)^p & 0 \\ 0 & (\alpha+\beta)^p \end{bmatrix}}{\begin{bmatrix} \alpha^p+\beta^p & 0 \\ 0 & \alpha^p+\beta^p \end{bmatrix}} \begin{bmatrix} \alpha^p+\beta^p & 0 \\ 0 & \alpha^p+\beta^p \end{bmatrix} \\ &\leq \frac{\begin{bmatrix} (\alpha+\beta)^p & 0 \\ 0 & (\alpha+\beta)^p \end{bmatrix}}{\begin{bmatrix} \alpha+\beta & 0 \\ 0 & \alpha+\beta \end{bmatrix}} \begin{bmatrix} \alpha^p+\beta^p & 0 \\ 0 & \alpha^p+\beta^p \end{bmatrix} \\ &= \begin{bmatrix} (\alpha+\beta)^{p-1} & 0 \\ 0 & (\alpha+\beta)^{p-1} \end{bmatrix} \begin{bmatrix} \alpha^p+\beta^p & 0 \\ 0 & \alpha^p+\beta^p \end{bmatrix}. \end{aligned}$$

Finally, we set $\alpha = \varsigma - \rho,\ \beta = \rho - \sigma$ and obtain

$$\begin{aligned} \begin{bmatrix} |\varsigma-\rho|^p & 0 \\ 0 & |\varsigma-\rho|^p \end{bmatrix} \leq & \begin{bmatrix} |\varsigma-\rho|^{p-1} & 0 \\ 0 & |\varsigma-\rho|^{p-1} \end{bmatrix} \Bigg(\begin{bmatrix} |\varsigma-\rho|^p & 0 \\ 0 & |\varsigma-\rho|^p \end{bmatrix} \\ & + \begin{bmatrix} |\varsigma-\rho|^p & 0 \\ 0 & |\varsigma-\rho|^p \end{bmatrix} \Bigg). \end{aligned}$$

Therefore,

$$d_\xi(\varsigma,\sigma) \leq \xi(\varsigma,\sigma)\big(d_\xi(\varsigma,\rho) + d_\xi(\rho,\sigma)\big).$$

Hence, $(A, \mathcal{A}, d_\xi)$ is a C^*-avEbMS. □

Remark 3.2. *Observe that* $\sup\{\xi(\varsigma,\sigma);\ \varsigma,\sigma \in \chi\} = \infty$. *Thus,* d_ξ *is not a* C^**-avbMS.*

Let $(A, \mathcal{A}, d_\xi)$ be a C^*-avEbMS. Then open ball of center $\varsigma \in A$ and radius $0_\mathcal{A} \prec \epsilon \in \mathcal{A}$ is defined by:

$$B_{d_\xi}(\varsigma, \epsilon) = \{\sigma \in A : d_\xi(\varsigma,\sigma) \prec \epsilon\}.$$

Similarly, the closed ball with center $\varsigma \in A$ and radius $\epsilon \succ 0$ is defined by:

$$B_{d_\xi}[\varsigma, \epsilon] = \{\sigma \in A : d_\xi(\varsigma,\sigma) \preccurlyeq \epsilon\}.$$

The family of open balls (for all $\varsigma \in A$ and $\epsilon \succ 0$)

$$\mathcal{U}_{d_\xi} = \{B_{d_\xi}(\varsigma,\epsilon) : \varsigma \in A, \epsilon \succ 0_\mathcal{A}\},$$

forms a basis of some topology τ_d on A.

Lemma 3.1. *Let* (A, τ_{d_ξ}) *be a topological space and* $f : A \to A$. *If* f *is continuous then every sequence* $\{\varsigma_n\} \subseteq A$ *such that* $\varsigma_n \to \varsigma$ *implies* $f\varsigma_n \to f\varsigma$. *The converse holds if* A *is metrizable.*

Definition 3.2. *A sequence* $\{\varsigma_n\}$ *in* $(A, \mathcal{A}, d_\xi)$ *is called convergent (with respect to* $\mathcal{A}$*), if for given* $\epsilon \succ 0_\mathcal{A}$*, there exists* $N \in \mathbb{N}$ *such that* $d_\xi(\varsigma_n, \varsigma) \prec \epsilon$*, for all* $n > N$. *We denote it by*

$$\lim_{n\to\infty} d_\xi(\varsigma_n, \varsigma) = 0_\mathcal{A}.$$

Definition 3.3. *A sequence* $\{\varsigma_n\}$ *in* $(A, \mathcal{A}, d_\xi)$ *is called Cauchy sequence (with respect to* $\mathcal{A}$*), if for given* $\epsilon \succ 0_\mathcal{A}$*, there exists* $N \in \mathbb{N}$ *such that* $d_\xi(\varsigma_n, \varsigma_m) \prec \epsilon$*, for all* $n, m > N$. *We denote it by*

$$\lim_{n\to\infty} d_\xi(\varsigma_n, \varsigma_m) = 0_\mathcal{A}.$$

Definition 3.4. *The triplet* $(A, \mathcal{A}, d_\xi)$ *is called complete* C^**-avEbMS if every Cauchy in* A *is convergent to a point* ς *in* A.

Observe that, in general a b-metric is not a continuous functional and so is a C^*-avEbMS.

Example 3.2. [14] *Let* $X = \mathbb{N} \cup \infty$ *and a mapping* $d : X \times X \to \mathbb{R}_+$ *defined by:*

$$d(\varsigma,\sigma) = \begin{cases} 0_\mathcal{A} & \textit{if } \varsigma = \sigma \\ |\frac{1}{\varsigma} - \frac{1}{\sigma}| & \textit{if } \varsigma,\sigma \textit{ are even or } \varsigma\sigma = \infty \\ 5 & \textit{if } \varsigma,\sigma \textit{ are odd or } \varsigma \neq \sigma \\ 5 & \textit{otherwise.} \end{cases}$$

Then (X, d) *is a b-metric space with* $s = 3$ *but it is not continuous.*

Lemma 3.2. *Let $(A, \mathcal{A}, d_\xi)$ be a C^*-avEbMS. If d_ξ is continuous then every convergent sequence has a unique limit.*

Our main result runs as follows:

Theorem 3.1. *Let $(A, \mathcal{A}, d_\xi)$ be complete C^*-avEbMS and $f : X \to X$ satisfies that the following:*

$$d_\xi(f\varsigma, f\sigma) \preccurlyeq c^* d_\xi(\varsigma, \sigma)c, \ \forall\ \varsigma, \sigma \in A. \tag{1}$$

where, $c \in \mathcal{A}$ with $\|c\| < 1$ and $\lim\limits_{n,m\to\infty} \xi(\varsigma_n, \varsigma_m)\|c\| \prec I$. Then f has a unique fixed point $\varsigma \in A$.

Proof. Choose $\varsigma_0 \in A$ and construct an iterative sequence $\{\varsigma_n\}$ by:

$$\varsigma_1 = f\varsigma_0,\ \varsigma_2 = f\varsigma_1 = f^2\varsigma_0,\ \varsigma_3 = f\varsigma_2 = f^3\varsigma_0, ..., \varsigma_n = f\varsigma_{n-1} = f^n\varsigma_0, \dots$$

Let, we denote $\Delta = d_\xi(\varsigma_0, \varsigma_1)$. Now, we assert that $\lim\limits_{n,m\to\infty} d_\xi(\varsigma_n, \varsigma_{n+1}) = 0_{\mathcal{A}}$. On setting $\varsigma = \varsigma_n$ and $\sigma = \varsigma_{n+1}$ in equation (1), we get

$$\begin{aligned}
d_\xi(\varsigma_n, \varsigma_{n+1}) &= d_\xi(f\varsigma_{n-1}, f\varsigma_n) = c^* d_\xi(\varsigma_{n-1}, \varsigma_n)c \\
&\preccurlyeq (c^*)^2 d_\xi(\varsigma_{n-2}, \varsigma_{n-1})c^2 \\
&\preccurlyeq \dots \\
&\preccurlyeq (c^*)^n d_\xi(\varsigma_0, \varsigma_1)c^n \\
&\preccurlyeq (c^*)^n \Delta c^n.
\end{aligned}$$

Now, we assert that $\{\varsigma_n\}$ is Cauchy sequence. For any $n, m \in \mathbb{N}$ such that $n < m$, we have

$$\begin{aligned}
d_\xi(\varsigma_n, \varsigma_m) &\preccurlyeq \xi(\varsigma_n, \varsigma_m)\big[d_\xi(\varsigma_n, \varsigma_{n+1}) + d_\xi(\varsigma_{n+1}, \varsigma_m)\big] \\
&\preccurlyeq \xi(\varsigma_n, \varsigma_m)d_\xi(\varsigma_n, \varsigma_{n+1}) + \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)d_\xi(\varsigma_{n+1}, \varsigma_{n+2}) + ... + \\
&\quad \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)\dots\xi(\varsigma_{m-2}, \varsigma_m)\xi(\varsigma_{m-1}, \varsigma_m)d_\xi(\varsigma_{m-1}, \varsigma_m) \\
&\preccurlyeq \xi(\varsigma_n, \varsigma_m)(c^*)^n\Delta c^n + \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)(c^*)^{n+1}\Delta c^{n+1} + ... + \\
&\quad \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)\dots\xi(\varsigma_{m-2}, \varsigma_m)\xi(\varsigma_{m-1}, \varsigma_m)(c^*)^{m-1}\Delta c^{m-1} \\
&= \xi(\varsigma_n, \varsigma_m)(c^*)^n\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}c^n + \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)(c^*)^{n+1}\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}c^{n+1} + ... + \\
&\quad \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)\dots\xi(\varsigma_{m-2}, \varsigma_m)\xi(\varsigma_{m-1}, \varsigma_m)(c^*)^{m-1}\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}c^{m-1} \\
&= \xi(\varsigma_n, \varsigma_m)(\Delta^{\frac{1}{2}}c^n)^*(\Delta^{\frac{1}{2}}c^n) + \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)(\Delta^{\frac{1}{2}}c^{n+1})^*(\Delta^{\frac{1}{2}}c^{n+1}) + ... + \\
&\quad \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)\dots\xi(\varsigma_{m-2}, \varsigma_m)\xi(\varsigma_{m-1}, \varsigma_m)(\Delta^{\frac{1}{2}}c^{m-1})^*(\Delta^{\frac{1}{2}}c^{m-1}) \\
&= \xi(\varsigma_n, \varsigma_m)|\Delta^{\frac{1}{2}}c^n|^2 + \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)|\Delta^{\frac{1}{2}}c^{n+1}|^2 + ... + \\
&\quad \xi(\varsigma_n, \varsigma_m)\xi(\varsigma_{n+1}, \varsigma_m)\dots\xi(\varsigma_{m-2}, \varsigma_m)\xi(\varsigma_{m-1}, \varsigma_m)|\Delta^{\frac{1}{2}}c^{m-1})|^2 \\
&= \sum_{i=0}^{m-1}|\Delta^{\frac{1}{2}}c^{n+i}|^2\prod_{j=0}^{i}\xi(\varsigma_{n+j}, \varsigma_m) \preccurlyeq \left\|\sum_{i=0}^{m-1}|\Delta^{\frac{1}{2}}c^{n+i}|^2\right\|\prod_{j=0}^{i}\xi(\varsigma_{n+j}, \varsigma_m) \\
&\preccurlyeq \sum_{i=0}^{m-1}\|\Delta\|\|c^{n+i}\|^2\prod_{j=0}^{i}\xi(\varsigma_{n+j}, \varsigma_m) \preccurlyeq \|\Delta\|\sum_{i=0}^{m-1}\|c^{n+i}\|^2\prod_{j=0}^{i}\xi(\varsigma_{n+j}, \varsigma_m),
\end{aligned}$$

Observe that, the above inequality is dominated by

$$\sum_{i=0}^{m-1}\|c^{n+i}\|^2\prod_{j=0}^{i}\xi(\varsigma_{n+j}, \varsigma_m) \preccurlyeq \sum_{i=0}^{m-1}\|c^i\|^2\prod_{j=0}^{i}\xi(\varsigma_j, \varsigma_m).$$

Now, by using the ratio test, we have

$$\lim_{i\to\infty}\frac{\|c^{i+1}\|^2\prod_{j=0}^{i+1}\xi(\varsigma_j, \varsigma_m)}{\|c^i\|^2\prod_{j=0}^{i}\xi(\varsigma_j, \varsigma_m)} \preccurlyeq \lim_{i\to\infty}\xi(\varsigma_i, \varsigma_m)\|c\|^2 \prec I.$$

Next, we say that (for all $m \geq 1$)

$$S_n = \sum_{i=0}^{n}\|c^i\|^2\prod_{j=0}^{i}\xi(\varsigma_j, \varsigma_m) \quad \text{and} \quad S = \sum_{i=0}^{\infty}\|c^i\|^2\prod_{j=0}^{i}\xi(\varsigma_j, \varsigma_m)$$

Consequently, we have

$$d_\xi(\varsigma_n, \varsigma_{n+p}) \preccurlyeq \|\Delta\|\|c^{2n}\|[S_{m-1} - S_n].$$

On making limit $n \to \infty$, we obtain that $\{\varsigma_n\}$ is a Cauchy sequence in A. Since, A is complete then there exists $a \in A$ such that

$$\lim_{n\to\infty} d_\xi(\varsigma_n, \varsigma) = 0_{\mathcal{A}}.$$

Now, we will show that a is a fixed point of f. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_\xi(f\varsigma, \varsigma) &\preccurlyeq \xi(f\varsigma, \varsigma)[d_\xi(f\varsigma, \varsigma_{n+1}) + d_\xi(\varsigma_{n+1}, \varsigma)] \\ &= \xi(f\varsigma, \varsigma)[d_\xi(f\varsigma, f\varsigma_n) + d_\xi(\varsigma_{n+1}, \varsigma)] \\ &\preccurlyeq \xi(f\varsigma, \varsigma)[c^* d_\xi(\varsigma, \varsigma_n)c + d_\xi(\varsigma_{n+1}, \varsigma)] \\ &\to 0_{\mathcal{A}} \text{ as } n \to \infty. \end{aligned}$$

Therefore, ς is a fixed point of f. For the uniqueness part, suppose that $\varsigma, \sigma \in A$ such that $fa = a$ and $f\sigma = \sigma$. Then by employing 3.1, we have

$$d_\xi(\varsigma, \sigma) = d_\xi(f\varsigma, f\sigma) \preccurlyeq c^* d_\xi(\varsigma, \sigma)c,$$

so that

$$\begin{aligned} \|d_\xi(\varsigma, \sigma)\| &= \|d_\xi(f\varsigma, f\sigma)\| \\ &\leq \|c^* d_\xi(\varsigma, \sigma)c\| \\ &\leq \|c^*\|\|d_\xi(\varsigma, \sigma)\|\|c\| \\ &= \|c\|^2\|d_\xi(\varsigma, \sigma)\| \\ &< \|d_\xi(\varsigma, \sigma)\| \end{aligned}$$

deals a contradiction. Hence, $\varsigma = \sigma$, that is, f has a unique fixed point. This completes the proof. □

Now, we furnish the following example which illustrates Theorem 3.1.

Example 3.3. *In Example 3.1, we define a map $f : A \to A$ by:*

$$f\varsigma = \frac{\varsigma}{5}, \text{ for all } \varsigma \in A.$$

Observe that, $d_\xi(f\varsigma, f\sigma) \preccurlyeq c^* d_\xi(\varsigma, \sigma)c$, (for all $\varsigma, \sigma \in A$) satisfies with

$$\rho = \begin{bmatrix} \frac{\sqrt{5}}{5} & 0 \\ 0 & \frac{\sqrt{5}}{5} \end{bmatrix} \in A \text{ and } \|\rho\| = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} < 1.$$

Thus, all the hypothesis of Theorem 3.1 are satisfied and $\varsigma = 0$ is unique fixed point of f.

Now, we obtain following corollaries:

Corollary 3.1. *Theorem 2.1 of Z. Ma et al.* [20] *is immediate from Theorem 3.1.*

Proof. By taking $\xi(\varsigma, \sigma) = 1$, for all $\varsigma, \sigma \in A$, we obtain required result, □

Corollary 3.2. *Theorem 2.1 of Z. Ma et al.* [21] *is immediate from Theorem 3.1.*

Proof. By taking $\xi(\varsigma, \sigma) = s$ (constant), for all $\varsigma, \sigma \in A$, we get required. □

4. **Application**

As an application of Theorem 3.1, we find the existence and uniqueness results for a type of following integral equation:

$$\varsigma(\mu) = \int_E G(\mu, \nu, \varsigma(\nu))d\nu + h(\mu), \;\; \mu, \nu \in E, \tag{2}$$

where E is a measurable set, $G : E \times E \times \mathbb{R} \to \mathbb{R}$ and $h \in L^\infty(E)$.

Let $A = L^\infty(E)$, $H = L^2(E)$ and $L(H) = \mathcal{A}$. Define $d_\xi : A \times A \to \mathcal{A}$ by (for all $h, k, I \in A$, $p \geq 1$ and $\|\rho\| = k < 1$):

$$d_\xi(h, k) = \pi_{|h-k|^p}$$

where $\pi_u : H \to H$ is the multiplicative operator defined by:

$$\pi_u(\psi) = u.\psi \;.$$

Now, define a mapping $\xi : A \times A \to \mathcal{A}$ by (for all $\varsigma, \sigma \in \chi$):

$$\xi(\varsigma, \sigma) = \begin{cases} \pi_{|h-k|^{p-1}} & \text{if } \varsigma \neq \sigma \\ I_{2\times 2} & \text{if } \varsigma = \sigma \end{cases}$$

where $I_{2\times 2}$ is an square identity matrix in $\mathcal{A}$ and $k > 0$ is a constant.
Note that, $(A, \mathcal{A}, d_\xi)$ is a complete C^*-avEbMS.

Now, we state and prove our result as follows:

Theorem 4.1. *Suppose that (for all $\varsigma, \sigma \in A$)*
(1) *there exist a continuous function $\psi : E \times E \to \mathbb{R}$ and $k \in (0, 1)$ such that*

$$| \; G(\mu, \nu, \varsigma(\nu)) - G(\mu, \nu, \sigma(\nu)) \; | \leq k \; | \; \psi(\mu, \nu)\big(\varsigma(\nu) - \sigma(\nu)\big) \; |,$$

for all $\mu, \nu \in E$.
(2) $\sup_{\mu \in E} \int_E | \; \psi(\mu, \nu) \; | \; d\nu \leq 1$.
Then the integral equation (2) *has a unique solution in A.*

Proof. Define $f : A \to A$ by:

$$f\varsigma(\mu) = \int_E G(\mu, \nu, \varsigma(\nu))d\nu + h(\mu), \;\; \forall \; \mu, \nu \in E.$$

Set $\rho = kI$, then $\rho \in \mathcal{A}$. For any $u \in H$ and $p \geq 1$, we have

$$\begin{aligned}
\|d_\xi(f\varsigma, f\sigma)\| &= \sup_{\|u\|=1} (\pi_{|f\varsigma - f\sigma|^p + I} u, u) \\
&= \sup_{\|u\|=1} \int_E \left[\left| \int_E G(\mu,\nu,\varsigma(\nu)) - G(\mu,\nu,\sigma(\nu)) d\nu \right|^p \right] u(\mu)\bar{u(\mu)} d\mu \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E \left| G(\mu,\nu,\varsigma(\nu)) - G(\mu,\nu,\sigma(\nu)) \right| d\nu \right]^p |u(\mu)|^2 d\mu \\
&\leq \sup_{\|u\|=1} \int_E \left[\int_E \left| k\psi(\mu,\nu)(\varsigma(\nu) - \sigma(\nu)) \right| d\nu \right]^p |u(\mu)|^2 d\mu \\
&\leq k^p \sup_{\|u\|=1} \int_E \left[\int_E |\psi(\mu,\nu)| d\nu \right]^p |u(\mu)|^2 d\mu \|\varsigma - \sigma\|_\infty^p \\
&\leq k \sup_{\mu \in E} \int_E |\psi(\mu,\nu)| d\nu \sup_{\|u\|=1} \int_E |u(\mu)|^2 d\mu \|\varsigma - \sigma\|_\infty^p \\
&\leq k \|\varsigma - \sigma\|_\infty^p \\
&= \|c\| \, \|d_\xi(\varsigma, \sigma)\|.
\end{aligned}$$

Since, $\|c\| < 1$, so it is verified that the mapping f meats all the requirements of Theorem 3.1. Hence, f has a unique fixed point, means that the Fredholm integral Equation (2) has a unique solution. □

5. Conclusions

As the C^*-avMS as well as C^*-avbMS are relatively new addition to the existing literature, therefore, we endeavor to further enrich this notion by introducing the idea of C^*-avEbMS wherein we replace the constant $s \geq 1$ by a function $\xi(\varsigma, \sigma)$. Our main result (i.e., Theorem 3.1) is an analogue of Banach contraction principle in C^*-avEbMS. An example is also adopted to highlight the realized improvements in our newly proved result. Finally, we apply Theorem 3.1 to examine the existence and uniqueness of solution for a system of Fredholm integral equation.

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