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CONSTRAINT FORCES COMPUTATION FOR DIAGONAL MASS MATRIX SYSTEMS USING GENERALIZED INVERSES

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Some of the fundamental issues of the constrained dynamics are briefly reviewed in the present work. Based on Gauss' least constraint principle a simplified and concise formulation of the constraint forces is developed. The solution is consistent for a wide class of problems. A comprehensive and straightforward algorithm for the computation of the constraint forces is described. The contact problem is solved by means of the generalized matrix inverse (GI). An application example relieves the efficiency of the method.

Keywords: constrained dynamics; non-smooth systems, generalized matrix inverse, dry friction.

1. Introduction

The evolution of the natural or technical systems is, in most cases, subjected to restrictions. The most common constraints are perceived as geometric limitations as, for instance, imposing the movement along a specific curve for a rigid body or require two bodies to remain a specified distance apart. The study of the constrained dynamics aims to connect Newton's laws and the geometric restrictions.

The first approach of the problem provided an approximate solution based on energy functions. Restrictions were imposed through rigid springs which accomplish a rough model of the constraint. In order to maintain a constraint, the spring constant should be large enough to produce large forces for small displacements. Such unbalance in the system parameters gives rise to stiff differential equations which are difficult or even impossible to integrate, see for instance [1].

Even more, the employment of additional energy in the system, known as penalty method, relates constraint forces with displacements. The displacements produced by applied forces act as signals that tell the constraint what restoring force is required. Therefore, apart from the stiffness introduced by the penalty constraints the system states will become subject to an erroneous drift, e.g., [1-3].

The alternative solution is to compute the forces required to maintain the constraints imposed, rather than using displacements and restoring forces. The function of these forces is to cancel the components of the applied forces which act against the constraints. Since forces produce acceleration, at this level

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constraint forces will assure the consistency between constraints and accelerations.

However, in order to this approach a series of problems has to be solved. In the rigid body assumption, constraint forces, i.e., normal or dry friction forces, are set-valued and do not fit in potential theory. Therefore, they have to be determined considering their relation with the motion of the dynamical system [4].

A major problem of the calculation methods applied to solve non-smooth systems is the uniqueness and existence of the solution. Indeterminacy and inconsistency have been observed in many planar rigid body problems, in the presence of dry friction. Some of them are known as “Painleve paradoxes”. The problem comes to a climax when the static friction forces (or other constraint types) are redundant. In this case, indeterminacy is attended by a rank-deficient constraint matrix.

Several computational methods have been developed, beginning with the regularization method which is completely determinate due to the “hard springs” added but brings the previously mentioned shortcomings. Fast and accurate methods using conditional statements to describe various modes of the system, as for instance the switch model [2], may not be used for a large number of contacts as the logical complexity increases in exponential rate.

Impulse velocity methods have been introduced and developed by [5]. Handling more than one impulse at a given moment was settled by [6]. The method proposed by [7] may even be used to model continuous contact. Also, in order to avoid inconsistent configurations, [8] solved the problem for impulsive forces and velocities (instead of forces and accelerations). Friction indeterminacy has also been approached as a probability problem by means of a statistical analysis of perturbed simulation results.

Convex analysis tools provide a general framework for problems involving set-valued laws. Inclusions may be solved using variational formulations, projections, proximations and associated optimization problems. Solutions are usually obtained iteratively solving projective equations by Jacobi or Gauss-Seidel algorithms, by quadratic programming or by solving complementarity problems. Detailed references may be found in [1, 4, 9].

In contact dynamics, models able to handle unilateral constraints have been extensively formulated as linear complementarity problem (LCP), e.g., [4, 5]. In the case of friction forces, complementarity constraints ensure that either static or kinetic friction is applied [10]. LCP solvers take advantage of the ample research works in mathematical programming. The point is that they allow the forces resolution through the complementarity equations and not by analyzing conditional statements.

It is emphasized, yet, that complementarity conditions and rigid body assumption confer a coherent mathematical framework but they may be still empirically corrected to improve their physical realism [5, 11] As available algorithms to solve complementarity problem based contact models are mainly linear, the friction cone has to be linearized by means of some special arrangement of constraints. A poor approximation is to the detriment of accuracy while a better one is computationally more expensive. Still, the friction forces negate the LCP convexity and the well-posedness of the problem may be assured, but only for small values of the friction coefficients. In a study regarding the validity of the complementarity conditions, [11] recalls possible physical inaccuracies behind the complementarity assumption and recounts that many authors [5, 8, 10] are aware of them.

It may thus be inferred that adequate methods should be tailored for each application [12]. Mathematically equivalent problems may be formulated in order to follow different computational procedures, e.g., the LCP formulated in the contact space.

The present work develops a method which does not involve complementarity conditions. The approach to the evaluation of the constraint forces is based on the Gauss' principle of least constraints and on the notion of generalized inverse (GI) of matrices. Practical applications of this method appear to be very sparse. Being very intuitive, the principle is sometimes used without even being mentioned [13, 14]. However, a slight resurgence of the subject may be noticed and applications may be found not only in physics and related domains but also in economy. A general outline of such method may be found in [15, 16] and related research is presented in [8, 6, 11, 14, 17-20]. The relations deduced assume a diagonal mass matrix, but a large number of interesting models studied in engineering and physics observe this condition.

2. A general solution for Constrained Dynamics

The constrained dynamics of mechanical multibody systems may be described by a Lagrangian equation embedding additional algebraic variables λ (Lagrangian multipliers) which define the motion constraints.

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{h}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - \sum_{i \in \boldsymbol{I_C}} w_i \lambda_i = 0 \quad (1)$$

The multipliers λ, usually forces or torques, are defined by set-valued laws and the constraint vectors $w \in \mathbb{R}^n$ specify how they are applied; $M \in \mathbb{R}^{n \times n}$ is the inertia matrix, $q \in \mathbb{R}^n$ the generalized coordinates vector and $h \in \mathbb{R}^n$ is the sum of the forces described by constitutive laws, e.g., viscous damping or elasticity.

The action of the constraint forces over the systems component masses is defined by the constraint matrix $W \in \mathbb{R}^{n \times m}$ ($W^T = J$, the Jacobian) which may be obtained by assembling the constraint vectors w. Using the the column vector of the constraint forces, $\lambda \in \mathbb{R}^m$ the equation may be written as:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{h}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{W}\boldsymbol{\lambda} = 0 \quad (2)$$

In order to introduce the formulation deduced by [15] we shall denote by

$$\boldsymbol{a} = \boldsymbol{M}^{-1}\boldsymbol{h}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{3}$$

the unconstrained acceleration of the system in the absence of the constraints, which results from the unconstrained equation of motion

$$\boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{h}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \tag{4}$$

Motion constraints may be expressed in a general form as

$$\boldsymbol{W}^T(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\ddot{\boldsymbol{q}} = \boldsymbol{b}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{5}$$

where $\boldsymbol{W} \in \mathbb{R}^{n\times m}$ is a generalized constrained matrix, the constraints expressions $\boldsymbol{b} \in \mathbb{R}^{m}$, and they may be ensured if additional generalized constraint forces $\boldsymbol{W\lambda}$ are applied.

Taking into account the Equations of constrained motion (1) and Equations (3-5), the constrained system may be described by the following differential-algebraic (DAE) equations. Because the equations use more coordinates than the underlying system, Equation (6) is known as the redundant coordinate system.

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{W} \\ \boldsymbol{W}^T & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{b} \end{bmatrix} \tag{6}$$

According to Gauss' principle the system's accelerations must fulfil sufficient optimality conditions for the "acceleration energy" which is a convex function and grants existence and uniqueness of the solution. The principle asserts that the values of the accelerations of a system subjected to constraints are the closest possible to the accelerations of the unconstrained system and, consequently, the constraints take the minimum possible values. Therefore, the resulting acceleration $\ddot{q}$ minimizes the function G over the set which satisfies the constraint Equation (5).

$$\min G(\ddot{\boldsymbol{q}}) = \frac{1}{2}(\ddot{\boldsymbol{q}} - \boldsymbol{a})^T \boldsymbol{M}(\ddot{\boldsymbol{q}} - \boldsymbol{a}) = \frac{1}{2}\|\ddot{\boldsymbol{q}} - \boldsymbol{a}\|_{\boldsymbol{M}}^2 \tag{7}$$

A similar minimum condition may be imposed for the constraints λ. This development is described by [15]. The expressions of $\ddot{q}$ and λ may be obtained formally inverting the matrix

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{W} \\ \boldsymbol{W}^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{M}^{-1} - \boldsymbol{M}^{-1}\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{W}^T\boldsymbol{M}^{-1} & \boldsymbol{M}^{-1}\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1} \\ (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}\boldsymbol{W}^T\boldsymbol{M}^{-1} & -(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1} \end{bmatrix} \tag{8}$$

which yields the generalized constraint accelerations and forces as

$$\ddot{\boldsymbol{q}} = \boldsymbol{a} + \boldsymbol{M}^{-1}\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{9}$$

$$\boldsymbol{\lambda} = (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{10}$$

Considering the substitution $J_M^T = M^{-\frac{1}{2}}W$ Equations (9) and (10) may be written as

$$\ddot{\boldsymbol{q}} = \boldsymbol{a} + \boldsymbol{M}^{-\frac{1}{2}}\left(\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{W}\right)\left(\boldsymbol{W}^T\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{W}\right)^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \\ = \boldsymbol{a} + \boldsymbol{M}^{-\frac{1}{2}}J_{\boldsymbol{M}}^T(J_{\boldsymbol{M}}J_{\boldsymbol{M}}^T)^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{11}$$

$$\boldsymbol{W\lambda} = \boldsymbol{M}^{\frac{1}{2}}\left(\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{W}\right)\left(\boldsymbol{W}^T\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{W}\right)^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \\ = \boldsymbol{M}^{\frac{1}{2}}J_{\boldsymbol{M}}^T(J_{\boldsymbol{M}}J_{\boldsymbol{M}}^T)^{-1}(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{12}$$

Even though the matrices involved in the above relations may not be invertible, the minimum condition expressed by the Gauss principle defines a unique solution. Yet the GI may be used for least square problems or for minimum norm solutions of linear systems [21, 22]. Therefore, employing the GI $J_M^+ = J_M^T(J_M J_M^T)^{-1}$ the constraint acceleration and force which satisfy Gauss' principle are explicitly given in Equations (13) and (14). Formulation simplicity and concision is the main argument for this approach.

$$\ddot{\boldsymbol{q}} = \boldsymbol{a} + \boldsymbol{M}^{-\frac{1}{2}}J_{\boldsymbol{M}}^+(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{13}$$

$$\boldsymbol{W\lambda} = \boldsymbol{M}^{\frac{1}{2}}J_{\boldsymbol{M}}^+(\boldsymbol{b} - \boldsymbol{W}^T\boldsymbol{a}) \tag{14}$$

2.1 A simplified formulation

While the above relations provide a general solution, further simplifications may be operated for particular sets of constrained systems. Hence if the inertia matrix M is diagonal, constraint λ, considering Equation (10), may be written as:

$$\boldsymbol{\lambda} = (\boldsymbol{W}^T\boldsymbol{W})^{-1}(\boldsymbol{Mb} - \boldsymbol{W}^T\boldsymbol{Ma}) \tag{15}$$

In the case of constraints which impose null relative accelerations, Equation (11 may be stated as:

$$\ddot{\boldsymbol{\xi}} = \boldsymbol{W}^T\ddot{\boldsymbol{q}} = \boldsymbol{W}^T\boldsymbol{a} + (\boldsymbol{W}^T\boldsymbol{M}^{-1}\boldsymbol{W})\boldsymbol{\lambda} = \boldsymbol{b} = 0 \tag{16}$$

Bearing in mind the development accomplished in the previous paragraphs, the constraint forces T should fulfil the minimum condition imposed to Equation (10) for the derived Equation (15) which gives

$$\boldsymbol{\lambda} = -(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{W}^T\boldsymbol{h} = -\boldsymbol{W}^+\boldsymbol{h} \tag{17}$$

This result may be regarded as an expression of the fact that sticking contacts of point masses eliminate relative motion between the elements in question and, therefore, the contact space Equation (16), subjected to the constraint λ becomes a static equilibrium.

Beside the concision of this formulation, Equation (17) improves the numerical integration efficiency. The computation of the unconstrained acceleration is not necessary. An important step may be removed from the method which shortens computation time and diminishes round-off errors.

It should be noted that there is a wide class of constrained systems which fulfil the conditions required by the Equation (17) and, in consequence, they may be approached as described above. Beyond its limpid formulation and solution consistency liability, the numerical algorithm based on the above reasoning requires minimal additional programming and, practically, no acquaintance with optimization theory.

3. A particularization: dry friction forces

The research of the phenomenon emphasized that the models of the dry friction, T, should be different for static friction and for kinetic friction [23, 24]. In the case of the kinetic friction, the slip force opposes the relative movement between the contact points (which may differ from the relative movement between the bodies). The stick force opposes to the resultant of all the other exerted forces (including inertia force, if applicable) and takes any value from zero up to the maximum stick force, T_{sMax}. It follows that the set-valued expression of the force law may be written as

$$T = \begin{cases} T_k(t, \xi_k, \dot{\xi}_k), & \dot{\xi}_k \neq 0, \quad slip \\ T_s(\Sigma F) \in [-T_{\boldsymbol{sMax}} \quad T_{\boldsymbol{sMax}}], & \dot{\xi}_s = 0, \quad stick \end{cases} \tag{18}$$

The indexes k and s are used for kinetic and static values, respectively, while ξ, $\dot{\xi}$ and $\ddot{\xi}$ denote the relative displacement, velocity and acceleration. The static friction force is a function of the applied forces and it exactly cancels them if T_s is smaller than the limit value T_{sMax}; otherwise, its limit value opposes to the external forces, ΣF, but a slip phase will begin. The following index sets may be used to define the system mode [9]:

$$\begin{aligned} \boldsymbol{I}_c &= \{1,2,\dots,m\} \\ \boldsymbol{I}_k &= \{k \in \boldsymbol{I}_c | \dot{\xi}_k \neq 0\} \\ \boldsymbol{I}_s &= \{s \in \boldsymbol{I}_c | \dot{\xi}_s = 0\} \end{aligned} \tag{19}$$

The set I_c contains the indexes of all the m contact points, while the complementary subsets I_k and I_s contain the indexes of the sliding contacts and, respectively, those of the potentially sticking contacts.

The metamorphism of the dry friction force may be summarized as it follows: When the other applied forces overcome the threshold value, the motion would commence. At this very moment, the value of the friction force is T_{SMax}, and its orientation opposes ΣF. Further, the kinetic friction force opposes relative velocity. There have been proposed many laws for its value - see for instance [23, 24]. Practical applications use Coulomb models with T_k depending on the relative velocity, such as the Stribeck curves Fig. 1. Hence, a general description of friction may be of the form:

$$\boldsymbol{T} = \begin{cases} \boldsymbol{T}(\boldsymbol{I}_s) = -\Sigma F sign(\Sigma F), & \dot{\xi}_s = 0, |\Sigma F| \leq T_{\boldsymbol{sMax}} \Leftrightarrow \ddot{\xi}_s = 0 \\ \boldsymbol{T}(\boldsymbol{I}_k) = \begin{cases} -T_{\boldsymbol{sMax}} sign(\Sigma F), & \dot{\xi}_k = 0, |\Sigma F| > T_{\boldsymbol{sMax}} \Leftrightarrow \ddot{\xi}_k \neq 0 \\ -T(\dot{\xi}_k) sign(\dot{\xi}_k), & \dot{\xi}_k \neq 0 \end{cases} \end{cases} \quad (20)$$

Such a friction law does not explicitly specify the friction force at zero velocity; the static force counteracts the external resultant below its maximum value and thus keeps objects in contact not to move relative to each other.

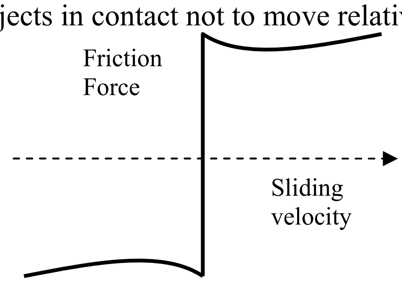

Fig. 1 Friction model

While transitions from stick to slip may not occur until T_{sMax} is reached, during slip to stick transitions, when $\dot{\xi}$ vanishes, the force value may jump from T_{sMax} anywhere in the set $[-T_{sMax} \quad T_{sMax}]$. This feature reveals discontinuity and hysteretic behaviour shrunk to one point, $\dot{\xi} = 0$, [4]. The vertical line drawn for $\dot{\xi} = 0$ expresses the fact that the static friction force may take any value within that set while the relative velocity remains unchanged. However, a unique solution may be found in most cases, as these multi-valued laws are used in conjunction with the equations of motion of the dynamical systems.

3.1 Static friction forces computation with the matrix GI

It is obvious that Equation (17) holds only for static friction forces. However, the system resolution may be done, considering kinetic frictions as applied forces, given by Equation (18). Such a method implies to change constraint matrix W according to the system mode. In the following, an algorithm to compute the static friction forces is developed.

The underlying idea is that the GI may be used for least square problems or for minimum norm solutions of linear systems. The aim of the model is to determine the static friction forces as the others, including kinetic friction forces, are functions of the system states or time and they are known. Therefore, the optimization problem may be formulated only for the static friction which occurs only in sticking contacts, i.e., when the relative velocity and acceleration vanishes.

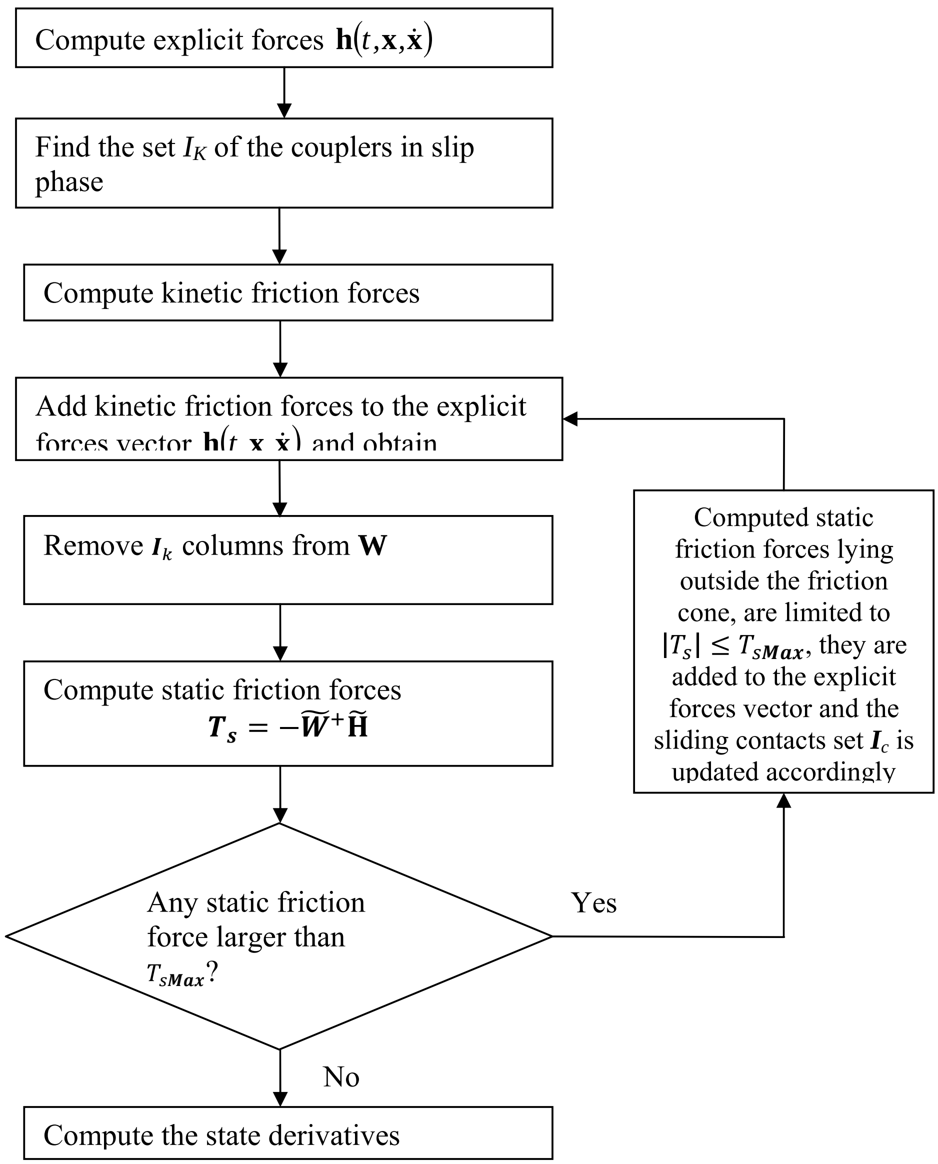

Fig. 2. Flowchart of the static friction forces computation scheme

If any of the computed forces are larger than the limit value, a switch from stick to slip occurs at the given contact. The force value is limited to the maximum stick force (but the force direction doesn't change). The mode configuration, and correspondingly, W and h matrices are modified. Constraints are computed iteratively until all of them lie inside the friction cone. The flowchart of the numerical scheme is pictured in Fig. 2.

The above procedure should be performed at each time step before the state vector is updated. Some particular tasks must be accomplished. The computing of the static forces may necessitate iteration. The decision to reiterate the computation may be taken by analyzing the I_k index set. Another specific task is to adjust the constraint matrix $\widetilde{W}$ and the known forces vector $\tilde{h}$ in accordance to the provisional sticking contacts.

As implied by Equation (17), static friction forces computation may be alternatively formulated either as $T_s = -(W^T W)^+ W^T h$ or $T_s = -W^+ h$. In the first case, both lines and columns corresponding to sliding contacts I_k are to be removed, and also the elements I_k from the vector $W^T h$. Even if mathematically equivalent, formulations and numerical techniques employed prove to be different from the computational perspective [16, 17, 25].

4. Application example: two blocks on a belt

The example below brings forward the benefits of using the GI algorithm for the computation of the constraint forces. It is based on the block on a belt system, a classical model in non-smooth dynamics, e.g., [2]. As pictured in Fig.2 the system comprises two masses laying on a belt which moves with a constant velocity v. The model parameters are given in Table 1. Both springs have the same stiffness.

Table 1

Parameter values for the two blocks on a belt model

Parameter	Value	Description	Units
m_1	4	Mass of block 1	kg
m_2	7	Mass of block 2	kg
T_{1Max}	4	Static friction limit	N
T_{2Max}	7	Static friction limit	N
T_{3Max}	5	Static friction limit	N
k	14	Springs stifness	N/m
v	0.3	Velocity of the belt	m/s

Three contact points where constraint forces may be acting. Between masses and belt, dry friction occurs, with friction forces T_1 and T_2 , respectively. Dry friction, T_3, is also present between the masses. These three forces become redundant when they all occur in stick phases.

The constraint forces which occur in the system studied are the friction forces, i.e., $\lambda = [T_1\ T_2\ T_3]^T$ and the constraint matrix is given in Equation (21a). The kinetic friction dependence on the relative velocity is given in Equation (21b), in order to allow alternative stick and slip phases for this particular system. The vector of the constitutive elastic forces is $h = [k \cdot (2q_1 - q_2),\ -k \cdot (q_1 - q_2)]^T$, where q_1 and q_2 are the displacements of the blocks.

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \tag{21a}$$

$$\boldsymbol{T_k} = -\frac{\boldsymbol{T_{SMax}} \cdot \boldsymbol{sgn}\left(\dot{\xi}\right)}{\left(1 + 3 \cdot \left|\dot{\xi}\right|\right)} \tag{21b}$$

The blocks velocity versus time is plotted in Fig. 3. Alternative stick and slip phases are recorded in the contacts where friction occurs. The sticking phases of the blocks on the belt are set apart by constant velocities (v = 0.3m/s). When the two blocks stick together, their velocities are equal. During simulation, relative rest occurs either simultaneously in all the three contact points or individually.

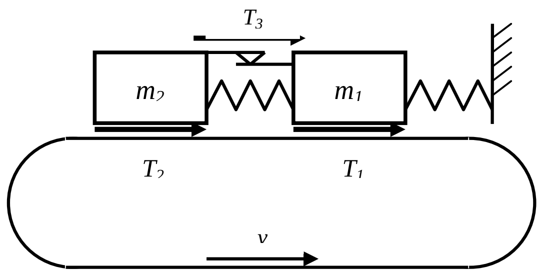

Fig. 3. Two blocks on a belt

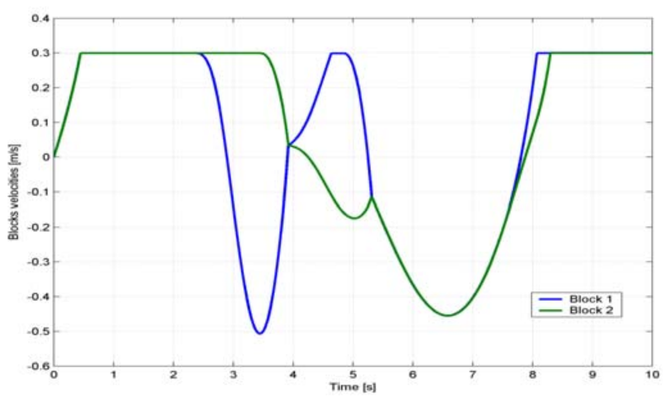

Fig. 4. Block velocities versus time

The plot of the friction forces, Fig. 4, depicts the algorithm ability to compute redundant static friction forces and presents specific features. The most striking aspects of their evolution are the step discontinuities which point out the slip to stick transitions, Fig. 5, or the inversion of the relative motion in the contact point. In the latter case the gap of the friction force is two times maximum stick force,

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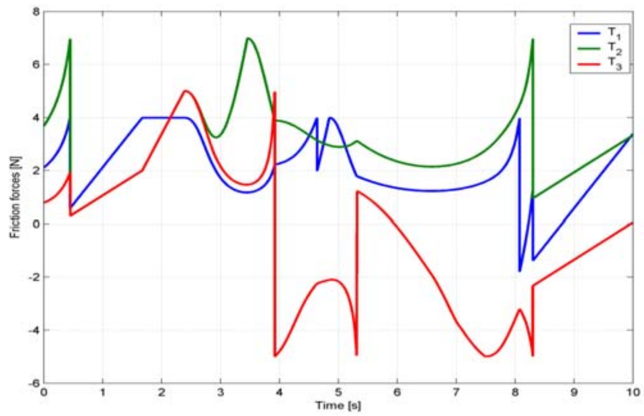

Fig.5 Friction forces versus time

Before the relative movement switches from slip, the friction attains its limit value T_{sMax} and, when the stick begins it sharply changes to equilibrate the other forces resultant ΣF which should be smaller than the maximum stick force to allow sticking; therefore, the friction gap will be smaller than $2T_{sMax}$.

5. Conclusions

A simplified equation of the constraint forces is developed in the present work. The solution may be applied for diagonal mass systems where constraints impose null relative accelerations in the contact points. A large number of important technical and physical models observe these conditions. The formulation deduced allows a convenient computer implementation, taking into account that mathematical packages contain GI computation subroutines. But its most remarkable features are that redundant sets of equations may be solved as systems of ordinary differential equations, degree of freedom changes do not involve any additional programming in the solution formulation, rheonomic and scleronomic constraints or forward and inverse dynamics are handled in a unified manner.

Based on Gauss' least constraint principle the matrix GI is employed to compute constraint forces. The formulation is far more concise than the general solution. Algorithms which determine matrices GIs use robust iterative schemes and do not have specific limitations for the input size. Hence the method may be used for models with a large number of degrees of freedom and contact points. Mathematical packages include routines for the pseudoinverses calculus, so they may be easily implemented in ordinary differential equations solvers. Some improvements regarding computational efficiency are also mentioned related to the GI employment, i.e., [11, 16, 21, 25, 26]. At last, the GI algorithm is simple, gives an intuitive description of the constrained system and may be applied without detailing the optimization methods.

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