SECOND ORDER DIFFERENTIAL INCLUSION WITH UNBOUNDED NONCONVEX MOVING SETS

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We prove, in this work, an existence result for a class of second order nonconvex sweeping processes depending on both time and state, subject to unbounded external forces known as perturbations, in infinite dimensional Hilbert spaces. The approach is based on Moreau's catching-up algorithm. In order to deal with a large class of unbounded nonconvex sets, we assume a truncated condition. An application for a class of quasi-variational inequality is given.

Keywords: Moreau's sweeping process, second order differential inclusion, prox regular sets, truncation

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1. Introduction

The study of "perturbed sweeping process" corresponds to the solving of the following differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + F(t, u(t)) \quad \text{a.e. in } [0, T], \\ \qquad u(t) \in C(t), \;\; u(0) = u_0 \in C(0) \end{cases}$$

where $N_{C(t)}(u(t))$ represents the normal cone to the nonempty closed moving set C and $F : [0, T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a perturbation. This kind of problems corresponds to several important mechanical problems, planning procedures in mathematical economy and nonsmooth dynamical systems. It was introduced by J. J. Moreau in the case where the sets $C(t)$ are assumed to be convex and without any perturbation ($F \equiv 0$) to study quasi-static evolution in elastoplasticity and friction dynamics. The approach used is a discretization scheme based on the catching-up algorithm. When external forces (perturbations) are applied to the system described by the sweeping process, the problem found many applications in resource allocation in economics, nonregular electrical circuits and crowd motion modeling.

The second-order sweeping process has been also considered by many authors. In [5], Castaing studied for the first time the case where the moving set depends on the state with convex compact values. Since then, various generalizations have been obtained, see e.g. [2, 4, 6, 19] and the references therein. When the moving sets C depends on time and state, one obtain a generalization of the classical sweeping process known as the state-dependent sweeping process. Such problems are motivated by parabolic quasi-variational inequalities arising e.g. in the evolution of sandpiles, and occur also in the treatment of 2-D or 3-D

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quasistatic evolution problems with friction, as well as in micro-mechanical damage models for iron materials with memory to describe the evolution of the plastic strain in presence of small damages. We refer to [13] for more details. This problem have been studied for the first time for convex sets $C(t,u)$ by [8] in $\mathbb{R}^3$, then by [14] in Hilbert spaces under some compactness condition. In [7] provided an other approach to prove the existence when $F \equiv \{0\}$ and $C(t,u(t))$ is prox regular and ball-compact, and for the perturbed problem (even in presence of a delay). They considered the possibly unbounded moving sets satisfying the classical Lipschitz continuity assumption with respect to Hausdorff distance. However, as pointed out by Tolstonogov [18], it is difficult for unbounded sets to hold this assumption since the Hausdorff distance of two unbounded sets may equal the infinity, for example, the case of rotating hyperplane. Recently, [1] proposed an implicit discretization scheme based on the Moreau's catching-up algorithm with different techniques to analyze the second-order sweeping processes under perturbation in Hilbert spaces. The moving set depends on the time, the state and is possibly unbounded. The set is supposed to be closed, convex and to have some Lipschitz variation. The perturbation force is supposed to be upper semicontinuous with convex and weakly compact values and satisfies a weak linear growth condition.

In this paper, we are mainly interested in the study of the existence of solution for the following perturbed second-order nonconvex sweeping process:

$$(\mathcal{P})\begin{cases}\ddot{u}(t) \in -N_{C(t,u(t))}(\dot{u}(t)) - F(t,u(t),\dot{u}(t)), \quad \text{a.e.} \quad t \in [0,T];\\ u(0)=u_0, \dot{u}(0)=v_0 \in C(0,u_0),\end{cases}$$

where C is an unbounded prox-regular set, and F is a nonempty closed convex set-valued mapping, scalarly upper semicontinuous and for some real $\alpha > 0$ we have $d_{F(t,u,v)}(0) \leq \beta(1+\|u\|+\|v\|)$, for all $t \in [0,T]$ and u, $v \in H$ with $v \in C(t,u)$. We extend the results obtained by [1] by adapting the implicit discretization scheme to the nonconvex case. The standard Lipschitz (or absolutely continuous) assumption is replaced by a truncated one, in order to deal with a large class of unbounded sets.

The paper is organized as follows: Section 2 is devoted to basic notions and some fundamental results in nonsmooth analysis needed in the proof and in Section 3, we will give our main existence. We present in Section 4, an application to a class of quasi-variational inequalities.

2. Notations and Preliminaries

In all the paper, H is a real separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The unit closed (resp., open) ball in H is denoted by $\overline{\mathbb{B}}$ (resp., $\mathbb{B}$). For a nonempty subset S of H, the support function and the distance function are defined respectively by

$$\sigma(x,S) := \sup_{y\in S}\langle x,y\rangle, \quad d_S(x) := \inf_{y\in S}\|x-y\| \quad \textit{for all } x \in H.$$

For all $x \in H$, the set of the nearest points of S to x is defined by

$$\mathrm{Proj}_S(x) := \{y \in S : d_S(x) = \|x-y\|\}.$$

Following [3], we define, for any $\rho \in]0,+\infty]$, the Hausdorff ρ-truncated distance between two nonempty subsets S and S' of H by:

$$\mathcal{H}_\rho(S,S') = \max\{exc_\rho(S,S'), exc_\rho(S',S)\},$$

where $exc_\rho(S, S') = \sup_{x\in S\cap\rho\overline{\mathbb{B}}} d_{S'}(x)$ is the ρ-truncated pseudo excess of S over S'. Moreover, the following inequality holds $\mathcal{H}_\rho(S, S') \le \hat{\mathcal{H}}_\rho(S, S')$, where $\hat{\mathcal{H}}_\rho(S, S') = \sup_{x\in\rho\overline{\mathbb{B}}} |d_S(x) - d_{S'}(x)|$. By convention we set $\rho\overline{\mathbb{B}} = H$ if $\rho = +\infty$. Note that

$$d_{S'}(x') \le d_{S\cap\rho\overline{\mathbb{B}}}(x') + exc_\rho(S, S'), \forall x' \in H. \tag{1}$$

The closed convex hull is denoted by $\overline{\text{co}}(S)$, and satisfies

$$\overline{\text{co}}(S) = \{x \in H, \forall \zeta \in H : \langle \zeta, x\rangle \le \sigma(\zeta, S)\}, \quad \text{for any} \quad S \subset H.$$

2.1. Subgradients and cones

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $x \in H$ be any point where $f(x)$ is finite.

- The Clarke normal subdifferential $\partial f(x)$ of f at x is the set of all $\xi \in H$ such that $\langle \xi, h\rangle \le f^\uparrow(x, h)$, $\forall h \in H$, where $f^\uparrow(x, h)$ is the upper subderivative of f at x with respect to h defined by:

$$f^\uparrow(x, h) = \limsup_{x'\to_f x, t\downarrow 0} \inf_{h'\to h} \frac{f(x' + th') - f(x')}{t},$$

  with $x' \to_f x \Leftrightarrow x' \to x$ *and* $f(x') \to f(x)$.
- The proximal subdifferential $\partial^p f(x)$ of f at x is the set of all $\xi \in H$ for wich there exists δ, $\lambda \ge 0$ such that, for all $x' \in x + \delta\overline{\mathbb{B}}$

$$\langle \xi, x' - x\rangle \le f(x') - f(x) + \lambda\|x' - x\|^2.$$

If f is not finite at x, we have the convention $\partial f(x) = \partial^p f(x) = \emptyset$. We always have $\partial^p f(x) \subset \partial f(x)$. Note that, if f is locally Lipschitz around x, then the upper derivative of f at x with respect to h coincides with the Clarke derivative $f^\circ(x, h)$ defined by $f^\circ(x, h) = \limsup_{x'\to x, t\downarrow 0} \frac{f(x' + th) - f(x')}{t}$.

Now let S be a nonempty closed set, the Clarke normal cone (resp., the proximal normal cone) of S at x is the Clarke subdifferential (resp., the proximal subdifferential) of the indicator function of the set S.

On can also defined $\xi \in N^p_S(x)$ (see [10]) by the existence of $\lambda > 0$ such that

$$\langle \xi, x' - x\rangle \le \lambda\|x' - x\|^2 \quad \forall x' \in S.$$

2.2. Prox regular sets

In this paper, we are interested in studying the existence of solutions for a variant of Moreau's sweeping process for prox regular set. Let us give some basic results about the prox regularity (see [9, 17]).

Definition 2.1. *Let $r > 0$ and S be a closed subset of H, we say that S is $r-$prox regular, if for any $x \in S$ and for any positive number $\xi \le r$, every proximal normal vector v to S at x with $\|v\| \le 1$ can be realized by a $\xi-$ball, which can be translated into the fact that for all $x \in S$ and $v \in N^p_S(x)$ with $\|v\| \le 1$, one has $x \in \text{Proj}_S(x + \xi v)$.*

We present now some properties concerning the prox regularity.

Proposition 2.1. *Let S be r-prox regular, then the following assertions hold:*

(a) *For all x, $x' \in S$ and $v \in N^p_S(x)$ we have $\langle v, x' - x\rangle \le \frac{1}{2r}\|v\|\|x' - x\|^2$.*

(b) *For any $x \in S$ we have $N^p_S(x) = N_S(x)$ and $\partial^p d_S(x) = \partial d_S(x)$.*

(c) *For any* $x \in H$ *such that* $d_S(x) < r$, *the set* $\mathrm{Proj}_S(x)$ *is a singleton.*

We refer the reader for more details on prox regularity to the survey [11].
The next proposition will give us a partial upper semi-continuity property. It is a variant of [15, Proposition 3.2].

Proposition 2.2. *Let* $r > 0$, *for any* $(t, x) \in [0, T] \times H$, $C(t, x)$ *be a nonempty closed subset of* H *which is r-prox regular. Assume that there exist* $\rho \in]0, +\infty]$ *and* $L \in [0, +\infty[$ *such that* $\mathcal{H}_\rho(C(\tau, x), C(t, y)) \leq L\Big(|\tau - t| + \|x - y\|\Big)$, *for all* τ, $t \in [0, T]$. *Let* $\overline{t} \in [0, T]$, $\overline{x} \in H$ *and* $\overline{y} \in C(\overline{t}, \overline{x}) \cap \rho\mathbb{B}$, $(t_n)_{n\in\mathbb{N}}$ *be a sequence from* $[\overline{t},\ T]$ *with* $t_n \to \overline{t}$, $(x_n)_{n\in\mathbb{N}}$ *be a sequence of* H *with* $x_n \to \overline{x}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be a sequence of* H *with* $y_n \to \overline{y}$ *and* $y_n \in C(t_n, x_n)$ *for all* $n \in \mathbb{N}$, *then, we have* $\limsup\limits_{n\to\infty} d^\circ_{C(t_n,x_n)}(y_n; h) \leq d^\circ_{C(\overline{t},\overline{x})}(y; h)$ *for all* $h \in H$.

3. Main Result

In this section, we are going to prove the main result of this paper.

Theorem 3.1. *let* $C : [0, T] \times H \rightrightarrows H$ *and* $F : [0, T] \times H \times H \rightrightarrows H$ *) two set valued mappings with nonempty closed values satisfying:*

($\mathscr{A}_F$) *F is convex and scalarly upper semicontinuous (i.e., for each* $y \in H$, *the function* $(t,\ u,\ v) \mapsto \sigma(y,\ F(t,\ u,\ v))$ *is upper semicontinuous) and for some real* $\beta > 0$

$$d_{F(t,u,v)}(0) \leq \beta(1 + \|u\| + \|v\|),\ \forall t \in [0, T]\ u,\ v \in H \text{ with } v \in C(t, u).$$

($\mathscr{A}_{C_1}$) *There is a constant* $r > 0$ *such that, for each* $t \in [0, T]$ *and each* $u \in H$, *the set* $C(t, u)$ *is r-prox regular.*

($\mathscr{A}_{C_2}$) *There exist a real* $L > 0$ *and an extended real* $\rho \geq \{\|v_0\|+\|u_0\|+2T(L+2\beta)\}e^{T(1+L+2\beta)}$ *such that for every* t, $s \in [0, T]$, x, $y \in H$

$$\mathcal{H}_\rho(C(t, x), C(s, y)) \leq L\Big(|t - s| + \|x - y\|\Big). \tag{2}$$

($\mathscr{A}_{C_3}$) *For any bounded subset* A *of* H *the set* $C(t, A)$ *is ball compact, i.e., the intersection of* $C([0, T] \times A)$ *with any closed ball of* H *is compact.*

Then, the problem $(\mathcal{P})$ *admits an absolutely continuous solution* $u : [0,\ T] \to H$.

Proof. Let $f(t, u, v)$ be the element of minimal norm of the closed convex set $F(t, u, v)$ of H defined as follows $f(t,\ u,\ v) = \mathrm{Proj}_{F(t,u,v)}(0)$. By ($\mathscr{A}_F$) we have

$$\|f(t,\ u,\ v)\| \leq \beta(1 + \|u\| + \|v\|). \tag{3}$$

Fix $n_0 \geq 1$ satisfying

$$\frac{(\rho + 1)(\beta + L)T}{n_0} < \frac{r}{2}. \tag{4}$$

Step 1. Consider for every $n \geq n_0$, a partition of $I = [0, T]$ defined by $t^n_i := ih_n$ for $0 \leq i \leq n$, where $h_n = \frac{T}{n}$. Put $u^n_0 = u_0$, $v^n_0 = v_0 \in C(t_0, u_0)$ and take $u^n_1 = u^n_0 + h_n v^n_0$. We construct u^n_0, $u^n_1, \ldots\ldots, u^n_n$ and v^n_0, $v^n_1, \ldots\ldots, v^n_n$ in H such that for each $i = \overline{0,\ n - 1}$, the following assertions hold:

$$v^n_{i+1} \in \mathrm{Proj}_{C(t^n_{i+1},u^n_{i+1})}\left(v^n_i - h_n f(t^n_i, u^n_i, v^n_i)\right). \tag{5}$$

$$v^n_{i+1} \in C(t^n_{i+1}, u^n_{i+1}) \text{ with } u^n_{i+1} = u^n_i + h_n v^n_i. \tag{6}$$

$$\|u^n_{i+1}\| + \|v^n_{i+1}\| \leq \rho. \tag{7}$$

Indeed, from (1), ($\mathscr{A}_{C_2}$), (3) and (4) one has

$$d_{C(t^n_1,u^n_1)}(v^n_0 - h_n f(t^n_0, u^n_0, v^n_0)) \leq d_{C(t^n_1,u^n_1)}(v^n_0) + h_n\|f(t^n_0, u^n_0, v^n_0)\|$$

$$\le d_{C(t_0^n,u_0^n)\cap\rho\overline{\mathbb{B}}}(v_0^n) + exc_\rho(C(t_0^n,u_0^n),C(t_1^n,u_1^n)) + h_n\|f(t_0^n,u_0^n,v_0^n)\|$$
$$\le exc_\rho(C(t_0^n,u_0^n),C(t_1^n,u_1^n)) + h_n\|f(t_0^n,u_0^n,v_0^n)\|$$
$$\le L(|t_1^n - t_0^n| + \|u_1^n - u_0^n\|) + h_n\beta(1+\|u_0^n\|+\|v_0^n\|)$$
$$\le h_n\beta + Lh_n(1+\|v_0^n\|) + h_n\beta(\|u_0^n\|+\|v_0^n\|) \le h_n(\beta+L)(1+\rho) < \frac{r}{2} < r.$$

Using the prox-regularity of the set $C(t_0^n, u_0^n)$, we conclude that the mapping $\mathrm{Proj}_{C(t_1^n,u_1^n)}(v_0^n - h_nf(t_0^n,\, u_0^n,\, v_0^n))$ is well-defined, so we can find a point $v_1^n \in C(t_1^n,u_1^n)$ such that $v_1^n \in \mathrm{Proj}_{C(t_1^n,u_1^n)}(v_0^n - h_nf(t_0^n,\, u_0^n,\, v_0^n))$, and

$$\|v_1^n - v_0^n + h_nf(t_0^n,u_0^n,v_0^n)\| \le d_{C(t_1^n,u_1^n)}(v_0^n - h_nf(t_0^n,u_0^n,v_0^n))$$
$$\le d_{C(t_1^n,u_1^n)}(v_0^n) + h_n\|f(t_0^n,u_0^n,v_0^n)\|$$
$$\le d_{C(t_0^n,u_0^n)\cap\rho\overline{\mathbb{B}}}(v_0^n) + exc_\rho(C(t_0^n,u_0^n),C(t_1^n,u_1^n)) + h_n\|f(t_0^n,u_0^n,v_0^n)\|$$
$$\le Lh_n(1+\|v_0^n\|) + h_n\|f(t_0^n,u_0^n,v_0^n)\|.$$

Then

$$\|v_1^n\| \le \|v_0^n\| + Lh_n(1+\|v_0^n\|) + 2h_n\beta(1+\|u_0^n\|+\|v_0^n\|). \tag{8}$$

On the other hand, we have

$$\|u_1^n\| \le \|u_0^n\| + h_n\|v_0^n\|. \tag{9}$$

Adding of (8) to (9)

$$\|u_1^n\| + \|v_1^n\| \le \|v_0^n\| + \|u_0^n\| + h_n(L+2\beta) + h_n(1+2\beta+L)\|v_0^n\| + 2h_n\beta\|u_0^n\|$$
$$\le \|v_0^n\| + \|u_0^n\| + 2T(L+2\beta) + T(1+2\beta+L)\|v_0^n\| + T(1+2\beta+L)\|u_0^n\|$$
$$\le \Big(\|v_0^n\| + \|u_0^n\|\Big)(1+T(L+2\beta+L)) + 2T(L+2\beta)$$
$$\le \Big(\|v_0^n\| + \|u_0^n\|\Big)e^{T+2\beta+L} + 2T(L+2\beta)e^{T+2\beta+L}$$
$$\le \Big(\|v_0^n\| + \|u_0^n\| + 2T(L+2\beta)\Big)e^{T+2\beta+L} \le \rho.$$

Suppose that the points $u_0^n,\, u_1^n,\, ...,\, u_{i+1}^n,\, v_0^n,\, v_1^n,\, ...,\, v_{i+1}^n$, have been constructed for $k \le i+1$, with $i \le n-1$, and take again $u_{i+2}^n = u_{i+1}^n + h_nv_{i+1}^n$, we have

$$d_{C(t_{i+2}^n,u_{i+2}^n)}(v_{i+1}^n - h_nf(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)) \le d_{C(t_{i+2}^n,u_{i+2}^n)}(v_{i+1}^n) + h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\|$$
$$\le d_{C(t_{i+1}^n,u_{i+1}^n)\cap\rho\overline{\mathbb{B}}}(v_{i+1}^n) + h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\| + exc_\rho(C(t_{i+1}^n,u_{i+1}^n),C(t_{i+2}^n,u_{i+2}^n))$$
$$\le exc_\rho(C(t_{i+1}^n,u_{i+1}^n),C(t_{i+2}^n,u_{i+2}^n)) + h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\|$$
$$\le h_n\beta + Lh_n(1+\|v_{i+1}^n\|) + h_n\beta(\|u_{i+1}^n\|+\|v_{i+1}^n\|) \le h_n(\beta+L)(1+\rho) < \frac{r}{2} < r.$$

According to the prox regularity of the set $C(t_{i+2}^n, u_{i+2}^n)$, the mapping $\mathrm{Proj}_{C(t_{i+2}^n,u_{i+2}^n)}\left(v_{i+1}^n - h_nf(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\right)$, is well-defined and has nonempty values. Then, we find a point v_{i+2}^n such that

$$v_{i+2}^n \in \mathrm{Proj}_{C(t_{i+2}^n,u_{i+2}^n)}\left(v_{i+1}^n - h_nf(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\right).$$

By construction we have $v_{i+2}^n \in C(t_{i+2}^n,\, u_{i+2}^n)$ and

$$\|v_{i+2}^n - v_{i+1}^n + h_nf(t_{i+1}^n,u_{i+1}^n,v_1^n)\| \le d_{C(t_{i+2}^n,u_{i+2}^n)}(v_{i+1}^n - h_nf(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n))$$
$$\le d_{C(t_{i+2}^n,u_{i+2}^n)}(v_{i+1}^n) + h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\|$$
$$\le d_{C(t_{i+1}^n,u_{i+1}^n)\cap\rho\overline{\mathbb{B}}}(v_{i+1}^n) + exc_\rho(C(t_{i+1}^n,u_{i+1}^n),C(t_{i+2}^n,u_{i+2}^n)) + h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\|$$

Hence

$$\|v_{i+2}^n - v_{i+1}^n\| \le Lh_n(1+\|v_{i+1}^n\|) + +h_n\|f(t_{i+1}^n,u_{i+1}^n,v_{i+1}^n)\|,$$

which implies that

$$\|v_{i+2}^n\| \leq \|v_{i+1}^n\| + Lh_n(1+\|v_{i+1}^n\|) + 2h_n\beta(1+\|u_{i+1}^n\| + \|v_{i+1}^n\|),$$
$$\leq \|v_{i+1}^n\| + h_n(L+2\beta)\|v_{i+1}^n\| + 2h_n\beta\|u_{i+1}^n\| + h_n(L+2\beta).$$

By induction one has

$$\|v_{i+2}^n\| \leq \|v_0^n\| + h_n(i+2)(L+2\beta) + h_n(L+2\beta)\sum_{j=0}^{i+1}\|v_j^n\| + 2h_n\sum_{j=0}^{i+1}\|u_j^n\|. \tag{10}$$

On the other hand, we have $\|u_{i+2}^n\| \leq \|u_{i+1}^n\| + h_n\|v_{i+2}^n\|$, so by induction we obtain

$$\|u_{i+2}^n\| \leq \|u_{i+1}^n\| + h_n\sum_{j=0}^{i+1}\|v_j^n\|. \tag{11}$$

Adding (10) to (11) we see that

$$\|v_{i+2}^n\| + \|u_{i+2}^n\| \leq \|v_0^n\| + \|u_0^n\| + 2T(L+2\beta) + h_n(1+L+2\beta)\sum_{j=0}^{i+1}\Big(\|v_j^n\| + \|u_j^n\|\Big).$$

Using discrete Gronwall's inequality Lemma, we conclude that

$$\|v_{i+2}^n\| + \|u_{i+2}^n\| \leq \Big\{\|v_0^n\| + \|u_0^n\| + 2T(L+2\beta)\Big\}e^{T(1+L+2\beta)} \leq \rho, \tag{12}$$

and consequently the sequences $(u_i^n)_i, (v_i^n)_i$ are uniformly bounded by ρ.
According to (3), we find that $\|f(t_i^n,\, u_i^n,\, v_i^n)\| \leq \beta(1+\rho) = \eta$. Furthermore by (12) we can deduce that

$$\left\|\frac{v_{i+1}^n - v_i^n}{h_n}\right\| \leq L(1+\|v_i^n\|) + 2\|f(t_i^n, u_i^n, v_i^n)\| \leq L(1+\rho) + 2\beta(1+\rho)$$
$$\leq (L+2\beta)(1+\rho) = \varsigma. \tag{13}$$

Step 2. Fix any integer $n \in \mathbb{N}^*$ and let us define on $[t_i^n, t_{i+1}^n]$ for $0 \leq i \leq n-1$ the mappings $u_n(\cdot)$, $v_n(\cdot)$ by

$$u_n(t) = u_i^n + \frac{t-t_i^n}{h_n}(u_{i+1}^n - u_i^n) \quad and \quad v_n(t) = v_i^n + \frac{t-t_i^n}{h_n}(v_{i+1}^n - v_i^n).$$

Then for all $t \in]t_i^n, t_{i+1}^n[$,

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{h_n} = v_i^n \in C(t_i^n, u_i^n), \tag{14}$$

and

$$\dot{v}_n(t) = \frac{v_{i+1}^n - v_i^n}{h_n}. \tag{15}$$

Let us set

$$\theta_n(t) = \begin{cases} t_{i+1}^n & if\ t \in [t_i^n, t_{i+1}^n[\\ t_n^n = T & if\ t = T \end{cases}, \gamma_n(t) = \begin{cases} t_i^n & if\ t \in [t_i^n, t_{i+1}^n[\\ t_{n-1}^n & if\ t = T \end{cases}. \tag{16}$$

It's easy to see that

$$\lim_{n\to+\infty}|\theta_n(t) - t| = \lim_{n\to+\infty}|\gamma_n(t) - t| = 0. \tag{17}$$

Combining (5),(15),(16), it results for almost all $t \in [0,T]$ that

$$-\dot{v}_n(t) \in N_{C(\theta_n(t), u_n(\theta_n(t)))}(v_n(\theta_n(t))) + f\left(\gamma_n(t), u_n(\gamma_n(t)), v_n(\gamma_n(t))\right) \tag{18}$$

From (13) and (15) we get

$$\|\dot{v}_n(t)\| \leq \varsigma, \tag{19}$$

and conclude that $(v_n(\cdot))_n$ is equi-continuous with ratio ς.
On the other hand, for each $t \in [t_i^n,\, t_{i+1}^n]$, we have

$$u_n(t) = u_i^n + \frac{t - t_i^n}{h_n}(u_{i+1}^n - u_i^n) = u_i^n + (t - t_i^n)v_i^n,$$

by iteration we obtain

$$u_n(t) = u_0 + \int_0^t v_n(\gamma_n(s))ds. \tag{20}$$

In view of (14) and (16) we have $v_n(\gamma_n(t)) \in C(\gamma_n(t), u_n(\gamma_n(t)))$, since $(u_i^n)_{i=0}^n$ and $(v_i^n)_{i=0}^n$ are uniformly bounded, we get $v_n(\gamma_n(t)) \in C([0,\, T], \rho\overline{\mathbb{B}}) \cap \rho\overline{\mathbb{B}}$. Then, $\Big(v_n(\gamma_n(t))\Big)_n$ is relatively compact in H, in view of $(\mathscr{A}_{C_3})$. Since

$$\|v_n(\gamma_n(t)) - v(t)\| \leq \int_t^{\gamma_n(t)} \|\dot{v}(s)\|ds \leq \varsigma(\gamma_n(t) - t) \to 0, \quad as \quad n \to \infty,$$

then the set $\{v_n(t),\, n \in \mathbb{N}\}$ is relatively compact in H. According to (19), $(v_n)_n$ is equicontinuous. So by Ascoli's Theorem, we conclude that $\Big(v_n(t)\Big)_n$ is relatively compact in $\mathscr{C}_H([0,T])$, so we can extract a subsequence, also denoted $(v_n(\cdot))_n$, which converges uniformly to $v \in \mathscr{C}_H([0,T])$. Using inequality (19) for the second time, we can extract a subsequence also denoted $(\dot{v}_n)_{n\in\mathbb{N}}$ which converges weakly in $L^\infty_H([0,T], (\sigma(L^\infty_H([0,T]), L^1_H([0,T]))$ to some mapping $w \in L^\infty_H([0,T])$ with $\|w(t)\| \leq \varsigma$ a.e. $t \in [0,T]$. Fixing now $t \in [0,T]$ and taking any $\xi \in L^\infty_H([0,T])$, from the weak convergence of $\dot{v}_n$ to w, we have

$$\lim_{n\to\infty} \langle \dot{v}_n(\cdot), \xi(\cdot)\rangle = \langle w(\cdot), \xi(\cdot)\rangle\,,$$

or equivalently

$$\lim_{n\to\infty} \int_0^t \langle \dot{v}_n(s), \xi(s)\rangle\, ds = \int_0^t \langle w(s), \xi(s)\rangle\, ds.$$

Taking $(e_j)_{j\in J}$ a basis of H, for each $j \in J$, the weak convergence gives us

$$\left\langle \lim_{n\to\infty} \int_0^t \dot{v}_n(s), e_j \right\rangle = \left\langle \int_0^t w(s)ds, e_j \right\rangle,\ thus\ \lim_{n\to\infty} \int_0^t \dot{v}_n(s)ds = \int_0^t w(s)ds,$$

and hence $\lim\limits_{n\to\infty} \Big[v_n(t) - v_n(0)\Big] = \int_0^t w(s)ds$. So $v(t) = v_0 + \int_0^t w(s)ds$ and $\dot{v} = w$, then $\dot{v}_n$ converges $\sigma(L^\infty_H([0,T]), L^1_H([0,T])$ to $\dot{v} \in L^\infty_H([0,T])$, so for all $\xi_1(\cdot) \in L^\infty_H([0,T]) \subset L^1_H([0,T])$, we have $\lim\limits_{n\longrightarrow 0} \langle \dot{v}_n, \xi_1(\cdot)\rangle = \langle \dot{v}(\cdot), \xi_1(\cdot)\rangle$. Then $(\dot{v}_n(\cdot))_n$ converges $\sigma(L^1_H([0,T]), L^\infty_H([0,T]))$ in $L^1_H([0,T])$ to $v(\cdot)$. From (17), (20) and the uniform convergence of $(v_n)_n$ to v, we conclude that $(u_n)_n$ converge uniformly to an absolutely continuous function u with $u(t) = u_0 + \int_0^t v(s)ds$.

Step3. Putting $z_n(t) = f(\gamma_n(t), u_n(\gamma_n(t)), v_n(\gamma_n(t)) \quad for\ all \quad t \in [0,T]$, then, $\|z_n(t)\| \leq \eta$. This implies that $(z_n(\cdot))_n$ is bounded. So taking a subsequence if necessary, we can deduce that the sequence $(z_n(\cdot))_n$ converges $\sigma(L^1_H([0,T]), L^\infty_H([0,T]))$ in $L^1_H([0,T])$ to a mapping $z \in L^1_H([0,T])$ with $\|z(t)\| \leq \eta$ a.e. $t \in [0,T]$.
Now we shall prove that, for all $t \in [0,T]$, we have $\dot{u}(t) \in C(t, u(t))$. Let $t \in [0,T]$, so

$$d_{C(t,u(t))}(v_n(\theta_n(t))) \leq d_{C(\theta_n(t),u_n(\theta_n(t)))\cap\rho\overline{\mathbb{B}}}(v_n(\theta_n(t)))$$

$$+exc_\rho\Big(C(\theta_n(t), u_n(\theta_n(t))), C(t, u(t))\Big) \leq L\{|\theta_n(t) - t| + \|u_n(\theta_n(t)) - u(t)\|\}.$$

We know that, if $n \to \infty$, $\{|\theta_n(t) - t| + \|u_n(\theta_n(t)) - u(t)\|\} \to 0$ and for every $t \in [0,T], v_n(\theta_n(t)) \underset{n\to\infty}{\longrightarrow} v(t) = \dot{u}(t)$. Since $C(t,u(t))$ is closed, we obtain $\dot{u}(t) \in C(t,u(t))$. Moreover, by inclusion (19) and the inequality

$$\|\dot{v}_n + z_n(t)\| \leq \|\dot{v}_n\| + \|z_n(t)\| \leq \varsigma + \eta = l, \text{ a.e.},$$

it follows that for a.e., $t \in [0,\, T]$

$$\dot{v}_n + z_n(t) \in -N_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t))) \cap l\overline{\mathbb{B}} = -l\partial d_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t))), \quad (21)$$

and

$$z_n(t) \in F(\gamma_n(t), u_n(\gamma_n(t)), v_n(\gamma_n(t)). \quad (22)$$

Using Mazur's Lemma and the weak convergence of $(\dot{v}_n + z_n, z_n)_n$ in $L^1_{H\times H}([0,T])$ to $(\dot{v} + z, z)$, there is a sequence $(\omega_n, \varpi_n)_n$ which converges strongly in $L^1_{H\times H}([0,T])$ to $(\dot{v} + z, z)$ with

$$\omega_n \in \overline{co}\{\dot{v}_m + z_m : m \geq n\} \quad \textit{and} \quad \varpi_n \in \overline{co}\{z_m : m \geq n\}. \quad (23)$$

So we can extract from $(\omega_n, \varpi_n)_n$ a subsequence which converges a.e. to $(\dot{v}+z, z)$. Therefore, there exists a Lebesgue negligible set $\mathcal{S} \subset [0,T]$ such that for every $t \in [0,\, T] \setminus \mathcal{S}$, we have

$$\dot{v}(t) + z(t) \subset \bigcap_{n\geq 0} \overline{co}\{\dot{v}_m(t) + z_m(t) : m \geq n\} \textit{ and } z(t) \subset \bigcap_{n\geq 0} \overline{co}\{z_m(t) : m \geq n\}.$$

It results from (21) and (22) that for any $n \in \mathbb{N}$, for any $t \in I$ and for any $y \in H$,

$$\langle y, \dot{v}_n(t) + z_n(t)\rangle \leq \sigma\left(y, -l\partial d_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t)))\right),$$

and

$$\langle y, z(t)\rangle \leq \sigma\left(y, F\left(\gamma_n(t), u_n(\gamma_n(t)), v_n(\gamma_n(t))\right)\right).$$

Further, for each $n \in \mathbb{N}$ and any $t \in [0,T] \setminus \mathcal{S}$ from (23) one has

$$\langle y, \dot{v}(t) + z(t)\rangle \leq \limsup_{n\to\infty} \langle y, \dot{v}_m(t) + z_m(t)\rangle$$

$$\leq \limsup_{n\to\infty} \sigma\left(y, -l\partial d_{C(\theta_n(t),u_n(\theta_n(t)))}(v_n(\theta_n(t)))\right),$$

and

$$\langle y, z(t)\rangle \leq \limsup_{n\to\infty} \langle y, z_m(t)\rangle \leq \limsup_{n\to\infty} \sigma\left(y, F\left(\gamma_n(t), u_n(\gamma_n(t)), v_n(\gamma_n(t))\right)\right),$$

By Proposition 2.2 and the scalar upper semicontinuity of F we get

$$\langle y, \dot{v}(t) + z(t)\rangle \leq \sigma\left(y, -l\partial d_{C(t,u(t))}(v(t))\right) \textit{ and } \langle y, z(t)\rangle \leq \sigma\left(y, F\left(t, u(t), v(t)\right)\right),$$

which ensures that

$$\dot{v}(t) + z(t) \in -l\partial d_{C(t,u(t))}(v(t)) \subset -N_{C(t,u(t))}(v(t)) \textit{ and } z(t) \in F(t, u(t), v(t)).$$

Consequently

$$\dot{v}(t) \in -N_{C(t,u(t))}(v(t)) - z(t) \textit{ and } z(t) \in F(t,u(t),v(t)) \quad \text{a.e.} \quad t \in [0,T],$$

with

$$\|\dot{v}(t) + z(t)\| \leq l \qquad \text{a.e.} \quad t \in [0,T].$$

This completes the proof. □

4. **Application for a class of quasi-variational inequalities**

We consider the following evolution variational inequalities: find $u : [0,T] \to H$ such that, for all $w \in D(u(t))$, $\lambda > 0$, we have

$$(\mathcal{AD})\begin{cases}\langle \ddot{u}(t)+g(t), w-\dot{u}(t)\rangle \geq \langle f(t,u(t),\dot{u}(t)), w-\dot{u}(t)\rangle + \lambda\|w-\dot{u}(t)\|^2,\\ \quad u(0)=u_0,\ \dot{u}(0)=v_0\in D(u(0)),\end{cases}$$

This class of problems find its motivation in the fact that it constitutes the variational formulation for linear elasticity problems with friction or unilateral constraints, and quasistatic frictional contact problems involving viscoelastic materials with short memory under state-dependent perturbation forces. For more details, we refer to [12], [16]. By the definition of proximal normal cone, this problem is equivalent to

$$\begin{cases}\ddot{u}(t)+g(t)\in -N^p_{D(u(t))}(\dot{u}(t))-f(t,u(t),\dot{u}(t)) \quad \text{a.e. in } [0,T]\\ \quad u(0)=u_0,\ \dot{u}(0)=v_0\in D(u(0)),\end{cases} \tag{24}$$

where D is a set valued mapping with nonempty closed values, $g \in L^1([0,T],H)$ and $f(t,u(t),\dot{u}(t)) \in F(t,u(t),\dot{u}(t))$.
We assume that $D(u(t))$ and $F(t,u(t),\dot{u}(t))$ satisfies the following assumptions

- $(\mathcal{S}_1)$ There is a constant $r > 0$ such that, for each $t \in [0,T]$ and each $u \in H$, the sets $D(u(t))$ are $r-$prox regular.

- $(\mathcal{S}_2)$ There exists a real $L_1 > 0$ and an extended real ρ positive such that for every $x,\ y \in H$,
$$\hat{\mathcal{H}}_\rho(D(x),D(y)) \leq L_1\|x-y\|.$$
- $(\mathcal{S}_3)$ For any bounded subset A of H the set $D(A)$ is ball compact.
- $(\mathcal{S}_4)$ F is scalarly upper semicontinuous and for some real $\beta_1 > 0$
$$d_{F(t,u,v)}(0) \leq \beta_1(1+\|u\|+\|v\|)\ \forall t\in[0,T]\ u,\ v\in H \ \textit{with}\ v\in D(u).$$

Theorem 4.1. *Under the assumptions* $(\mathcal{S}_1),(\mathcal{S}_2),(\mathcal{S}_3)$ *and* $(\mathcal{S}_4)$, *for each* $u_0 \in H$, *the problem* $(\mathcal{AD})$ *admits an absolutely continuous solution.*

Proof. Putting for all $t,\ s$ in $[0,T]$,

$$x(t)=u(t)+\int_0^t\int_0^s g(\tau)d\tau ds \ \text{ and } C(t,x)=D\big(x-\int_0^t\int_0^s g(\tau)d\tau ds\big)+\int_0^t g(s)ds.$$

Using Proposition 2.1 and (24), we find

$$\begin{aligned}\ddot{x}(t)\in &-N_{C(t,x(t))-\int_0^t g(s)ds}\big(\dot{x}(t)-\int_0^t g(s)ds\big)-F(t,x(t),\dot{x}(t)),\\ \in &-N_{C(t,x(t))}\big(\dot{x}(t)\big)-F(t,x(t),\dot{x}(t)),\end{aligned}$$

with $x(0)=x_0$ and $\dot{x}(0)=y_0\in C(0,x_0)$. Obviously, the set-valued map $C(t,x(t))$ satisfies conditions $(\mathscr{A}_{C_1})$, $(\mathscr{A}_{C_3})$. Indeed, for all $t,t'\in[0,T]$, one has

$$\mathcal{H}_\rho(C(t,x),C(t',y)) \leq \hat{\mathcal{H}}_\rho(C(t,x),C(t',y))$$

$$= \sup_{z\in\rho\overline{\mathbb{B}}}\big|d_{D(x-\int_0^t\int_0^s g(\tau)d\tau ds)}(z-\int_0^t g(s)ds)-d_{D(y-\int_0^{t'}\int_0^s g(\tau)d\tau ds)}(z-\int_0^{t'} g(s)ds)\big|$$

$$\leq \sup_{z\in\rho\overline{\mathbb{B}}}\big|d_{D(x-\int_0^t\int_0^s g(\tau)d\tau ds)}(z-\int_0^t g(s)ds)-d_{D(y-\int_0^{t'}\int_0^s g(\tau)d\tau ds)}(z-\int_0^t g(s)ds)\big|$$

$$+\|\int_0^t g(s)ds - \int_0^{t'} g(s)ds\|$$

$$\leq L_1\big\|x - \int_0^t\int_0^s g(\tau)d\tau ds - y + \int_0^{t'}\int_0^s g(\tau)d\tau ds\big\| + \int_{t'}^t \|g\|_{L^\infty([0,T],H)}ds$$

$$\leq L_1\|x-y\| + L_1\|\int_0^s\int_t^{t'} g(\tau)d\tau\| + \int_{t'}^t \|g\|_{L^\infty([0,T],H)}ds$$

$$\leq L_1\|x-y\| + L_1|t-t'|\big(L_1\frac{t+t'}{2}+1\big)\|g\|_{L^\infty([0,T],H)} \leq L(|t-t'| + \|x-y\|)$$

with $L = (L_1T+1)\|g\|_{L^\infty([0,T],H)}$.
On the other hand F is scalarly upper semicontinuous. Moreover, for all $x, y \in H$, one has $d_{F(t,x,y)}(0) \leq \beta\big(1+\|x\|+\|y\|\big)$, where $\beta = \beta_1(1+T\|g\|_{L^\infty([0,T],H)} + \frac{T^2}{2}\|g\|_{L^\infty([0,T],H)})$. Thus all assumptions of Theorem 3.1 are satisfied and so we have the existence of solution for the problem $(\mathcal{AD})$.

Example 4.1. *In the convex case, we consider the following quasistatic frictional contact problem involving viscoelastic materials with short memory: find a displacement field* $u : [0,T]\times\Omega \to \mathbb{R}^d$ *and a stress field* σ *such that*

$$(\mathcal{ADA})\begin{cases}\dfrac{\partial u_i}{\partial t^2} + \dfrac{\partial a_{ijkh}}{\partial x_j}a_{ijkh}\varepsilon_{kh}(u) = g_{0_i} \ \ in \ \ ]0,T[\times\Omega \\ u_i = U_i \ \ on \ \ ]0,T[\times\Gamma_U \\ \sigma_\nu = g_2 \ \ on \ \ ]0,T[\times\Gamma_N \\ \|\sigma_\tau\| < h \Longrightarrow \dfrac{\partial u_\tau}{\partial t} = 0, \\ \|\sigma_\tau\| = h \Longrightarrow \exists\alpha > 0 \ such\ that\ \dfrac{\partial u_\tau}{\partial t} = -\alpha\sigma_\tau \\ u(x,0) = u_0(x), \ \ \dfrac{\partial u(x,0)}{\partial t} = u_1(x) \ \ x\in\Omega\end{cases}$$

Here Ω *is an open set in* $\mathbb{R}^d$ $(d = 2,3)$, *its boundary* $\partial\Omega = \Gamma$, *which is composed of two parts* $\Gamma = \Gamma_U \cup \Gamma_N$ *(we consider the case with friction on a part* Γ_N *and we assume that the displacement is given on* Γ_U*) and* $\{U_i\}$ *the vector field prescribed on* Γ_U. *The vector* $g = \{g_{0_i}\}$ *represents a volume density of prescribed forces. The coefficient* a_{ijkh} *play the role of elasticity coefficient and* ε_{kh} *the linearized strain tensors. We also denote* σ_ν *and* σ_τ *the normal and tangential components of the stress field* σ. *Similarly the normal and tangential components is denoted by* v_ν *and* v_τ. *We refer* [12, 16] *for more details.*
Let us introduce the following notations, for the variational formulation

$$H = (H^1(\Omega))^3$$

$$\mathcal{U}_{ad}(t) = \{v \in (H^1(\Omega))^3 \cap D(u) \ : \ v = \dot{U}(t) \ on \ \Gamma_U\}$$

$$a(u,v) = \int_\Omega a_{ijkh}\varepsilon_{kh}\varepsilon_{ij}(u)dx$$

$$j(v) = \int_\Gamma h\|v_\tau\|d\Gamma \quad h\in L^\infty(\Gamma)$$

$$\langle g(t), v\rangle = \int_\Gamma g_0(t)vdx + \int_{\Gamma_N} g_2(t)vd\Gamma \quad g_0 \in L^2(\Omega), \ g_2\in L^2(\Gamma),$$

where D *is a set valued mapping with nonempty closed convex values and satisfies the assumptions* $(\mathcal{S}_2)$, $(\mathcal{S}_3)$. *Then the preceding problem can be formulated as*

$$\langle \ddot{u}(t), v-\dot{u}(t)\rangle + a(u(t), v-\dot{u}(t)) + j(v) - j(\dot{u}(t)) \geq \langle g(t), v-\dot{u}(t)\rangle \ \ \forall v\in\mathcal{U}_{ad}(t)$$

which can be written in the following form

$$-\ddot{u}(t) - g(t) \in N_{D(u(t))}(\dot{u}(t)) + \partial J(\dot{u}(t)) + Au(t) \;\; a.e.\ t \in [0,T]$$

where $A : H \to H$ is the linear bounded operator defined by

$$\langle Au, v\rangle = a(u,v).$$

Therefore, the previous inclusion is equivalent to

$$\begin{cases} -\ddot{u}(t) - g(t) \in N_{D(u(t))}(\dot{u}(t)) + F(t,u(t),\dot{u}(t) \;\; a.e.\ t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = v_0 \in D(u(t)). \end{cases}$$

where $F(t,u,v) = \{\partial J(v) + Au\}$. It is clear that F satisfies the condition $(\mathcal{S}_4)$. Thus, we have the solution of problem $(\mathcal{ADA})$.

□

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