MODIFICATION DESIGN AND ANALYSIS OF THE PITCH CURVE WITH CONVEX POINTS DRIVEN BY ROTATION TIME CONSTRAINT FOR *N*-LOBED NONCIRCULAR GEAR

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In order to improve the transmission smoothness of noncircular gear, a novel pitch curve modification model for the pitch curve with convex points of N-lobed noncircular gear (N-LNG) is proposed. The model is based on fundamentals of the calculus of variations and considers minimal mean kinetic energy under the prescribed time constraint. Additionally, the modification algorithm for the pitch curve with convex points and the relation between prescribed time and modification accuracy are deduced and analyzed, respectively. Modification example demonstrates the validity of the proposed modification model and algorithm. Additionally, according to the analysis of modification results, when the rotation time constraint is given, the modification accuracy of minimal mean kinetic energy modification method proposed in this paper is higher to that of the previous proposed modification method.

Keywords: *N*-lobed noncircular gear, pitch curve design, rotation time constraint, minimal mean kinetic energy modification.

1. Introduction

Noncircular gears has been used in printing machine, slotting machine, flow meters and other mechanical products to replace the traditional slot wheel, cam, ratchet, or connecting-rod mechanism, to realize the variable speed , differential speed, or intermittent movement of mechanical products, differential or intermittent motion because of compact structure , stable transmission , high precision and easy to achieve dynamic balance [1-4]. Specially, a method of constructing *N*-lode noncircular gear pitch curves with Bézier, B-spline, minimal rotary inertia and minimal mean kinetic energy characteristics have been proposed in Ref. [5-7]. Conjugate pitch curves of external and internal meshing noncircular gears by the common plane curve with unidirectional continuous rotary characteristics (ellipse, eccentric circle, pascal curve, etc.) as the pitch curve of noncircular gear have been given by Liu, Wu and Li [8]. A general generation method of pitch curves for *N*-lobed elliptical gears resorting to the basic ellipse have been proposed by Figliolini, Lanni and Ceccarelli [9], so that the number of speed cycles per revolution can be increased.

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Above pitch curves of noncircular gears were constructed by existing plane curves, it has been unable to meet design requirements of specific non-uniform speed transmission with the extensive application of noncircular gears in mechanical products [10-11]. This limitation has gradually been found by some researchers: A novel design method of pitch curve with epicycle constraint constructed by plane regular *N*-curved polygon for *N*-lobed noncircular gear have been presented by Yao [12]. The pitch curves with steepest rotation characteristic for noncircular gears have been synthesized, and this research has been applied to modify the pitch curve with discontinuity points [13-14].

However, according to the former researches, we know that the difficulty in manufacturing is increased for *N*-lobed noncircular gears that possess the pitch curves with convex (or concave) points [6-7, 12-13, 15-16]. In this paper, therefore, the calculus of variations is used to establish a pitch curve modification model and algorithm for the pitch curve with convex points for *N*-LNG, which considers minimal mean kinetic energy with prescribed time constraint because smaller mean kinetic energy can improve the transmission smoothness of noncircular gears. Numerical example is shown to validate the proposed modification model and algorithm.

2. Modification model for the pitch curve with convex points based on calculus of variations

As shown in Fig. 1, the fixed coordinate system $\Gamma(o\text{-}xy)$ rigidly connected with the rotation center o of the pitch curve $r(\theta)$ with convex points c_1, c_2, ..., c_N for *N*-LNG, and the polar angle θ is measured counterclockwise from the positive direction x-axis.

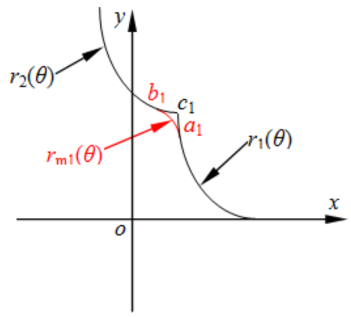

Fig. 1. Sketch of the modification model for pitch curve with convex points for *N*-LNG

Further, the polar equation for the pitch curve $r(\theta)$ with convex points can be obtained [14]

$$\begin{cases} r(\theta)=\begin{cases} r_1(\theta),\ \theta\in[0,2\pi/N] \\ r_2(\theta)=r_1(\theta-2\pi/N),\ \theta\in[2\pi/N,4\pi/N] \\ \vdots \\ r_k(\theta)=r_1(\theta-\frac{2(k-1)\pi}{N}),\ \theta\in[\frac{2(k-1)\pi}{N},\frac{2k\pi}{N}] \end{cases} \\ s.t.\ r_k(\frac{2k\pi}{N})=r_{(k+1)}(\frac{2k\pi}{N}),\ r_k'(\frac{2k\pi}{N})>0,\ r_{(k+1)}'(\frac{2k\pi}{N})<0 \end{cases} \tag{1}$$

where N is the number of lobes of pitch curve $r(\theta)$, k=1,2,...,N and the following discussion relating to subscript k, if k=N, then (k+1)≡1.

Referring to Fig. 1, in order to ensure that the modification pitch curve $r_{m1}(\theta)$ possesses the characteristic of minimal mean kinetic energy E_{min} under the prescribed rotation time constraint T, the pitch curve $r_{m1}(\theta)$ should satisfy the following expressions based on principles of mechanics

$$\begin{cases} E_{\min}=\min\int_{\theta_{a1}}^{\theta_{b1}}\frac{M\omega r_{m1}(\theta)\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}{2T}d\theta \\ s.t.\begin{cases} T=\int_{\theta_{a1}}^{\theta_{b1}}\frac{\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}{\omega r_{m1}(\theta)}d\theta \\ r_{m1}(\theta_{a1})=r_1(\theta_{a1}),\ r_{m1}'(\theta_{a1})=r_1'(\theta_{a1}) \\ r_{m1}(\theta_{b1})=r_1(\theta_{b1}),\ r_{m1}'(\theta_{b1})=r_1'(\theta_{b1}) \end{cases} \end{cases} \tag{2}$$

where $E_{\min}$ is refers to the minimal mean kinetic energy of noncircular gear through the modification pitch curve $r_{m1}(\theta)$ under the prescribed time T, ω and M are respectively angular velocity and mass of the noncircular gear.

In order to obtain the solution of Eq. (2), the auxiliary function can be established according to the calculus of variations [19]

$$E_{\min}^*(\theta)=\min\int_{\theta_{a1}}^{\theta_{b1}}\left(\frac{M\omega r_{m1}(\theta)\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}{2T}+\lambda\frac{\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}{\omega r_{m1}(\theta)}\right)d\theta \tag{3}$$

where λ is Lagrange multiplier.

Meanwhile, the Euler-Lagrange Equation with first integral of Eq. (3) can be obtained

$$\frac{M\omega r_{m1}^3(\theta)}{2T\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}+\frac{\lambda r_{m1}^2(\theta)}{\omega r_{m1}(\theta)\sqrt{r_{m1}^2(\theta)+r_{m1}'^2(\theta)}}=h \tag{4}$$

where h is an integral constant.

Assuming

$$r_{m1}'(\theta)=r_{m1}(\theta)\tan\mu,\ \mu\in(-\frac{\pi}{2},\frac{\pi}{2}) \tag{5}$$

where μ is an undetermined parameter.

Substituting Eq. (5) into Eq. (4), one obtains

$$r_{\mathrm{m1}}^2(\theta)=\frac{\omega f_1 \sec\mu - f_2}{\omega^2},\ \mu\in(-\frac{\pi}{2},\ \frac{\pi}{2}) \tag{6}$$

where f_1 and f_2 are undetermined parameters and they can be expressed as

$$f_1=\frac{2T}{M}h,\ f_2=\frac{2T}{M}\lambda \tag{7}$$

Differentiating both sides of Eq. (6), we can obtain

$$dr_{\mathrm{m1}}(\theta)=\frac{f_1}{2\omega r_{\mathrm{m1}}(\theta)}\sec\mu\tan\mu\, d\mu \tag{8}$$

Along with Eq. (5) and Eq. (6), one obtains

$$d\theta=\frac{dr_{\mathrm{m1}}(\theta)}{r'_{\mathrm{m1}}(\theta)}=\frac{\omega f_1}{2}\cdot\frac{1}{\omega f_1 - f_2\cos\mu}d\mu \tag{9}$$

Integrating both sides of Eq. (9), we can obtain

$$\theta(\mu)=\begin{cases}\dfrac{\omega f_1}{\omega f_1 - f_2}\sqrt{\dfrac{\omega f_1 - f_2}{\omega f_1 + f_2}}\arctan(\sqrt{\dfrac{\omega f_1 + f_2}{\omega f_1 - f_2}}\tan\dfrac{\mu}{2})+f_3, \left(\dfrac{\omega f_1}{f_2}\right)^2>1\\ \dfrac{\omega f_1}{2(\omega f_1 - f_2)}\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}\ln\left|\dfrac{\tan\dfrac{\mu}{2}+\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}{\tan\dfrac{\mu}{2}-\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}\right|+f_3, \left(\dfrac{\omega f_1}{f_2}\right)^2<1\end{cases} \tag{10}$$

where f_3 is an integral constant.

Along with Eq. (2) and Eq. (6), the minimal mean kinetic energy modification model driven by rotation time T for the pitch curve with convex points and its constraint conditions can be expressed as

$$\begin{cases} r_{\mathrm{m1}}(\mu)=\sqrt{\dfrac{\omega f_1\sec\mu - f_2}{\omega^2}}\\ \theta(\mu)=\begin{cases}\dfrac{\omega f_1}{\omega f_1 - f_2}\sqrt{\dfrac{\omega f_1 - f_2}{\omega f_1 + f_2}}\arctan(\sqrt{\dfrac{\omega f_1 + f_2}{\omega f_1 - f_2}}\tan\dfrac{\mu}{2})+f_3, \left(\dfrac{\omega f_1}{f_2}\right)^2>1\\ \dfrac{\omega f_1}{2(\omega f_1 - f_2)}\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}\ln\left|\dfrac{\tan\dfrac{\mu}{2}+\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}{\tan\dfrac{\mu}{2}-\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}\right|+f_3, \left(\dfrac{\omega f_1}{f_2}\right)^2<1\end{cases}\end{cases} \tag{11}$$

and

$$\begin{cases} T=\displaystyle\int_{\theta_{\mathrm{a1}}}^{\theta_{\mathrm{b1}}}\frac{\sqrt{r_{\mathrm{m1}}^2(\theta)+{r'_{\mathrm{m1}}}^2(\theta)}}{\omega r_{\mathrm{m1}}(\theta)}d\theta\\ r_{\mathrm{m1}}(\theta_{\mathrm{a1}})=r_1(\theta_{\mathrm{a1}}),\ r_{\mathrm{m1}}'(\theta_{\mathrm{a1}})=r_1'(\theta_{\mathrm{a1}})>0\\ r_{\mathrm{m1}}(\theta_{\mathrm{b1}})=r_1(\theta_{\mathrm{b1}}),\ r_{\mathrm{m1}}'(\theta_{\mathrm{b1}})=r_1'(\theta_{\mathrm{b1}})<0\end{cases} \tag{12}$$

Assuming μ_{a1} and μ_{b1} are undetermined values of parameter μ corresponding to polar angles θ_{a1} and θ_{b1}, then Eq. (12) can be rearranged

$$\begin{cases} \theta(\mu_{a1}) = \theta_{a1},\ \theta(\mu_{b1}) = \theta_{b1} \\ r_{m1}(\mu_{a1}) = r_1(\theta_{a1}),\ r_{m1}{}'(\mu_{a1}) = r_1'(\theta_{a1}) > 0 \\ r_{m1}(\mu_{b1}) = r_1(\theta_{b1}),\ r_{m1}{}'(\mu_{b1}) = r_1'(\theta_{b1}) < 0 \\ T = \dfrac{f_2}{\omega^2 f_1}(\theta_{b1} - \theta_{a1}) + \dfrac{1}{2\omega}\ln\left|\dfrac{\sec\mu_{b1} + \tan\mu_{b1}}{\sec\mu_{a1} + \tan\mu_{a1}}\right| \end{cases} \tag{13}$$

3. Analysis of undetermined parameters

According to Eq. (5) and Eq. (13), in order to ensure that the constraints $r'_{m1}(\theta_{a1})>0$ and $r'_{m1}(\theta_{b1})<0$ can be satisfied, the ranges of undetermined parameters μ_{a1} and μ_{b1} can be obtained

$$\mu_{a1} \in (0,\ \frac{\pi}{2}),\ \mu_{b1} \in (-\frac{\pi}{2},\ 0) \tag{14}$$

Along with Eq. (6), the following constraints can be obtained according to the characteristic of modification pitch curve $r_{m1}(\theta)$ is greater than 0

$$(\omega f_1 \sec\mu - f_2)_{\min} > 0,\ \mu \in [\mu_{a1},\ \mu_{b1}] \tag{15}$$

Furthermore, we can obtain

$$\frac{d(\omega f_1 \sec\mu - f_2)}{d\mu} = \omega f_1 \sec\mu \tan\mu \tag{16}$$

Along with Eq. (15), undetermined parameters f_1 and f_2 should satisfy the following constraints

$$\begin{cases} f_1 > 0 \\ \omega f_1 \sec\mu_{a1} - f_2 > 0 \\ \omega f_1 \sec\mu_{b1} - f_2 > 0 \end{cases},\ \text{or}\ \begin{cases} f_1 < 0 \\ \omega f_1 - f_2 > 0 \end{cases} \tag{17}$$

In order to ensure that the polar angle function $\theta(\mu)$ is strictly monotone increase, undetermined parameters f_1 and f_2 should satisfy the following condition

$$\left(\frac{\omega f_1}{2}\cdot\frac{1}{\omega f_1 - f_2\cos\mu}\right)_{\max} < 0,\ \mu \in [\mu_{a1},\ \mu_{b1}] \tag{18}$$

Meanwhile, we can be obtain

$$\frac{d\left(\dfrac{\omega f_1}{2}\cdot\dfrac{1}{\omega f_1 - f_2\cos\mu}\right)}{d\mu} = \frac{\omega f_1}{2}\cdot\frac{-f_2\sin\mu}{(\omega f_1 - f_2\cos\mu)^2} \tag{19}$$

Along with Eq. (18), undetermined parameters f_1 and f_2 should satisfy the following constraints

$$\begin{cases} f_1 f_2 > 0 \\ f_1(\omega f_1 \sec\mu_{a1} - f_2) < 0 \\ f_1(\omega f_1 \sec\mu_{b1} - f_2) < 0 \end{cases} \tag{20}$$

Referring to Fig. 1, it is only when the curvature radius ρ is outer convex that the modification pitch curve $r_{m1}(\theta)$ can be used to modify the pitch curve with convex points within a small error range, therefore, the curvature radius ρ should satisfy the following condition

$$\rho = \frac{r_{m1}(\mu)\sec^2\mu}{\frac{2f_2}{\omega f_1} - \sec\mu} > 0,\ \forall\mu \in [\mu_{a1},\ \mu_{b1}] \tag{21}$$

In order to ensure that the Eq. (21) is permanent, one obtains

$$\left(\frac{2f_2}{\omega f_1} - \sec\mu\right) > 0,\ \forall\mu \in [\mu_{a1},\ \mu_{b1}] \tag{22}$$

Meanwhile, we can be obtain

$$\frac{d\left(\frac{2f_2}{\omega f_1} - \sec\mu\right)}{d\mu} = -\sec\mu\tan\mu \tag{23}$$

Along with Eq. (22), undetermined parameters f_1 and f_2 should satisfy the following constraint

$$\frac{2f_2}{\omega f_1} > \sec\mu_{a1},\ \frac{2f_2}{\omega f_1} > \sec\mu_{b1} \tag{24}$$

Therefore, along with Eq. (17) and Eq. (20), the ranges of undetermined parameters f_1 and f_2 can be obtained

$$\begin{cases} f_1 < 0, f_2 < 0 \\ \frac{f_2}{\omega f_1} > \sec\mu_{a1},\ \frac{f_2}{\omega f_1} > \sec\mu_{b1} \end{cases} \tag{25}$$

Along with Eq. (11), Eq. (13) and Eq. (14), the modification model and its constraint conditions can be simplified

$$\begin{cases} r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{\dfrac{\omega f_1\sec\mu-f_2}{\omega^2}} \\ \theta(\mu)=\begin{cases} \dfrac{\omega f_1}{\omega f_1-f_2}\sqrt{\dfrac{\omega f_1-f_2}{\omega f_1+f_2}}\arctan(\sqrt{\dfrac{\omega f_1+f_2}{\omega f_1-f_2}}\tan\dfrac{\mu}{2})+f_3 \\ \dfrac{\omega f_1}{2(\omega f_1-f_2)}\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}\ln\left|\dfrac{\tan\dfrac{\mu}{2}+\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}{\tan\dfrac{\mu}{2}-\sqrt{\dfrac{f_2-\omega f_1}{f_2+\omega f_1}}}\right|+f_3 \end{cases} \end{cases} \\ s.t.\ \theta\in[\theta_{a1},\ \theta_{b1}]\ \text{and}\begin{cases} \mu\in[\mu_{a1},\ \mu_{b1}],\ \mu_{a1}\in(0,\ \dfrac{\pi}{2}),\ \mu_{b1}\in(-\dfrac{\pi}{2},0) \\ f_1<0, f_2<0,\ \dfrac{f_2}{\omega f_1}>\sec\mu_{a1},\ \dfrac{f_2}{\omega f_1}>\sec\mu_{b1} \end{cases} \end{cases} \tag{26}$$

and

$$T=\frac{f_2}{\omega^2 f_1}(\theta_{b1}-\theta_{a1})+\frac{1}{2\omega}\ln\left|\frac{\sec\mu_{b1}+\tan\mu_{b1}}{\sec\mu_{a1}+\tan\mu_{a1}}\right| \tag{27}$$

Along with Eq. (1), the modified pitch curve $R(\theta)$ and its conjugated external meshing noncircular gear pitch curve $R_{Ee}(\theta_{Ee})$ can be obtained

$$\begin{cases} R(\theta)=\begin{cases} R_1(\theta)=\begin{cases} r_1(\theta),\ \theta\in[\theta_{b1}-2\pi/N,\ \theta_{a1}] \\ r_{m1}(\theta),\ \theta\in[\theta_{a1},\ \theta_{b1}] \end{cases} \\ R_2(\theta)=\begin{cases} r_2(\theta)=r_1(\theta-2\pi/N),\ \theta\in[\theta_{b1},\ \theta_{a1}+2\pi/N] \\ r_{m2}(\theta)=r_{m1}(\theta-2\pi/N),\ \theta\in[\theta_{a1}+2\pi/N,\ \theta_{b1}+2\pi/N] \end{cases} \\ \vdots \\ R_k(\theta)=\begin{cases} r_k(\theta)=r_1(\theta-\dfrac{2(k-1)\pi}{N}),\ \theta\in[\theta_{b1}+\dfrac{2(k-2)\pi}{N},\ \theta_{a1}+\dfrac{2(k-1)\pi}{N}] \\ r_{mk}(\theta)=r_{m1}(\theta-\dfrac{2(k-1)\pi}{N}),\ \theta\in[\theta_{a1}+\dfrac{2(k-1)\pi}{N},\ \theta_{b1}+\dfrac{2(k-1)\pi}{N}] \end{cases} \end{cases} \\ R_{Ee}(\theta_{Ee})=\begin{cases} R_{Ee}(\theta_{Ee})=a_E-R(\theta) \\ \theta_{Ee}=\displaystyle\int_0^{\theta}\frac{R(\theta)}{a_E-R(\theta)}d\theta \end{cases},\ s.t.\ \dfrac{2\pi}{N_E}=\displaystyle\int_0^{\frac{2\pi}{N}}\frac{R_1(\theta)}{a_E-R_1(\theta)}d\theta \end{cases} \tag{28}$$

where a_E is the center distance of the conjugated external noncircular gears, θ_{Ee} and N_E are the polar angle and the number of lobes for the pitch curve $R_{Ee}(\theta_{Ee})$ respectively.

4. Modification algorithm and examples

The minimal mean kinetic energy modification algorithm with prescribed time for the pitch curve with convex points is depicted in Fig. 2. The initial condition is the N-lobed noncircular gear pitch curve $r(\theta)$ with convex points. Convex point c_1 can be chosen as the modification cusp by designer, meanwhile, undetermined parameters f_1, f_2, f_3, θ_{a1}, θ_{b1}, μ_{a1} and μ_{b1} can be obtained by resorting to Eq. (1), Eq. (26) and Eq. (27). Furthermore, the modification and modified pitch curves $r_{m1}(\theta)$ and $R(\theta)$ can be obtained by Eq. (26) and Eq. (28).

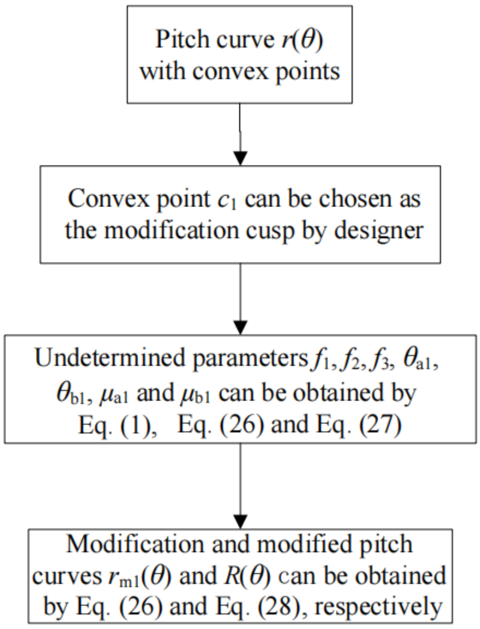

Fig. 2. Minimal mean kinetic energy modification algorithm with prescribed time for the pitch curve with convex points

In order to compare and analyze the modification results of two different modification methods proposed in this paper and Ref. [19], the numerical example of 3-lobed noncircular gear pitch curve $r_v(\theta)$ with convex cusps and the given transmission ratio $i_v(\theta)$ can also be expressed as

$$r_v(\theta)=\begin{cases} r_1(\theta)=4-\sqrt{3}\sin\theta-\cos\theta,\ \theta\in[0,\ 2\pi/3] \\ r_2(\theta)=r_1(\theta-2\pi/3),\ \theta\in[2\pi/3,\ 4\pi/3] \\ r_3(\theta)=r_1(\theta-4\pi/3),\ \theta\in[4\pi/3,\ 2\pi] \end{cases} \quad (29)$$

$$i_{\mathrm{v}}(\theta)=\begin{cases}\dfrac{r_1(\theta)}{4.7741-r_1(\theta)},\theta\in[0,2\pi/3]\\ \dfrac{r_2(\theta)}{4.7741-r_2(\theta)},\theta\in[2\pi/3,4\pi/3]\\ \dfrac{r_3(\theta)}{4.7741-r_3(\theta)},\theta\in[4\pi/3,2\pi]\end{cases} \tag{30}$$

Referring to Ref. [19], the conjugated external meshing 3-lobed noncircular gear pitch curves $r_{\mathrm{v}}(\theta)$, $r_{\mathrm{e}}(\theta_{\mathrm{e}})$ and the transmission ratio $i_{\mathrm{v}}(\theta)$ of Eq. (30) are respectively depicted in Fig. 3 and Fig. 4. According to the proposed modification algorithm in this paper, the minimal mean kinetic energy modification parameters with rotation time constraint for the pitch curve $r_{\mathrm{v}}(\theta)$ with convex cusps are listed in Table 1. The center distance a_{E} and the modified transmission ratio function $i_{\mathrm{Ev}}(\theta)$ can be solved by the noncircular gear meshing principle and listed in Table 2. Without additional explanation, the units of parameters in this paper are adopted by standard international unit (SIU).

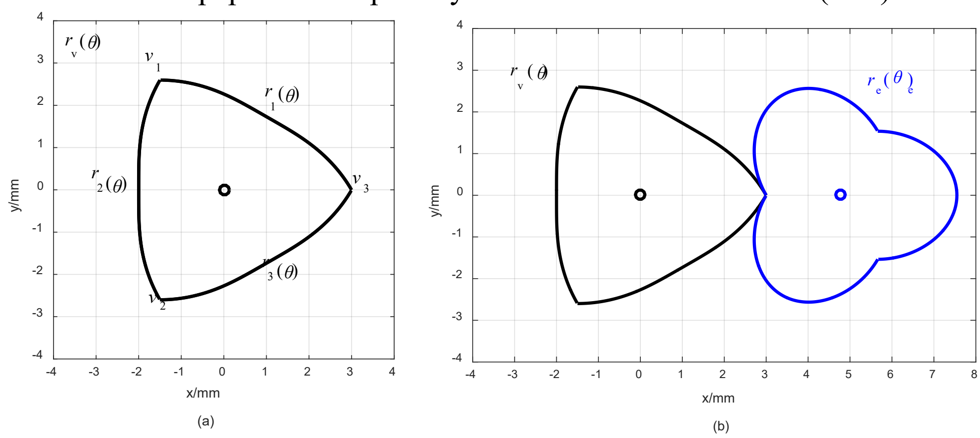

Fig. 3. Conjugated external meshing 3-lobed noncircular gear pitch curves $r_{\mathrm{v}}(\theta)$, $r_{\mathrm{e}}(\theta_{\mathrm{e}})$ in Ref. [19]

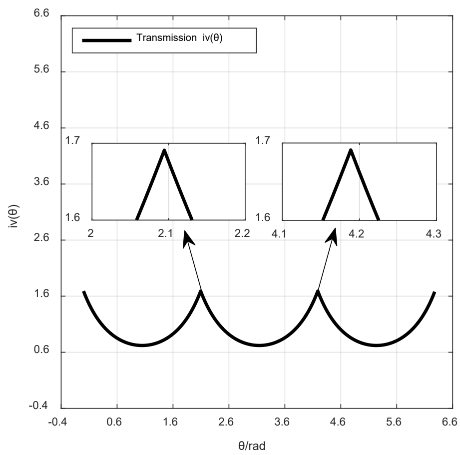

Fig. 4. Given transmission ratio $i_{\mathrm{v}}(\theta)$ of Eq. (30) in Ref. [19]

Table 1

Minimal mean kinetic energy modification parameters with rotation time constraint

Constraint T and angular velocity ω	Undetermined parameters f_1, f_2, f_3, θ_{a1}, θ_{b1}, α_{a1} and α_{b1}	Minimal rotary inertia modified model for the pitch curve with convex cusps
$\begin{cases} T = 0.08 \\ \omega = 1 \end{cases}$	$\begin{cases} f_1 = -1.2117, f_2 = -10.0120, f_3 = \frac{2\pi}{3} \\ \theta_{a1} = \frac{2\pi}{3} - 0.0380, \theta_{b1} = \frac{2\pi}{3} + 0.0380 \\ \mu_{a1} = 0.5232\ , \mu_{b1} = -0.5232 \end{cases}$	$r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{-1.2117\sec\mu + 10.0120} \\ \theta(\mu)= -0.0610\ln\left\lvert\frac{\tan\frac{\mu}{2} + 0.8855}{\tan\frac{\mu}{2} - 0.8855}\right\rvert + \frac{2\pi}{3} \end{cases}$
$\begin{cases} T = 0.09 \\ \omega = 1 \end{cases}$	$\begin{cases} f_1 = -1.3584, f_2 = -10.1324, f_3 = \frac{2\pi}{3} \\ \theta_{a1} = \frac{2\pi}{3} - 0.0428, \theta_{b1} = \frac{2\pi}{3} + 0.0428 \\ \mu_{a1} = 0.5231\ , \mu_{b1} = -0.5231 \end{cases}$	$r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{-1.3584\sec\mu + 10.1342} \\ \theta(\mu)= -0.0676\ln\left\lvert\frac{\tan\frac{\mu}{2} + 0.8738}{\tan\frac{\mu}{2} - 0.8738}\right\rvert + \frac{2\pi}{3} \end{cases}$
$\begin{cases} T = 0.1 \\ \omega = 1 \end{cases}$	$\begin{cases} f_1 = -1.5042, f_2 = -10.2556, f_3 = \frac{2\pi}{3} \\ \theta_{a1} = \frac{2\pi}{3} - 0.0476, \theta_{b1} = \frac{2\pi}{3} + 0.0476 \\ \mu_{a1} = 0.5229\ , \mu_{b1} = -0.5229 \end{cases}$	$r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{-1.5042\sec\mu + 10.2556} \\ \theta(\mu)= -0.0741\ln\left\lvert\frac{\tan\frac{\mu}{2} + 0.8627}{\tan\frac{\mu}{2} - 0.8627}\right\rvert + \frac{2\pi}{3} \end{cases}$
$\begin{cases} T = 0.11 \\ \omega = 1 \end{cases}$	$\begin{cases} f_1 = -1.6491, f_2 = -10.3760, f_3 = \frac{2\pi}{3} \\ \theta_{a1} = \frac{2\pi}{3} - 0.0523, \theta_{b1} = \frac{2\pi}{3} + 0.0523 \\ \mu_{a1} = 0.5228\ , \mu_{b1} = -0.5228 \end{cases}$	$r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{-1.6491\sec\mu + 10.3760} \\ \theta(\mu)= -0.0805\ln\left\lvert\frac{\tan\frac{\mu}{2} + 0.8519}{\tan\frac{\mu}{2} - 0.8519}\right\rvert + \frac{2\pi}{3} \end{cases}$
$\begin{cases} T = 0.12 \\ \omega = 1 \end{cases}$	$\begin{cases} f_1 = -1.7930, f_2 = -10.4956, f_3 = \frac{2\pi}{3} \\ \theta_{a1} = \frac{2\pi}{3} - 0.0571, \theta_{b1} = \frac{2\pi}{3} + 0.0571 \\ \mu_{a1} = 0.5226\ , \mu_{b1} = -0.5226 \end{cases}$	$r_{m1}(\theta)=\begin{cases} r_{m1}(\mu)=\sqrt{-1.7930\sec\mu + 10.4956} \\ \theta(\mu)= -0.0867\ln\left\lvert\frac{\tan\frac{\mu}{2} + 0.8415}{\tan\frac{\mu}{2} - 0.8415}\right\rvert + \frac{2\pi}{3} \end{cases}$

Table 2

Solved center distance a_E

Constraint T and angular velocity ω	$\begin{cases} T = 0.08 \\ \omega = 1 \end{cases}$	$\begin{cases} T = 0.09 \\ \omega = 1 \end{cases}$	$\begin{cases} T = 0.1 \\ \omega = 1 \end{cases}$	$\begin{cases} T = 0.11 \\ \omega = 1 \end{cases}$	$\begin{cases} T = 0.12 \\ \omega = 1 \end{cases}$
External meshing maximum polar angle θ_{Ee} of one lobed noncircular gear pitch curve	$\begin{cases} \theta_{Ee} = \frac{2\pi}{3} = \int_0^{\frac{2\pi}{3}} \frac{R_1(\theta)}{\alpha_E - R_1(\theta)} d\theta \\ R_1(\theta) = \begin{cases} r_1(\theta) = 4 - \sqrt{3}\sin\theta - \cos\theta, \theta \in [\theta_{b1} - \frac{2\pi}{N}, \theta_{a1}] \\ r_{m1}(\theta), \theta \in [\theta_{a1}, \theta_{b1}] \end{cases} \end{cases}$				
Solved center distance a_E	4.7729	4.7724	4.7718	4.7716	4.7710
Modified transmission ratio function $i_{Ev}(\theta)$, $\theta \in [0, 2\pi/3]$	$\frac{R_1(\theta)}{4.7729 - R_1(\theta)}$	$\frac{R_1(\theta)}{4.7729 - R_1(\theta)}$	$\frac{R_1(\theta)}{4.7718 - R_1(\theta)}$	$\frac{R_1(\theta)}{4.7716 - R_1(\theta)}$	$\frac{R_1(\theta)}{4.7710 - R_1(\theta)}$

Referring to Table 1 and Table 2, when the given constraint T is equal 0.08, the modified pitch curve $R(\theta)$ and its conjugate external meshing pitch curve $R_{Ee}(\theta_{Ee})$ are depicted in Fig. 5(a)~(b). And the pitch curve equations $R(\theta)$ and $R_{Ee}(\theta_{Ee})$ are shown in Eq. (31) and Eq. (32), respectively.

$$R(\theta)=\begin{cases}\begin{cases}r_1(\theta)=4-\sqrt{3}\sin\theta-\cos\theta,\theta\in[0.0380,\frac{2\pi}{3}-0.0380]\\ r_{m1}(\theta),\theta\in[\frac{2\pi}{3}-0.0380,\frac{2\pi}{3}+0.0380]\end{cases}\\ \begin{cases}r_2(\theta)=r_1(\theta-\frac{2\pi}{N}),\theta\in[\frac{2\pi}{3}+0.0380,\frac{4\pi}{3}-0.0380]\\ r_{m2}(\theta)=r_{m1}(\theta-\frac{2\pi}{N}),\theta\in[\frac{4\pi}{3}-0.0380,\frac{4\pi}{3}+0.0380]\end{cases}\\ \begin{cases}r_3(\theta)=r_1(\theta-\frac{4\pi}{3}),\theta\in[\frac{4\pi}{3}+0.0380,2\pi-0.0380]\\ r_{m3}(\theta)=r_{m1}(\theta-\frac{4\pi}{3}),\theta\in[2\pi-0.0380,2\pi+0.0380]\end{cases}\end{cases} \tag{31}$$

$$\begin{cases}R_{Ee}(\theta_{Ee})=4.7729-R(\theta)\\ \theta_{Ee}=\int_0^{\theta}\frac{R(\theta)}{4.7729-R(\theta)}d\theta\end{cases} \tag{32}$$

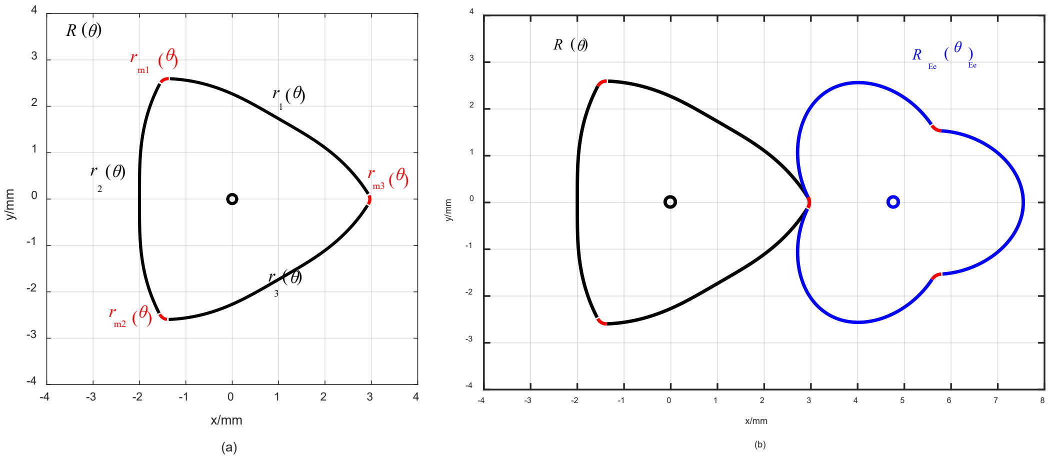

Fig. 5. Minimal mean kinetic energy modified pitch curve $R(\theta)$ and its conjugated external meshing noncircular gear pitch curve $R_{Ee}(\theta_{Ee})$

Referring to Tables 1~2 and Ref. [19], the changes of $\Delta\theta$ and modified transmission ratio functions for the minimal mean kinetic energy modification proposed in this paper and the minimal rotary inertia modification proposed in Ref. [19] with different rotation time constraints of pitch curve with convex cusps are respectively depicted in Fig. 6 and Fig. 7.

By comparing Fig. 6 and Fig. 7, we know that the minimal mean kinetic energy modification proposed in this paper has less error than the minimal rotary inertia modification proposed in Ref. [19] for the pitch curve defect with convex cusps when the rotation time constraint is equal, this result can clearly be represented by Fig. 8(a). Namely, when the rotation time constraint is given and equal, the modification accuracy of minimal mean kinetic energy modification method proposed in this paper is higher to that of the minimal rotary inertia modification method proposed in Ref. [19]. Fig. 8(b) is the comparisons of the transmission ratio function between the original 3-lobed noncircular gears and the modified 3-lobed noncircular gears with the rotation time constraint (T=0.08). It shows that the transmission stability of the minimal mean kinetic energy modification proposed in this paper is superior to that of the minimal rotary inertia modification proposed in Ref. [19].

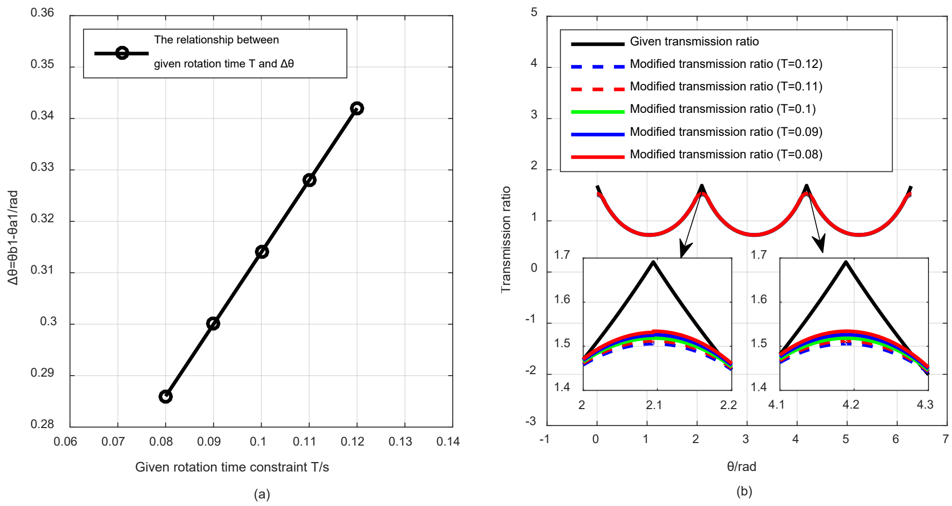

Fig. 6. Changes of $\Delta\theta$ and modified transmission ratio functions for the minimal rotary inertia modification proposed in this paper

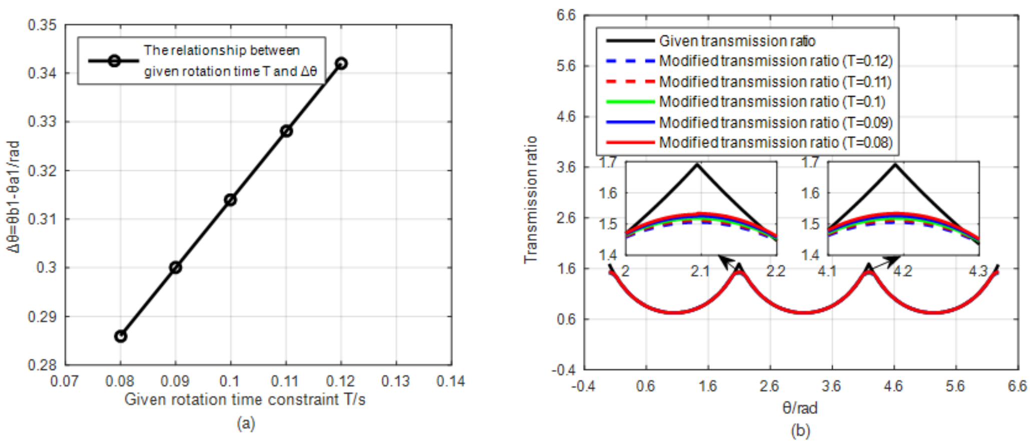

Fig. 7. Changes of $\Delta\theta$ and modified transmission ratio functions for the minimal rotary inertia modification proposed in Ref. [19]

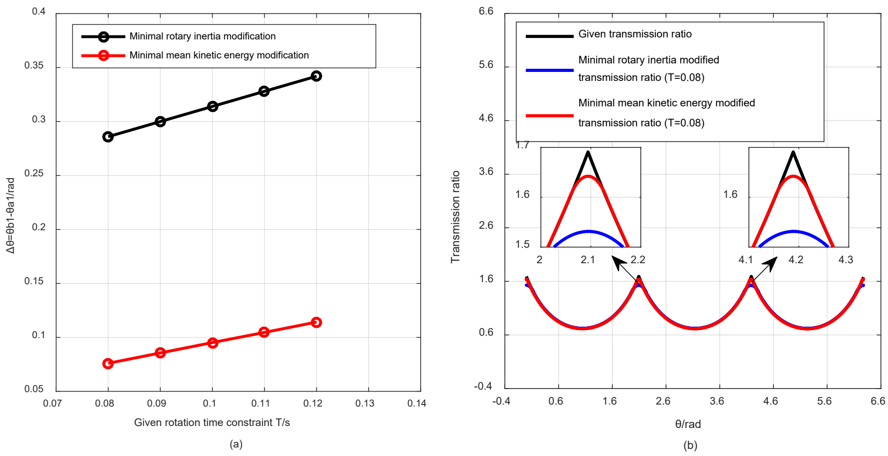

Fig. 8. Comparisons of two modification methods

5. Conclusions

The minimal mean kinetic energy modification method of pitch curve design defect with convex cusps for *N*-lobed noncircular gear with rotation time constraint are proposed. A modification example is plotted for illustration and its results demonstrate the validity of the proposed modification method.

And when the rotation time constraint is given and equal, the modification accuracy of minimal mean kinetic energy modification method is higher than that of the previous proposed modification method. Namely, the transmission stability of minimal mean kinetic energy modified *N*-lobed noncircular gears for the pitch curve design with convex cusps is superior to that of the previous proposed modification method.

According to different application environments and constraint conditions of multi-lobed noncircular gears, the modification accuracy is higher with the decrease of the rotation time constraint. Therefore, in the actual modification process, the arbitrary modification accuracy requirement can be guaranteed by choosing appropriate rotation time constraint.

Although the method proposed in this paper has many advantages, the computational complexity of the proposed method and its corresponding tooth profile calculation will increase significantly with the increase of the number of noncircular gears lobes, reduce the design efficiency of noncircular gears and the strength of the effect will be analyzed in future studies.

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R E F E R E N C E S

[1]. *F. L. Litvin and A. Fuentes*, (Gear geometry and applied theory), Cambridge University Press, Cambridge, UK, 2004.

[2]. *G. Figliolini and J. Angeles*, "The synthesis of elliptical gears generated by shaper-cutters", Journal of Mechanical Design, Transactions of the ASME, **vol. 125**, no. 4, 2003, pp. 793-801.

[3]. *Z. Stefano*, *M. F. Christian and F. Marco*, "A method for the design of ring dampers for gears in aeronautical applications", Journal of Mechanical Design, Transactions of the ASME, **vol. 134**, no. 9, 2012, pp. 091003.1- 091003.10.

[4]. *G. Figliolini*, *C. Lanni and M. Ceccarelli*, "On the kinematic synthesis of non-circular gears", International Conference on Gears & Transmissions and mechanical Systems, **vol. 3**, 2000, pp. 90-98.

[5]. *F. Q. R. Hector*, *C. F. Salvador and J. N. Liuisa*, "The Synthesis of a *N*-lobe Noncircular Gear Using Bézier and B-spline Nonparametric Curve on the Design of Its Displacement Law", Journal of Mechanical Design, Transactions of the ASME, **vol. 129**, no. 9, 2007, pp. 982-985.

[6]. *X. Zhang*, *S. W. Fan*, "Study on the pitch curve with minimal rotary inertia for the noncircular gears", Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, **vol. 232**, no. 15, 2018, pp. 2666-2673.

[7]. *X. Zhang*, *S. Fan*, "Generation of the pitch curves with minimal mean kinetic energy characteristic for noncircular gears", UPB Scientific Bulletin, Series D: Mechanical Engineering, **vol. 84**, no. 4, 2019, pp. 151-160.

[8]. *Y. P. Liu*, *X. T. Wu and H. Q. Li*, "Study on some common close convex pitch curves of noncircular gear pair", Transactions of the Chinese Society for Agricultural Machinery, **vol.** 38, no. 6, 2007, pp. 143-146.

[9]. *G. Figliolini*, *C. Lanni and M. Ceccarelli*, "On the kinematic synthesis of non-circular gears", International Conference on Gears & Transmissions and mechanical Systems, **No. 3**, 2000, pp. 283–292.

[10]. *G. H. Xu*, *J. N. Chen and G. F. Zhang*, "Design and performance analysis of Fourier non-circular gear-driven differential pump", Transactions of the Chinese Society for Agricultural Machinery, **vol.** 45, no. 12, 2014, pp. 80-87.

[11]. *G. Danieli*, "Analytical description of meshing of constant pressure angle teeth profiles on a variable radius gear and its applications", Journal of Mechanical Design, Transactions of the ASME, **vol. 122**, no. 1, 2000, pp. 123–129.

[12]. *W. X. Yao*, (Noncircular gears design), China Machine Press, Beijing, China, 2013.

[13]. *X. Zhang and S. W. Fan*, "Synthesis of the Steepest Rotation Pitch Curve Design for Noncircular Gear", Mechanism and Machine Theory, **vol. 102**, 2016, pp. 16-35.

[14]. *X. Zhang*, *S. W. Fan*, "Classification identification and modification of the pitch curve with nondifferentiable points for *N*-lobed noncircular gear", Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, **vol. 230**, no. 19, 2016, pp. 3556-3567.

[15]. *G. Lv*, *S. W. Fan and X. Zhang*, "Study on the design and the shaping of the pitch surface of a multi-lobed non-circular bevel gear", Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering, **vol. 230**, no. 4, 2016, pp. 542-553.

[16]. *X. Zhang*, *S. Fan*, "Modification design with minimal rotary inertia for the pitch curve with concave cusps of *N*-LNG", Academic Journal of Manufacturing Engineering, **vol. 17**, no. 3, 2019, pp. 113-121.

[17]. *X. Zhang*, *S. Fan*, "Minimal mean kinetic energy modification for the design defect of pitch curve with concave cusps of *N*-lobe noncircular gear", UPB Scientific Bulletin, Series D: Mechanical Engineering, **vol. 82**, no. 4, 2020, pp. 191-198.

[18]. *D. Z. Lao*, (Fundamentals of the calculus of variations), National Defense Industry Press, Beijing, China, 2015.

[19]. *X. Zhang*, "The minimal rotary inertia modified model with rotation time constraint for the pitch curve with convex cusps defect for multi-lobe non-circular gear", UPB Scientific Bulletin, Series D: Mechanical Engineering, **vol. 83**, no. 4, 2021, pp. 211-222.