QUASI-F-POWER INCREASING SEQUENCES AND THEIR NEW APPLICATIONS

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In this paper, a known result dealing with an application of quasi-f-power increasing sequences has been proved under less and weaker conditions. Some new results have also been obtained.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n^α nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is $t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v$, where $A_n^\alpha = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), A_{-n}^\alpha = 0$, for $n > 0$.

The series $\sum a_n$ is said to be sumable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if (see [5]) $\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^\alpha \mid^k < \infty$. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence $X = (X_n)$ is said to be a quasi σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ holds for all $n \geq m \geq 1$ (see [6]). It should be noted that every almost increasing sequence is a quasi- σ-power increasing sequence for any nonnegative σ, but the converse may not be true as can be seen by taking an example, say $X_n = n^{-\sigma}$ for $\sigma > 0$. A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} \mid \lambda_n - \lambda_{n+1} \mid < \infty$. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$, holds for $n \geq m \geq 1$, where $f = (f_n) = [n^\sigma(\log n)^\gamma$, $\gamma \geq 0$, $0 < \sigma < 1]$ (see [8]). It should be noted that if we take γ=0 , then we get a quasi-σ-power increasing sequence. In [3], we have proved the following theorem dealing with an application of a quasi- σ-power increasing sequences.

Theorem A. Let $(\lambda_n) \in \mathcal{BV}$ and (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$. Suppose also that there exist sequences (β_n) and (λ_n), such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \quad \beta_n \to 0 \quad as \quad n \to \infty \tag{1}$$

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$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \quad \mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2}$$

If the sequence (u_n^α) defined by (see [7])

$$u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\le v\le n} |t_v^\alpha|, & 0 < \alpha < 1 \end{cases} \tag{3}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{4}$$

then the series $\sum a_n\lambda_n$ is sumable $\mid C, \alpha; \delta \mid_k$, $k \ge 1$ and $0 \le \delta < \alpha \le 1$.

2. **The main result.**

The aim of this paper is to prove Theorem A under less and weaker conditions. Now, we shall prove the following more general theorem.

Theorem . Let (X_n) be a quasi-f-power increasing sequence. If conditions from (1) to (2) are satisfied and if

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n\ X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{5}$$

satisfies, then the series $\sum a_n\lambda_n$ is sumable $\mid C, \alpha; \delta \mid_k$, $k \ge 1$ and $0 \le \delta < \alpha \le 1$.

Remark. It should be noted that condition (5) is the same as condition (4) when k=1. When $k > 1$ condition (5) is weaker than condition (4). But the converse is not true. As in [9] we can show that if (4) is satisfied , then get that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n\ X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n} = O(X_m).$$

If (5) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} n^{\delta k} \frac{(w_n^\alpha)^k}{n} = \sum_{n=1}^{m} \frac{(u_n^\alpha)^k}{n\ X_n^{k-1}} X_n^{k-1} = O(X_m^{k-1}) \sum_{n=1}^{m} n^{\delta k} \frac{(u_n^\alpha)^k}{n\ X_n^{k-1}} = O(X_m^k) \ne O(X_m).$$

Also it should be that the condition $(\lambda_n) \in \mathcal{BV}$ has been removed.
We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). Under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem , we have the following :

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \quad nX_n\beta_n = O(1).$$

Lemma 2 ([4]). If $0 < \alpha \le 1$ and $1 \le v \le n$, then $\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \mid \le \max_{1\le m\le v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \mid$.

3. **Proof of the theorem .** Let (T_n^α) be the nth (C, α), with $0 < \alpha \le 1$, mean

of the sequence $(na_n\lambda_n)$. Then, we have $T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v$. Applying Abel's transformation first and then using Lemma 2, we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

$$\begin{aligned}
\mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \mid \\
&\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid u_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha \quad .
\end{aligned}$$

To complete the proof of the theorem, Minkowski's inequality, it is enough to show that $\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^\alpha \mid^k < \infty \quad for \quad r = 1, 2$. Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} \mid T_{n,1}^\alpha \mid^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1} (A_n^\alpha)^{-k} \{\sum_{v=1}^{n-1} (A_v^\alpha)^k (u_v^\alpha)^k \mid \Delta\lambda_v \mid^k\} \\
&\times \{\sum_{v=1}^{n-1} 1\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k-2+k-\alpha k} \{\sum_{v=1}^{n-1} v^{\alpha k} (u_v^\alpha)^k \beta_v^k\} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k \beta_v^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2+(\alpha-\delta-1)k}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k \beta_v^k \int_v^\infty \frac{dx}{x^{2+(\alpha-\delta-1)k}} \\
&= O(1)\sum_{v=1}^{m} (u_v^\alpha)^k \beta_v \beta_v^{k-1} v^{\delta k+k-1} \\
&= O(1)\sum_{v=1}^{m} (u_v^\alpha)^k \beta_v \left(\frac{1}{vX_v}\right)^{k-1} v^{\delta k+k-1} \\
&= O(1)\sum_{v=1}^{m} v\beta_v v^{\delta k} \frac{(u_v^\alpha)^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{\delta k} \frac{(u_r^\alpha)^k}{rX_r^{k-1}} \\
&+ O(1) m\beta_m \sum_{v=1}^{m} v^{\delta k} \frac{(u_v^\alpha)^k}{vX_v^{k-1}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1)m\beta_m X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of Lemma 1 and under the hypotheses of the theorem . Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\delta k} \frac{(u_n^{\alpha})^k}{n} \\
&= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{\delta k} \frac{(u_n^{\alpha})^k}{n X_n^{k-1}} \\
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\delta k} \frac{(u_v^{\alpha})^k}{v X_v^{k-1}} \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\delta k} \frac{(u_n^{\alpha})^k}{n X_n^{k-1}} \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 1. This completes the proof of the theorem . It should be noted that, if we take $\delta = 0$ (resp. $\alpha = 1$), then we get a new result for $\mid C,\alpha \mid_k$ (resp. $\mid C,1;\delta \mid_k$) summability. Also, if we take $\gamma = 0$, then we get another new result.

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