A COMBINATORIAL CHARACTERIZATION OF TERNARY DIAGONAL ALGEBRAS

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In this paper algebras with two binary diagonal fundamental operations are studied and conditions providing their term equivalence to ternary diagonal algebras are indicated. We show that the number of ternary term operations alone determines the structure of the algebras.

Keywords: n-ary diagonal algebra, p_n-sequence, minimal extension property
MSC2010: 20N 15, 20M 07.

1. Introduction

Many important types of algebras can be defined by axiom systems containing only identities. For example, groups considered as algebras $(G, \cdot, {}^{-1}, e)$ of type $(2, 1, 0)$ are defined by three simple identities, but groups also can be defined as algebras with one binary operation satisfying one (rather complicated) identity (see [13]). On the other hand, some varieties of algebras can by characterized by finite or infinite sequences $(p_0, p_1, p_2, \ldots)$, where p_n determines the number of all distinct n-ary term operations depending on every variable defined over any nontrivial algebra from such varieties (for details the reader is refereed to Section 2). Moreover, these sequences are very useful also in some algebraic constructions (for example see [3], [4], [27], [28], [29]).

A *diagonal semigroup* (or a *rectangular band*) is an idempotent semigroup $(G, \cdot)$ satisfying the identity $xyz = xz$. Equivalently, it can be characterized as a semigroup satisfying the identity $xyx = x$ (see [17]). Diagonal semigroups seem to be firstly investigated by F. Klein-Barmen in [22], where distinct possible values of the product aba for semigroup elements a and b were discussed. Nowadays they are studied by many authors in various directions (see, e.g., [14], [18], [23], [35]) and have many important applications (see, e.g., [32], [36]).

As a generalization to the case of an n-ary algebra (i.e., an algebra with one n-ary fundamental operation for $n > 1$), J. Płonka introduced a notion of an *n-ary*

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diagonal algebra which is an idempotent n-ary algebra (A, f) satisfying the identity

$$f(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{n1},\ \ldots,\ x_{nn})) = f(x_{11},x_{22},\ldots,x_{nn})$$

(see [33]). Then, K. Urbanik proved that every diagonal algebra is term equivalent to a binary algebra with finitely many fundamental operations (see [37]). In particular, it follows from Urbanik's construction that the clone of a ternary diagonal algebra (A, f) is generated by two distinct diagonal semigroup operations, call them $\cdot$ and $\circ$, such that the operation $\cdot$ does not coincide with $\circ$ or its dual. That is to say, the algebras (A, f) and $(A, \cdot, \circ)$ are term equivalent. Therefore only algebras of the form $(A, \cdot, \circ)$ are here investigated.

The aim of this paper is to characterize the variety of ternary diagonal algebras by their ternary clones. We show the number of ternary term operations alone determines the structure of the algebras. The following four identities play a special role in our considerations.

$$(xy) \circ z = x(y \circ z), \tag{1}$$

$$(xy) \circ z = (x \circ z)y, \tag{2}$$

$$x \circ (yz) = (x \circ y)z, \tag{3}$$

$$x \circ (yz) = y(x \circ z). \tag{4}$$

Our main result is given by the following statement.

Characterization Theorem. *Let $(A, \cdot, \circ)$ be an algebra with two semigroup fundamental operations. Then the following conditions are equivalent:*

- (a) $(A, \cdot, \circ)$ *is term equivalent to an essentially ternary diagonal algebra,*
- (b) $p_3(A, \cdot, \circ) = 6$,
- (c) $(A, \cdot, \circ)$ *satisfies exactly one of the identities* (1) – (4).

Then the following classic combinatorial characterization of ternary diagonal algebras (cf. [11]) is directly derivable from Characterization Theorem.

Corollary 1.1. *The sequence* $\mathbf{a}^* = (0, 1, 6, 6, 0, 0, \ldots)$ *is the minimal extension of the sequence* $\mathbf{a} = (0, 1, 6)$ *and an arbitrary universal algebra* $\mathfrak{A}$ *represents the sequence* $\mathbf{a}^*$ *if and only if* $\mathfrak{A}$ *is term equivalent to an essentially ternary diagonal algebra.*

Recall that each diagonal semigroup $(A, \cdot)$ is isomorphic to some semigroup $(X \times Y, \cdot)$ with multiplication defined by $(x_1, y_1) \cdot (x_2, y_2) = (x_1, y_2)$. If elements (x_1, y_1) and (x_2, y_2) are viewed as opposite vertices of a rectangle in $X \times Y$, then the products $(x_1, y_1) \cdot (x_2, y_2)$ and $(x_2, y_2) \cdot (x_1, y_1)$ are the remaining vertices of this rectangle, see e.g. [2]. Following this, we get a nice geometrical interpretation of our result. Consider a binary algebra with two diagonal semigroup operations $(A, \cdot, \circ)$ whose reducts $(A, \cdot)$ and $(A, \circ)$ are decomposed to $(X_1 \times Y_1, \cdot)$ and $(X_2 \times Y_2, \circ)$, respectively. It follows from Characterization Theorem that the algebra $(A, \cdot, \circ)$ is term equivalent to a ternary diagonal algebra (A, f) if and only if these two decompositions have a common refinement, that is $(A, \cdot, \circ)$ can be decomposed to a product $X \times Y \times Z$ with two binary operations satisfying exactly one of (1)–(4).

In particular, for $X_1 = X$, $Y_1 = Y \times Z$, $X_2 = X \times Y$ and $Y_2 = Z$ we obtain a decomposition corresponding to the identity (1). Then the binary operations $\cdot$ and $\circ$ can be described as follows:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1, y_2, z_2) \quad \text{and} \quad (x_1, y_1, z_1) \circ (x_2, y_2, z_2) = (x_1, y_1, z_2).$$

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are viewed as points of the 3-dimensional real space $\mathbf{R}^3$, then the operation $\cdot$ represents the orthogonal projection of the point (x_2, y_2, z_2) onto the plane $x = x_1$, whereas the operation $\circ$ can be viewed as the orthogonal projection of the point (x_1, y_1, z_1) onto the plane $z = z_2$. Therefore the identity (1) states that the superposition of these two projections is commutative.

The remaining identities, characterizing ternary diagonal algebras, correspond to other groupings of the sets X, Y and Z and have similar geometrical interpretations.

2. **Notation and terminology**

An n-ary operation f of the set A is said to *depend* on the variable x_i, if there exist $a_1, \ldots, a_n, b \in A$ such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

If f depends on every its variable, then f is called *essentially n-ary*. Following this, an n-ary diagonal algebra (A, f) with essentially n-ary fundamental operation f will be called an *essentially* n-ary diagonal algebra.

For an universal algebra $\mathfrak{A}$, let $p_n(\mathfrak{A})$ for $n \geq 1$ denote the number of all distinct essentially n-ary term operations of $\mathfrak{A}$ and let $p_0(\mathfrak{A})$ stands for the number of all distinct constant unary term operations of $\mathfrak{A}$. Then, the sequence

$$\mathbf{p}(\mathfrak{A}) = (p_0(\mathfrak{A}), p_1(\mathfrak{A}), \ldots, p_n(\mathfrak{A}), \ldots)$$

is called the *p_n-sequence* of the algebra $\mathfrak{A}$ (see [10]). We say that an algebra $\mathfrak{A}$ *represents* a (finite or infinite) sequence $(a_0, \ldots, a_n, \ldots)$, if $p_n(\mathfrak{A}) = a_n$ for every n. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a finite sequence of non-negative integers. We say that the sequence $\mathbf{a}$ has the *minimal extension property* if there exists an algebra $\mathfrak{A}$ representing the sequence $\mathbf{a}$ such that for every algebra $\mathfrak{B}$ representing $\mathbf{a}$, we have $p_n(\mathfrak{B}) \geq p_n(\mathfrak{A})$ for all $n > k$. Then the p_n-sequence of $\mathfrak{A}$

$$\mathbf{a}^\star = (a_0, a_1, \ldots, a_k, p_{k+1}(\mathfrak{A}), p_{k+2}(\mathfrak{A}), \ldots)$$

is known as the *minimal extension* of the sequence $\mathbf{a}$ (for more details see [11]).

The theory of p_n-sequences of universal algebras was founded by E. Marczewski and his *Wrocław School* back in the sixties. It began from E. Marczewski's idea of the characterization of algebras by their clones [31]. This – together with the notion of a p_n-sequence introduced by G. Grätzer [10] – originated a method of identifying varieties of algebras with a numeric function p_n, the function invariant under the clone equivalence of algebras. As E. Marczewski expected, there are many varieties of algebras uniquely determined by their p_n-sequences (at least in some classes of algebras), e.g., the variety of semilattices, distributive lattices or Boolean algebras (see [6], [25]). In the most spectacular case only one element of a p_n-sequence uniquely determines a variety of algebras (it is so, e.g., for the variety of distributive lattices, some affine spaces or Steiner quasigroups, see [6], [7], [8], [24]). The problem of characterization of these sequences, which uniquely identify some varieties of algebras, is still open. But in many papers numbers implying algebraic structures are indicated and p_n-sequences of algebras are studied from distinct points of view (see, e.g., [1], [16], [26]). There are two main general problems in the theory of p_n-sequences. The first is to describe all sequences which can be represented as p_n-sequences of algebras of a certain kind (see, [3], [12], [16], [21]). The second is to determine which properties of algebras can be deduced from their p_n-sequences (see [4], [5], [20], [34]).

For a given algebra $\mathfrak{A} = (A, F)$, the smallest set containing all projections and all elements of F that is closed under superpositions of functions is called the set of *term operations* of $\mathfrak{A}$, or the *clone* of $\mathfrak{A}$ (for details see [30]). Two algebras defined on the same set are *term equivalent* if their clones are equal. Such algebras have the same p_n-sequences.

3. **Auxiliary results**

We consider here clones of algebras $(A, \cdot, \circ)$, which both reducts $(A, \cdot)$ and $(A, \circ)$ are not term equivalent essentially diagonal semigroups. The following four ternary operations seem to be especially important.

$$f_1(x, y, z) = (xy) \circ z, \qquad f_2(x, y, z) = x \circ (yz),$$
$$f_3(x, y, z) = (x \circ y)z, \qquad f_4(x, y, z) = x(y \circ z).$$

The proof of Characterization Theorem is based on the following three main statements.

Proposition 3.1. *If* $(A, \cdot, \circ)$ *satisfies one of the identities* (1) *and* (2)*, then* $(A, \star)$*, where* $x \star y = (xy) \circ y$*, is a diagonal semigroup and* (A, f_1)*, where* $f_1(x, y, z) = (xy) \circ z$*, is a ternary diagonal algebra term equivalent to the algebra* $(A, \cdot, \circ)$*.*

Proposition 3.2. *If* $(A, \cdot, \circ)$ *satisfies one of the identities* (3) *and* (4)*, then* $(A, \star)$*, where* $x \star y = y \circ (xy)$*, is a diagonal semigroup and* (A, f_2)*, where* $f_2(x, y, z) = x \circ (yz)$*, is a ternary diagonal algebra term equivalent to the algebra* $(A, \cdot, \circ)$*.*

Proposition 3.3. *For the algebra* $(A, \cdot, \circ)$ *we have*

$$p_2(A, \cdot, \circ) \geq 6 \quad \textit{and} \quad p_3(A, \cdot, \circ) \geq 6.$$

The proofs of these three propositions are the key part of this section. But now, let us begin with the following observation.

Lemma 3.1. *If* $(A, \cdot, \circ)$ *is an algebra with two idempotent essentially binary operations such that* xy *is not equal to* $x \circ y$ *nor* $y \circ x$*, then at least one of the term operations* f_1 *and* f_2 *is essentially ternary. The same is true for the term operations* f_3 *and* f_4*.*

Proof. First observe that $f_1(x,y,z) = (xy) \circ z$ depends on z and also on at least one of the variables x, y. Clearly, none of f_1 and f_2 is equal to a variable. Assume that both f_1 and f_2 are essentially binary (and not essentially ternary), i.e., assume that $f_1(x,y,z) = x \circ z$ or $f_1(x,y,z) = y \circ z$ and also, independently, $f_2(x,y,z) = x \circ y$ or $f_2(x,y,z) = x \circ z$. Then $xy = (xy) \circ (xy) = x \circ (xy)$ or $xy = y \circ (xy)$ which yield immediately $xy = x \circ x = x$, $xy = x \circ y$, $xy = y \circ x$ or $xy = y \circ y = y$, a contradiction. The dual statement for the term operations f_3 and f_4 can be proved analogously. □

Let $g(x_1, x_2, \ldots, x_n) = f(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$ for some fixed n-ary term operation f and a permutation σ of variables $x_1, x_2, \ldots, x_n$. Such defined term operation g is denoted by f^{σ}. In the case $f = f^{\sigma}$ we say that the permutation σ is *admissible* by the operation f. The set of all admissible permutations for a fixed term operation f forms a group which is called the *symmetry group* of f.

In the further considerations we assume that the operations $\cdot$ and $\circ$ are not commutative.

Lemma 3.2. *The symmetry groups of the term operations f_1, f_2, f_3, f_4 have only one element.*

Proof. Assume that the term operation $f_1(x,y,z) = (xy) \circ z$ admits the transposition (x, y), i.e., the identity $(xy) \circ z = (yx) \circ z$ holds. Replacing in this identity x by xy we get $((xy)y) \circ z = (y(xy)) \circ z$. Since $(A, \cdot)$ is a diagonal semigroup, the last identity implies $(xy) \circ z = y \circ z$. Consequently, we have $y \circ z = (xy) \circ z = (yx) \circ z = x \circ z$ and finally, by idempotence of $\circ$, the identity $x \circ z = y \circ z$ leads to $x \circ y = y$, a contradiction. Using again idempotence of the operations $\cdot$ and $\circ$, we infer that for every permutation σ of variables x, y, z such that $\sigma(z) \neq z$, the identity $f_1(x,y,z) = f_1(\sigma(x), \sigma(y), \sigma(z))$ implies $z = \sigma(z)$, a contradiction. Therefore the symmetry group of the term operation f_1 contains only the identity permutation.

For the term operations f_2, f_3 and f_4 the proof is similar. □

If the symmetry group of an essentially n-ary term operation f of an algebra $\mathfrak{A}$ has k elements, then the clone of $\mathfrak{A}$ contains exactly $\frac{n!}{k}$ distinct essentially n-ary term operations obtained from f by permuting of its variables. Therefore we have the following.

Corollary 3.1. *Every essentially ternary term operation f_i, where $i \in \{1, \ldots, 4\}$, induces six distinct essentially ternary term operations obtained by permuting of its variables.*

Lemma 3.3. *If the term operations f_1 and f_2 both are essentially ternary, then $p_3(A, \cdot, \circ) \geq 12$. The same is true for f_3 and f_4.*

Proof. According to Lemma 3.2, the symmetry groups of term operations f_1 and f_2 have only one element. So, it suffices to prove that

$$(xy) \circ z \notin \{x \circ (yz),\ x \circ (zy),\ y \circ (xz),\ y \circ (zx),\ z \circ (xy),\ z \circ (yx)\}.$$

Then every of f_1 and f_2, generates 6 essentially ternary term operations which are pairwise distinct and, consequently, $p_3(A,\cdot,\circ) \geq 12$.

Obviously, the assumption $(xy)\circ z \in \{z\circ(xy),\, z\circ(yx)\}$, leads to the commutativity of $\circ$, a contradiction. Let $(xy)\circ z = x\circ(yz)$. Then $xy = (xy)\circ(xy) = x\circ(y(xy)) = x\circ y$, which is impossible. If $(xy)\circ z = x\circ(zy)$, then $(xy)\circ z = ((xy)\circ z)\circ z = (x\circ(zy))\circ z = x\circ z$ and hence $(xy)\circ z$ is not essentially ternary despite the assumption. More general, if $(xy)\circ z = \sigma(z)\circ(\sigma(x)\sigma(y))$ for an arbitrary permutation σ of the set $\{x,y,z\}$, then $(xy)\circ z = ((xy)\circ z)\circ z = \big(\sigma(z)\circ(\sigma(x)\sigma(y))\big)\circ z = \sigma(z)\circ z$, so $(xy)\circ z$ is not essentially ternary despite the assumption.

The dual statement for the term operations f_3 and f_4 holds analogously. This completes the proof. □

Lemma 3.4. *The term operation $x*y = f_1(x,y,x)$ is not commutative and $x*y \notin \{y,\, yx,\, x\circ y\}$. Moreover,*

(*a*) $x*y = x \Longleftrightarrow f_1(x,y,z) = x\circ z$,

(*b*) $x*y = xy \Longleftrightarrow f_2(x,y,z) = x\circ y$,

(*c*) $x*y = y\circ x \Longleftrightarrow f_1(x,y,z) = y\circ z$.

Proof. All the statements are easy to prove using diagonality of the operations $\cdot$ and $\circ$. □

Lemma 3.5. $p_2(A,\cdot,\circ) \geq 6$.

Proof. Let $x*y = f_1(x,y,x)$. By Lemma 3.4, $x*y \notin \{y,\, yx,\, x\circ y\}$. Suppose that $x*y \in \{x,\, xy,\, y\circ x\}$.

Case 1. $x*y = x$. Then $f_1(x,y,z) = x\circ z$ which means that f_1 is not essentially ternary. So, f_2 must be essentially ternary (Lemma 3.1). Let $x\bullet y = f_2(x,y,x) = x\circ(yx)$. Since f_2 is essentially ternary $x\bullet y \neq x$. In the case $x\bullet y = y$, by diagonality of $\circ$, we obtain $y = y\circ y = (x\bullet y)\circ y = (x\circ(yx))\circ y = x\circ y$, a contradiction. Hence $x\bullet y \notin \{x,\, y\}$. Also $x\bullet y \notin \{x\circ y,\, y\circ x\}$ because f_2 is essentially binary. If $x\bullet y = xy$, then $z\circ(xy) = z\circ(x\bullet y) = z\circ(x\circ(yx)) = z\circ(yx)$, which contradicts Lemma 3.2. Let now $x\bullet y = yx$. Then $(yx)\circ z = (x\bullet y)\circ z = (x\circ(yx))\circ z = x\circ z = (xy)\circ z$, by Lemma 3.4 (*a*). This also contradicts Lemma 3.2. So, $x\bullet y \notin \{xy,\, yx,\, x\circ y,\, y\circ x\}$. Assume that $x\bullet y = y\bullet x$. Since $(A;\circ)$ is an essentially diagonal semigroup, we get $x = x\circ x = (x\circ(yx))\circ x = (x\bullet y)\circ x = (y\bullet x)\circ x = (y\circ(xy))\circ x = y\circ x$, a contradiction. Therefore $x\bullet y$ is not commutative.

Case 2. $x*y = xy$. Then $f_2(x,y,z) = x\circ y$, by Lemma 3.4 (*b*), which means that f_1 is essentially ternary (Lemma 3.1). Consider the operation $x\star y = f_1(x,y,y) = (xy)\circ y$. Obviously $x\star y \notin \{x,\, y, x\circ y,\, y\circ x\}$. If $x\star y = xy$, then $z\circ x = f_2(z,x,y) = z\circ(xy) = z\circ(x\star y) = z\circ((xy)\circ y) = z\circ y$, which implies $z\circ x = z$, a contradiction. For $x\star y = yx$ we have $f_1(x,y,z) = (y\star x)\circ z = f_1(y,x,z)$. This contradicts Lemma 3.2. So, $x\star y \notin \{xy,\, yx,\, x\circ y,\, y\circ x\}$. Assume $x\star y = y\star x$. Since $(A;\circ)$ is an essentially diagonal semigroup, we get $y = y\circ y = y\circ((xy)\circ y) =$

$y \circ (x \star y) = y \circ (y \star x) = y \circ ((yx) \circ x) = y \circ x$, a contradiction. Therefore $x \star y$ is not commutative.

CASE 3. $x * y = y \circ x$. Then, by Lemma 3.4 (c), we have $f_1(x, y, z) = y \circ z$ which shows that f_1 is not essentially ternary. Hence f_2 is essentially ternary. Let $x \odot y = f_2(x, x, y) = x \circ (xy)$. If $x \odot y \in \{x, y\}$, then $z \circ x = z \circ (xy) = z \circ y$, a contradiction. If $x \odot y = x \circ y$, then $f_2(x, y, z) = x \circ yz = x \odot yz = x \circ (x(yz)) = x \circ xz = x \circ z = x \odot z = f_2(x, x, z)$ which is impossible because f_2 is essentially ternary. By the same reason $x \odot y \neq y \circ x$. If $x \odot y = xy$, then $x \circ z = (xy) \circ z = y \circ z$, a contradiction. For $x \odot y = yx$ we obtain $f_2(z, x, y) = f_2(z, y, x)$, which also is impossible (Lemma 3.2). Hence $x \odot y \notin \{xy,\ yx,\ x \circ y,\ y \circ x\}$. Assume $x \odot y = y \odot x$. Then $x = x \circ x = (x \circ (yx)) \circ x = (x \odot y) \circ x = (y \odot x) \circ x = (y \circ (yx)) \circ x = y \circ x$, a contradiction. Therefore $x \odot y$ is not commutative.

Summarizing, in any case we have at least six essentially binary term operations. Therefore $p_2(A, \cdot, \circ) \geq 6$, as required. □

Lemma 3.6. *If the algebra $(A, \cdot, \circ)$ satisfies the identity $f_1 = f_3^{\sigma}$, then $\sigma(y) = z$ and $\sigma(z) = y$, i.e., this algebra satisfies* (2) *and $x \circ (yz) = x \circ y$.*

Proof. Indeed, in the case $(xy) \circ z = (y \circ z)x$ we also have $(xy) \circ y = yx$ and, consequently, $(yx) \circ z = (xy) \circ z$, which contradicts the statement of Lemma 3.2. In the case $(xy) \circ z = (z \circ y)x$ we have $(xy) \circ y = yx$ and as above $(yx) \circ z = (xy) \circ z$, a contradiction. If $(xy) \circ z = (z \circ x)y$, then $(xy) \circ x = xy$ and hence $z \circ (xy) = z \circ x$. But then we have $(xy) \circ z = ((z \circ (xy))y$ and, consequently, $x \circ z = (z \circ x)x$. This implies $(x \circ z)y = ((z \circ x)x)y = (z \circ x)y$, which contradicts Lemma 3.2. If $(xy) \circ z = (x \circ y)z$ or $(xy) \circ z = (y \circ x)z$, then $x \circ z = xz$ which is impossible because, by assumption, algebras $(A, \cdot)$ and $(A, \circ)$ are not term equivalent. Therefore, $(xy) \circ z = (x \circ z)y$ and hence also $(xy) \circ x = xy$, $z \circ (xy) = z \circ x$ and consequently $x \circ (yz) = x \circ y$, as required. □

Lemma 3.7. *If the algebra $(A, \cdot, \circ)$ satisfies the identity $f_1 = f_4^{\sigma}$, then σ is the identity permutation, i.e., this algebra satisfies* (1) *and $x \circ (yz) = x \circ z$.*

Proof. $(A, \cdot)$ and $(A, \circ)$ are not term equivalent, thus $(xy) \circ z \notin \{z(x \circ y),\ z(y \circ x)\}$. If $(xy) \circ z = y(x \circ z)$, then $(xy) \circ x = yx$ and $(xy) \circ z = (yx) \circ z$, a contradiction with Lemma 3.2. If $(xy) \circ z = y(z \circ x)$, then $(xy) \circ x = yx$ and, consequently, $(yx) \circ z = (xy) \circ z$, a contradiction. If $(xy) \circ z = x(z \circ y)$, then as above we get $z \circ (xy) = z \circ y$. But then $(xy) \circ z = x(z \circ (xy))$ and, consequently, $x \circ z = x(z \circ x)$. This implies $y(x \circ z) = y(z \circ x)$, which contradicts Lemma 3.2. Therefore must be $(xy) \circ z = x(y \circ z)$. Then also $(xy) \circ y = xy$, $z \circ (xy) = z \circ y$ and consequently $x \circ (yz) = x \circ z$, as required. □

Analogously one can prove the following.

Lemma 3.8.

(*a*) *If the algebra $(A, \cdot, \circ)$ satisfies the identity $f_2 = f_3^{\sigma}$, then σ is the identity permutation, i.e., this algebra satisfies* (3) *and $(xy) \circ z = x \circ z$.*

(b) *If the algebra* $(A,\cdot,\circ)$ *satisfies the identity* $f_2 = f_4^\sigma$, *then* $\sigma(x) = y$ *and* $\sigma(y) = x$, *i.e., this algebra satisfies* (4) *and* $(xy)\circ z = y\circ z$.

Lemma 3.9. *If* $p_3(A,\cdot,\circ) = 6$, *then exactly one of operations* f_1, f_2 *is an essentially ternary diagonal operation and the algebra* $(A,\cdot,\circ)$ *is term equivalent to this algebra* (A,f_i) *which is essentially ternary.*

Proof. The term operations f_1, f_2, f_3, f_4 are all idempotent. By Corollary 3.1, every essentially ternary operation f_i induces six distinct ternary operations f_i^σ obtained by permuting of variables in f_i. Lemmas 3.1 and 3.3 together with the assumption $p_3(A,\cdot,\circ) = 6$ show that exactly one of the operations f_1, f_2 is essentially ternary. So, if f_1 is essentially ternary, then for some permutation σ of variables x,y,z we have either $f_1 = f_3^\sigma$ or $f_1 = f_4^\sigma$. Similarly, if f_2 is essentially ternary, then either $f_2 = f_3^\sigma$ or $f_2 = f_4^\sigma$. This, by Lemmas 3.6 and 3.7, means that in $(A,\cdot,\circ)$ exactly one of the identities (1) – (4) is satisfied.

If $(A,\cdot,\circ)$ satisfies one of identities (1) and (2), then the operation f_1 is idempotent and essentially ternary. It is also diagonal. In fact, in the case when (1) is satisfied we get

$$\begin{aligned}
&f_1(f_1(x_{11},x_{12},x_{13}),f_1(x_{21},x_{22},x_{23}),f_1(x_{31},x_{32},x_{33}))\\
&\quad= \big(((x_{11}x_{12})\circ x_{13})((x_{21}x_{22})\circ x_{23})\big)\circ((x_{31}x_{32})\circ x_{33})\\
&\quad= \big(((x_{11}x_{12})\circ x_{13})((x_{21}x_{22})\circ x_{23})\big)\circ x_{33}\\
&\quad= ((x_{11}x_{12})\circ x_{13})\big(((x_{21}x_{22})\circ x_{23})\circ x_{33}\big)\\
&\quad= ((x_{11}x_{12})\circ x_{13})((x_{21}x_{22})\circ x_{33}) = (x_{11}(x_{12}\circ x_{13}))(x_{21}(x_{22}\circ x_{33}))\\
&\quad= x_{11}(x_{22}\circ x_{33}) = (x_{11}x_{22})\circ x_{33} = f_1(x_{11},x_{22},x_{33}),
\end{aligned}$$

In the case of (2) we have

$$\begin{aligned}
&f_1(f_1(x_{11},x_{12},x_{13}),f_1(x_{21},x_{22},x_{23}),f_1(x_{31},x_{32},x_{33}))\\
&\quad= \big(((x_{11}x_{12})\circ x_{13})((x_{21}x_{22})\circ x_{23})\big)\circ((x_{31}x_{32})\circ x_{33})\\
&\quad= \big(((x_{11}x_{12})\circ x_{13})((x_{21}x_{22})\circ x_{23})\big)\circ x_{33}\\
&\quad= \big(((x_{11}\circ x_{13})x_{12})((x_{21}\circ x_{23})x_{22})\big)\circ x_{33}\\
&\quad= \big((x_{11}\circ x_{13})(x_{12}(x_{21}\circ x_{23})x_{22})\big)\circ x_{33} = ((x_{11}\circ x_{13})(x_{12}x_{22}))\circ x_{33}\\
&\quad= ((x_{11}\circ x_{13})x_{22})\circ x_{33} = ((x_{11}x_{22})\circ x_{13})\circ x_{33} = (x_{11}x_{22})\circ x_{33}\\
&\quad= f_1(x_{11},x_{22},x_{33}).
\end{aligned}$$

as required. For the identities (3) and (4) we consider the operation f_2. The argumentation is very similar, so we omit it.

Since every essentially ternary term operation f_i easily induces both fundamental operations $\cdot$ and $\circ$, we get the term equivalence between $(A,\cdot,\circ)$ and (A,f_i). □

Proof of Proposition 3.1. If at least one of the fundamental diagonal operations of $(A,\cdot,\circ)$ is commutative, then $|A| = 1$. Thus the assumption that both $(A,\cdot)$ and

$(A, \circ)$ are essentially diagonal semigroups, implies that both $\cdot$ and $\circ$ are not commutative.

It is not difficult to verify that $x \star y = (xy) \circ y = f_1(x, y, y)$ is an essentially binary diagonal operation. Since f_1 is defined as a combination of the fundamental operations of $(A, \cdot, \circ)$ which are both idempotent, it is enough to note that $xy = f_1(x, y, xy)$ and $x \circ y = f_1(x, x, y)$. The rest is a consequence of Lemma 3.9. □

Proof of Proposition 3.2. is dual to the proof of Proposition 3.1. □

Proof of Proposition 3.3. It follows from Lemmas 3.1 and 3.2 for essentially ternary term operations and from Lemma 3.5 for essentially binary ones. □

4. **The main results**

This section is devoted predominantly to the proof of our main results. We recall them for the convenience of the reader. Then we present some examples and pose some problems.

Theorem 4.1. *Let* $(A, \cdot, \circ)$ *be an algebra with two semigroup fundamental operations. Then the following conditions are equivalent:*

- (a) $(A, \cdot, \circ)$ *is term equivalent to an essentially ternary diagonal algebra,*
- (b) $p_3(A, \cdot, \circ) = 6$*,*
- (c) $(A, \cdot, \circ)$ *satisfies exactly one of the identities* (1) – (4).

Proof. $(a) \Rightarrow (b)$ Assume that $(A, \cdot, \circ)$ is (up to the term equivalence) an essentially ternary diagonal algebra. Then the number of distinct essentially ternary term operations of $(A, \cdot, \circ)$ equals 6 (routine calculations are here omitted).

$(b) \Rightarrow (c)$ Assume that $p_3(A, \cdot, \circ) = 6$. By Lemma 3.1, at least one of f_1 and f_2 is essentially ternary (the same is true for f_3 and f_4). If both f_1 and f_2 are essentially ternary, then according to Lemma 3.3 the number of distinct essentially ternary term operations of $(A, \cdot, \circ)$ is not less than 12 (the same we have for f_3 and f_4). Therefore exactly one of f_1 and f_2 (and also one of f_3 and f_4) is essentially ternary. By Lemmas 3.6 – 3.8, one of the identities (1) – (4) holds.

$(c) \Rightarrow (a)$ If one of the identities (1) – (4) holds, then by Propositions 3.1 and 3.2, the algebra $(A, \cdot, \circ)$ is term equivalent to an essentially ternary diagonal algebra, completing the proof. □

Corollary 4.1. *The sequence* $\mathbf{a}^* = (0, 1, 6, 6, 0, 0, \ldots)$ *is the minimal extension of the sequence* $\mathbf{a} = (0, 1, 6)$ *and an arbitrary universal algebra* $\mathfrak{A}$ *represents the sequence* $\mathbf{a}^*$ *if and only if* $\mathfrak{A}$ *is term equivalent to an essentially ternary diagonal algebra.*

Proof. (a) Consider the class of binary idempotent algebras $(A, \cdot, \circ)$, such that $(A, \cdot)$ and $(A, \circ)$ are non-term equivalent essentially diagonal semigroups. Assume that $(A, \cdot, \circ)$ represents the sequence $\mathbf{a} = (0, 1, 6)$. Then Proposition 3.3 implies that $p_3(A, \cdot, \circ) \geq 6$. Moreover, by Characterization Theorem, $p_3(A, \cdot, \circ) = 6$ if and only if it is term equivalent to an essentially ternary diagonal algebra $\mathfrak{A}$ (according to the following Example 4.1 such algebra really exists). By Characterization Theorem again, $\mathfrak{A}$ represents the sequence $\mathbf{a}^* = (0, 1, 6, 6, 0, 0, \ldots)$. Thus we have $p_n(A, \cdot, \circ) \geq p_n(\mathfrak{A})$ for every nonnegative integer n. Therefore $\mathbf{a}^*$ is the minimal extension of the sequence $\mathbf{a}$ in this class.

More general, assume that an arbitrary universal algebra $\mathfrak{A}$ represents the sequence $\mathbf{a}$. In particular we have $p_0(\mathfrak{A}) = 0$ and $p_1(\mathfrak{A}) = 1$. Therefore $\mathfrak{A}$ is idempotent. It follows that either its p_n-sequence $\mathbf{p}(\mathfrak{A})$ is strictly increasing or $\mathfrak{A}$ is term equivalent to a diagonal algebra (A, f) (see [19], see also [37]). Then the assumption $p_2(\mathfrak{A}) = 6$ implies that (A, f) is an essentially ternary diagonal algebra. Thus we have $p_n(\mathfrak{A}) \geq p_n(A, f)$ for every n, and consequently, the $\mathbf{p}(A, f) = \mathbf{a}^*$ is the minimal extension of the sequence $\mathbf{a}$.

(b) If an arbitrary universal algebra $\mathfrak{A}$ represents the sequence $\mathbf{a}^*$, by [37] again, we infer $\mathfrak{A}$ is an essentially ternary diagonal algebra. Conversely, consider an essentially n-ary diagonal algebra (A, f). It is clear that, by the assumptions of the idempotence and the diagonality of (A, f), every its term operation reduces to the form $f(x_{i_1}, \ldots, x_{i_n})$ for some $x_{i_1}, \ldots, x_{i_n} \in \{x_1, \ldots, x_n\}$. Note that every term operation of this form depends on every its variable. Indeed, consider a term operation $\varphi(x_1, \ldots, x_k) = f(x_{i_1}, \ldots, x_{i_n})$, where $\{x_{i_1}, \ldots, x_{i_n}\} = \{x_1, \ldots, x_k\}$ for some $k \leq n$, and assume that $i_s = j$ for some $s = 1, \ldots, k$. Suppose that $\varphi(x_1, \ldots, x_k)$ does not depend on the variable x_j. Then we have

$$\begin{aligned} f(y_1, \ldots, y_{s-1}, \varphi(x_1, \ldots, x_k), y_{s+1}, \ldots, y_n) \\ = f(y_1, \ldots, y_{s-1}, x_j, y_{s+1}, \ldots, y_n), \end{aligned}$$

and therefore $f(y_1, \ldots, y_{s-1}, x_j, y_{s+1}, \ldots, y_n)$ does not depend on the variable x_j, a contradiction.

Note also that two arbitrary term operations of this form are distinct if only their variables differ in any place. Indeed, take two term operations $\varphi_1 = f(x_{i_1}, \ldots, x_{i_n})$ and $\varphi_2 = f(x_{j_1}, \ldots, x_{j_n})$ such that $x_{i_1}, \ldots, x_{i_n}$ and $x_{j_1}, \ldots, x_{j_n}$ are variables from the set $\{x_1, \ldots, x_n\}$ satisfying $x_{i_s} \neq x_{j_s}$ for some $s \in \{1, \ldots, n\}$. Assume that $\varphi_1 = \varphi_2$. Then the following holds:

$$\begin{aligned} f(y_1, \ldots, y_{s-1}, f(x_{i_1}, \ldots, x_{i_s}, \ldots, x_{i_n}), y_{s+1}, \ldots, y_n) = \\ = f(y_1, \ldots, y_{s-1}, f(x_{j_1}, \ldots, x_{j_s}, \ldots, x_{j_n}), y_{s+1}, \ldots, y_n) \end{aligned}$$

and hence

$$f(y_1, \ldots, y_{s-1}, x_{i_s}, y_{s+1}, \ldots, y_n) = f(y_1, \ldots, y_{s-1}, x_{j_s}, y_{s+1}, \ldots, y_n),$$

a contradiction.

Therefore the number of distinct essentially k-ary term operations of (A, f) for any $k \leq n$ can be counted as the number of distinct factorizations of the n-element set onto k nonempty partitions. So, $p_k(A, f) = \left\{ {n \atop k} \right\} \cdot k!$ for every $0 \leq k \leq n$ and $p_k(A, f) = 0$ for every $k > n$, where $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind (cf. [11]). In particular, for $n = 3$ it follows that the algebra (A, f) represents the sequence $(0, 1, 6, 6, 0, 0, \ldots)$. □

Remark 4.1. Simple combinatorial arguments show that for every essentially n-ary diagonal algebra $\mathfrak{D}_n$ we have $p_0(\mathfrak{D}_n) = 0$, $p_1(\mathfrak{D}_n) = 1$, $p_m(\mathfrak{D}_n) = 0$ for every $m > n$ and for every k such that $2 \le k \le n$ the following formula holds:

$$p_k(\mathfrak{D}_n) = k^n - \sum_{l=1}^{k-1} \binom{k}{l} p_l(\mathfrak{D}_n).$$

In particular, for small k we have

$$p_2(\mathfrak{D}_n) = (2^{n-1} - 1) \cdot 2! = 2^n - 2,$$

$$p_3(\mathfrak{D}_n) = (3^n - 3) - \tbinom{3}{2}(2^n - 2) = 3^n + 3 - 3 \cdot 2^n,$$

$$p_4(\mathfrak{D}_n) = 4^n + 6 \cdot 2^n - 4 \cdot 3^n - 4$$

and also

$$p_{n-1}(\mathfrak{D}_n) = \tfrac{1}{2} n!(n-1),$$

$$p_n(\mathfrak{D}_n) = n!$$

For $n = 2$ we get

$$\mathbf{p}(\mathfrak{D}_2) = (0,\ 1,\ 2,\ 0,\ 0,\ 0,\ \ldots),$$

so we obtain the well known fact for diagonal semigroups. Proper 3, 4 and 5-dimensional diagonal algebras are characterized by the p_n-sequences:

$$\mathbf{p}(\mathfrak{D}_3) = (0,\ 1,\ 6,\ 6,\ 0,\ 0,\ 0,\ \ldots)$$

$$\mathbf{p}(\mathfrak{D}_4) = (0,\ 1,\ 14,\ 36,\ 24,\ 0,\ 0,\ 0,\ \ldots)$$

$$\mathbf{p}(\mathfrak{D}_5) = (0,\ 1,\ 30,\ 150,\ 240,\ 120,\ 0,\ 0,\ 0,\ \ldots)$$

Thus the p_n-sequences of essentially n-ary diagonal algebras are increasing at beginnings and then decreasing. Since the p_n-sequences of idempotent algebras are strictly increasing (except some known varieties of algebras, for details see [19]), the p_n-sequences of essentially diagonal algebras are very unusual.

Example 4.1. Let $(G, +)$ be an Abelian group of exponent 30. Then (G, f), where $f(x, y, z) = 6x + 15y + 10z$ is an essentially ternary diagonal algebra. Putting

$$xy = f(x, y, y), \quad x \star y = f(y, x, y) \quad \text{and} \quad x \circ y = f(y, y, x)$$

we obtain three essentially binary operations induced by f. It is not difficult to see that $(G, \cdot)$, $(G, \star)$, $(G, \circ)$ are essentially binary diagonal semigroups and every of the algebras $(G, \cdot, \star)$, $(G, \star, \circ)$, $(G, \cdot, \circ)$ is term equivalent to (G, f). Moreover, these algebras satisfy (1), (3) and (4), respectively. The identity (2) is satisfied by the algebra $(G, \cdot, \circ)$, where $xy = f(y, x, x)$ and $x \circ y = f(x, x, y)$. Thus $p_2(G, \cdot, \circ) = p_3(G, \cdot, \circ) = 6$ and, consequently, $p_2(G, f) = p_3(G, f) = 6$. Since (G, f)

is idempotent, $p_0(G, f) = 0$ and $p_1(G, f) = 1$. The diagonal law of f implies that every n-ary term operation with $n > 3$ reduces to the fundamental operation f involving only 3 (not necessarily distinct) variables. Therefore $p_n(G, f) = 0$ for every $n > 3$ and, consequently, the algebra (G, f) represents the sequence $(0, 1, 6, 6, 0, 0, \ldots)$.

Example 4.2. Let $(G, +)$ be an Abelian group of exponent 6. Defining on this group a ternary operation $f(x, y, z) = x + 3y + 3z$ we obtain an idempotent essentially ternary algebra (G, f) representing the sequence $(0, 1, 2, 3, \ldots, n, \ldots)$ (see [34]). Therefore it is not a ternary diagonal algebra. But the groupoid $(G, \cdot)$ with the operation $xy = f(x, x, y)$ is an essentially diagonal semigroup. It represents the sequence $(0, 1, 2, 0, 0, \ldots)$. The groupoid $(G, \circ)$, where $x \circ y = 2x + 5y$, is not a diagonal semigroup and it has exactly 4 distinct essentially binary term operations xy, yx, $x \circ y$ and $y \circ x$. Since $(x \circ y) \circ y = xy$, the algebra $(G, \cdot, \circ)$ is term equivalent to the idempotent groupoid $(G, \circ)$ and hence it represents the sequence $(0, 1, 4)$. So, it is not induced from (G, f).

Problem 1. *Characterize all diagonal algebras obtained from Abelian groups and also from groups.*

Below we present a method of construction of essential ternary diagonal algebras based on Cantor identities described in [9].

Example 4.3. Consider a variety $\mathcal{V}$ of algebras with one binary $\cdot$ and two unary fundamental operations $'$, *, satisfying

$$(xy)' = x, \quad (xy)^* = y \quad \text{and} \quad (x^*)(x') = x.$$

A model of these identities is a set A (empty, of one element or infinite) with a multiplication which is a bijection between $A \times A$ and A. Then * and $'$ are the two components of the inverse to multiplication. These identities have been first considered by B. Jónsson and A. Tarski in [15] and recently in [9] under a name *Cantor identities*, since G. Cantor may have been the first to recognize a bijection between a set and its square.

Let $\mathfrak{A} = (A, \cdot, ', ^*)$ be a proper algebra from the variety $\mathcal{V}$. Then A is infinite and free algebras with 1-element and n-element set of free generators are isomorphic for every $n > 1$. Then there exists an n-ary term operation $f = f(x_1, \ldots, x_n)$ with $n > 1$ and unary term operations $f_1, \ldots, f_n$ such that the algebra $(A, f, f_1, \ldots, f_n)$ satisfies

$$f(f_1(x), \ldots, f_n(x)) = x \quad \text{and} \quad f_i(f(x_1, \ldots, x_n)) = x_i,\ i = 1, \ldots, n.$$

Put $\delta = \delta(x_1, x_2, \ldots, x_n) = f(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$. Then the algebra (A, δ) is an essential n-dimensional diagonal algebra (for details and a generalization see [9]). In particular, for $n = 3$ we get a method of construction of essential ternary diagonal algebras.

Varieties of diagonal algebras are uniquely determined by their p_n-sequences. Moreover, in the class of binary algebras $(A, \cdot, \circ)$ with two diagonal semigroup fundamental operations the number $p_3(A, \cdot, \circ) = 6$ alone indicates ternary diagonal algebras.

Problem 2. *Find another varieties uniquely defined by their p_n-sequences. In particular, consider varieties of algebras with two binary fundamental operations.*

Problem 3. *Is the variety of normalizations of distributive lattices uniquely determined by its p_n-sequence?*

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