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A FIXED POINT THEOREM FOR QUASI-CONTRACTION MAPPINGS IN PARTIALLY ORDER MODULAR SPACES WITH AN APPLICATION

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In this paper, we prove a new fixed point theorem for quasi-contraction mappings in partially order modular spaces without the Δ_2-condition. As an application of the main result, we prove the generalized Hyers–Ulam stability of Cauchy mappings in modular spaces endowed with the partial order without the $\Delta 2$-condition.

Keywords: Fixed point; quasi-contraction; partial order; modular space; stability.

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1. Introduction and preliminaries

A problem that mathematicians has dealt with, for almost fifty years, is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and integral equations with kernels of nonpower types. The more abstract generalization was given by Nakano [1] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called modular. This idea, which was the basis of the theory of modular spaces and initiated by Nakano in connection with the theory of the order space, was refined and generalized by Musielak and Orlicz [2] in 1959. Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis.

It is well known that fixed point theory is one of the powerful tools in solving integral and differential equations. Banach's contraction principle is one of the pivotal results in fixed point theory and it has a board set of applications. Recently, Khamsi et al. [3] investigated the fixed point results in modular function spaces. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces (see, for instance, [4], [5], [6], [7], [8]).

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In [10], Khamsi proved the following theorem:

Theorem 1.1. *Let (X,ρ) be a modular space such that ρ satisfies the Fatou property. Let C be a ρ-complete nonempty subset of $X\rho$ and let $T : C \to C$ be a quasi-contraction. Let $x \in C$ be such that $\delta_\rho(x) < \infty$. Then $T^n(x)$ is ρ-converges to $\omega \in C$ and, if $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$. Then ω is a unique fixed point of T.*

In this paper, we present a new extension of Theorem 1.1 to partially ordered sets and, by using our fixed point theorem, we prove the generalized Hyers–Ulam stability of Cauchy mappings in modular spaces endowed with partial order.

We begin by recalling some basic concepts of modular spaces. For more information, we refer to the book by Musielak [9].

Definition 1.2. Let X be an arbitrary vector space over K ($\mathbb{R}$ or $\mathbb{C}$).

(1) A function $\rho : X \to [0,+\infty]$ is said to be modular if
 (a) $\rho(x) = 0$ if and only if $x = 0$;
 (b) $\rho(\alpha x) = \rho(x)$ for all scaler α with $|\alpha| = 1$;
 (c) for all $x, y \in X$, $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ for any $\alpha, \beta \ge 0$;

(2) If (c) is replaced by
 (c') $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ for any $\alpha, \beta \ge 0$,
 then we say that ρ is convex modular;

(3) A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

In general, the modular ρ does not behave as a norm or a distance because it is not sub-additive. But one can associate to a modular the F-norm (see [9]).

Definition 1.3. The modular space X_ρ can be equipped with the $F-$norm defined by

$$|x|_\rho = \inf\Big\{\alpha > 0 : \rho\big(\frac{x}{\alpha}\big) \le \alpha\Big\}.$$

Namely, if ρ be convex, then the functional

$$\|x\|_\rho = \inf\Big\{\alpha > 0 : \rho\big(\frac{x}{\alpha}\big) \le 1\Big\}$$

is a norm, which is called the Luxemburg norm in X_ρ. This is equivalent to the F-norm $|\cdot|_\rho$.

Definition 1.4. Let X_ρ be a modular space.

(1) A sequence $\{x_n\}$ in X_ρ is said to be:
 (a) ρ-convergent to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$;
 (b) a ρ-Cauchy sequence if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$;

(2) X_ρ is said to be $\rho-$complete if every ρ-Cauchy sequence is $\rho-$convergent;

(3) A subset $B \subseteq X_\rho$ is said to be ρ-closed if, for any sequence $\{x_n\} \subset B$ with $x_n \to x$, then $x \in B$;

(4) A subset $B \subseteq X_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup\{\rho(x - y) : x, y \in B\} < \infty,$$

where $\delta_\rho(B)$ is called the $\rho-$diameter of B;

(5) We say that ρ has the Fatou property if

$$\rho(x-y) \leq \underline{\lim}_{n\to\infty} \rho(x_n - y_n)$$

whenever $\rho(x_n - x) \to 0$ and $\rho(y_n - y) \to 0$ as $n \to \infty$;

(6) ρ is said to satisfy the $\triangle_2$-condition if

$$\rho(2x_n) \to 0$$

as $n \to \infty$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

It is easy to check that, for every modular ρ and $x, y \in X_\rho$,

(1) $\rho(\alpha x) \leq \rho(\beta x)$ for all $\alpha, \beta \in \mathbb{R}^+$ with $\alpha \leq \beta$;

(2) $\rho(x+y) \leq \rho(2x) + \rho(2y)$.

Example 1.5. (1) The Orlicz modular is defined for every measurable real function f by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|)d\mu(t),$$

where μ denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0, \infty)$ is continuous. We also assume that $\varphi(u) = 0$ if and only if $u = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

The modular space induced by the Orlicz modular is a modular function space, which is called the Orlicz space.

(2) The Musielak–Orlicz modular spaces (see [2]). Let

$$\rho(f) = \int_{\Omega} \varphi(\omega, |f(\omega)|)d\mu(\omega),$$

where μ is a $\sigma-$finite measure on Ω and $\varphi : \Omega \times \mathbb{R} \to [0, \infty)$ satisfy the following conditions:

(a) $\varphi(\omega, u)$ is a continuous even function of u which is non-decreasing for any $u > 0$ such that $\varphi(\omega, 0) = 0$, $\varphi(\omega, u) > 0$ for $u \neq 0$ and $\varphi(\omega, u) \to \infty$ as $u \to \infty$;

(b) $\varphi(\omega, u)$ is a measurable function of ω for any $u \in \mathbb{R}$;

(c) $\varphi(\omega, u)$ is a convex function of u for any $\omega \in \Omega$.

It is easy to check that ρ is a convex modular function and the corresponding modular space is called the Musielak–Orlicz spaces, which is denoted by L^φ.

Definition 1.6. Let (X, ρ) be a modular space and $\leq$ be a partial order on X. Let C be a nonempty subset of X_ρ. The self-mapping $T : C \to C$ is said to be an ordered quasi-contraction if there exists $k < 1$ such that

$$\begin{aligned} &\rho(T(x) - T(y)) \\ &\leq k \max\{\rho(x-y), \rho(x - T(x)), \rho(T(y) - y), \rho(x - T(y)), \rho(T(x) - y)\} \end{aligned}$$

for all $x, y \in C$ with $x \leq y$.

2. **Main results**

Our starting point is the following proposition:

For any $x \in C$, define the orbit $O(x) = \{x, T(x), T^2(x), \cdots\}$ and its ρ-diameter by

$$\delta_\rho(x) = \operatorname{diam} O(x) = \sup\{\rho(T^n(x) - T^m(x)) : n, m \in \mathbb{N}\}.$$

Lemma 2.1. *Let (X, ρ) be a modular space and let $\leq$ be a partial order on X. Let C be a nonempty subset of $X\rho$ and $T : C \to C$ be a nondecreasing ordered quasi-contraction. Let $x \in C$ such that $x \leq T(x)$ and $\delta_\rho(x) < \infty$. Then, for any $n \geq 1$, one has*

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x),$$

where k is the constant associated with the ordered quasi-contraction definition of T. Moreover, one has

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x),$$

for all $n, m \in \mathbb{N}$.

Proof. Let $n, m \in \mathbb{N}$. Since T is nondecreasing, we have $T^n(x) \leq T^m(x)$ or $T^m(x) \leq T^n(x)$ and

$$\begin{aligned}
&\rho(T^n(x) - T^m(y)) \\
&\leq k \max\{\rho(T^{n-1}(x) - T^{m-1}(y)), \rho(T^{n-1}(x) - T^n(x)), \rho(T^m(y) - T^{m-1}(y)), \\
&\rho(T^{n-1}(x) - T^m(y)), \rho(T^n(x) - T^{m-1}(y))\}
\end{aligned}$$

for any $x, y \in C$ with $x \leq y$. This obviously implies that

$$\delta_\rho(T^n(x)) \leq k\delta_\rho(T^{n-1}(x))$$

for all $n \geq 1$. Hence, for all $n \geq 1$, we have

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x).$$

Moreover, for all $n, m \in \mathbb{N}$, we have

$$\rho(T^n(x) - T^{n+m}(x)) \leq \delta_\rho(T^n(x)) \leq k^n \delta_\rho(x).$$

This completes the proof. □

The next lemma is helpful to prove the main result of this paper.

Lemma 2.2. *Let (X, ρ) be a modular space such that ρ satisfies the Fatou property and let $\leq$ be a partial order on X. Let C be a ρ-complete nonempty subset of $X\rho$ and let $T : C \to C$ be a nondecreasing and ordered quasi-contraction. Let $x \in C$ be such that $x \leq T(x)$ and $\delta_\rho(x) < \infty$. Then $T^n(x)$ is ρ-converges to a point $\omega \in C$. Moreover, one has*

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x)$$

for all $n \geq 1$.

Proof. From Lemma 2.1, we know that $\{T^n(x)\}$ is nondecreasing and a ρ-Cauchy sequence. Since C is ρ-complete, then there exists $\omega \in C$ such that $\{T^n(x)\}$ is ρ-converges to ω. Since

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x)$$

for all $n, m \in \mathbb{N}$ and ρ satisfies the Fatou property, letting $m \to \infty$, we have

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x).$$

This completes the proof. □

Theorem 2.3. *Let C, T, ω and x be as in Lemma 2.2. If T is ρ-continuous, then T has a fixed point.*

Proof. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \rho(T^n(x) - \omega) = 0$ and T is ρ-continuous, we have

$$\lim_{n\to\infty} \rho(T^{n+1}(x) - T(\omega)) = 0.$$

Thus there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\rho(T^n(x) - \omega) < \frac{\epsilon}{2}$$

for all $n > N_1$ and

$$\rho(T^{n+1}(x) - T(\omega)) < \frac{\epsilon}{2}$$

for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then, for all $n > N$, we have

$$\rho\Big(\frac{T(\omega) - \omega}{2}\Big) \leq \rho(T^{n+1}(x) - T(\omega)) + \rho(T^{n+1}(x) - \omega) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so $\rho(\frac{T(\omega)-\omega}{2}) = 0$ and $T(\omega) = \omega$. This completes the proof. □

Theorem 2.4. *Let C, T, and x be as in Lemma 2.2. Assume that $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$. If if $\{y_n\}$ is a nondecreasing sequence in C and $\rho(y_n - y) \to 0$ as $n \to 0$, then $y_n \leq y$ for all $n \in \mathbb{N}$, then the ρ-limit of $\{T^n(x)\}$ is a fixed point of T, that is, $T(\omega) = \omega$.*

Proof. Since T is nondecreasing, the sequence $\{T^n(x)\}$ is nondecreasing and, by hypothesis, $T^n(x) \leq \omega$ for all $n \in \mathbb{N}$. Also, we have

$$\begin{aligned} &\rho(T(x) - T(\omega)) \\ &\leq k \max\{\rho(x - \omega), \rho(x - T(x)), \rho(T(\omega) - \omega), \rho(x - T(\omega)), \rho(T(x) - \omega)\} \end{aligned}$$

and so

$$\rho(T(x) - T(\omega)) \leq k \max\{\delta_\rho(x), \rho(T(\omega) - \omega), \rho(x - T(\omega))\}.$$

Assume that, for any $n \geq 1$,

$$\rho(T^n(x) - T(\omega)) \leq \max\{k^n \delta_\rho(x), k\rho(T(\omega) - \omega), k^n \rho(x - T(\omega))\}.$$

Since T is a quasi-contraction, we have

$$\begin{aligned} &\rho(T^{n+1}(x) - T(\omega)) \\ &\leq k \max\{\rho(T^n(x) - \omega), \rho(T^n(x) - T^{n+1}(x)), \rho(T(\omega) - \omega), \\ &\qquad \rho(T^{n+1}(x) - \omega), \rho(T^n(x) - T(\omega))\} \end{aligned}$$

and so

$$\rho(T^{n+1}(x) - T(\omega)) \leq k \max\{k^n \delta_\rho(x), \rho(T(\omega) - \omega), \rho(T^n(x) - T(\omega))\}.$$

Thus, by our assumption, we get

$$\rho(T^{n+1}(x) - T(\omega)) \leq \max\{k^{n+1}\delta_\rho(x), k\rho(T(\omega) - \omega), k^{n+1}\rho(x - T(\omega))\}.$$

Therefore, by induction, we have

$$\rho(T^n(x) - T(\omega)) \leq \max\{k^n \delta_\rho(x), k\rho(T(\omega) - \omega), k^n \rho(x - T(\omega))\}$$

for all $n \geq 1$. Thus it follows that

$$\limsup_{n\to\infty} \rho(T^n(x) - T(\omega)) \leq k\rho(T(\omega) - \omega).$$

Since ρ has the Fatou property, we get

$$\rho(T(\omega) - \omega) \leq \liminf_{n\to\infty} \rho(T^n(x) - T(\omega)) \leq k\rho(T(\omega) - \omega).$$

Since $k < 1$, we have $T(\omega) = \omega$. This completes the proof. □

3. **Applications**

We say that a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to a true solution of (ξ).

The stability of functional equations was first introduced by Ulam [11] in 1940. In 1941, Hyers [12] gave a partial solution of Ulam's problem for the case of approximate additive mappings in the context of Banach spaces. In 1978, Th.M. Rassias [13] generalized the theorem of Hyers. The phenomenon of stability that was introduced by Th.M. Rassias [13] is called the Hyers–Ulam–Rassias stability or the generalized Hyers–Ulam stability (see also [14, 15]). In 1994, Găvruta [16] generalized the Th.M. Rassias theorem by using a general control function.

Recently, Sadeghi [17] presented a fixed point method to prove the generalized Hyers–Ulam stability of functional equations in modular spaces with the Δ_2-condition.

In this section, by using our fixed point theorem in modular spaces, we prove the generalized Hyers–Ulam stability of Cauchy mapping in modular spaces endowed with partial order. It is very important to note that we remove the Δ_2-condition in this theorem.

From now on, we suppose that $(\mathcal{E}, \leq_1)$ is a partially ordered real or complex linear space with following conditions:

(a) if $x \leq_1 y$ for all $x, y \in \mathcal{E}$, then $rx \leq_1 ry$ for all $r \in \mathbb{R}^+$;

(b) for all $x, y \in \mathcal{E}$, there exists $z \in \mathcal{E}$ such that z is comparable to x and y.

From now on, we assume that ρ is a convex modular on $\mathcal{X}$ with the Fatou property, $\mathcal{X}$ is endowed with the partial order $\leq_2$ and satisfies the condition (a) and the following conditions:

(c) for all $x, y \in \mathcal{X}$, there exists $z \in \mathcal{X}$ such that z is an upper bound of $\{x, y\}$;

(d) if $\{x_n\}$ is a nondecreasing sequence in $\mathcal{X}$ and $x_n \to x$, then $x \geq x_n$ for all $n \in \mathbb{N}$.

In this section, we consider $0 \times \infty = 0$. Now, we prove our main theorem.

Theorem 3.1. *Suppose that $f : \mathcal{E} \to \mathcal{X}$ is a mapping satisfying the following:*

$$2f(x) \leq_2 f(2x); \tag{3.1}$$

$$\rho(2^k f(2x) - 2^{k+1} f(x)) \leq 2^k \rho(f(2x) - 2f(x)); \tag{3.2}$$

$$\rho(f(x+z+y-w) - f(x) - f(z) - f(y) + f(w)) \leq \phi(x,z) + \phi(y,w) \tag{3.3}$$

for all $k \in \mathbb{N}$ and $x, y, z, w \in \mathcal{E}$ which x is comparable to z and y is comparable to w, where $\phi : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ is a function satisfying $\phi(0,0) = 0$ and the following condition:

$$\phi(x,y) \leq 2\ L\ \phi(\frac{x}{2}, \frac{y}{2}) \tag{3.4}$$

for all $x, y \in \mathcal{E}$ with x comparable to y, where $L \in (0,1)$ is a constant. Then there exists a unique additive mapping $T : \mathcal{E} \to \mathcal{X}$ such that

$$\rho(T(x) - f(x)) \leq \frac{1}{2(1-L)} \phi(x,x) \tag{3.5}$$

for all $x \in \mathcal{E}$.

Proof. We consider the set

$$\mathcal{M} = \{g : \mathcal{E} \to \mathcal{X}_\rho,\ g(0) = 0\}$$

and introduce the convex modular $\widetilde{\rho}$ on $\mathcal{M}$ as follows:

$$\widetilde{\rho}(g) = \inf\{c > 0 : \rho(g(x)) \leq c\phi(x,x),\ \forall x \in \mathcal{E}\}.$$

Define the partial order $\leq$ on $\mathcal{M}$ as follows:

$$h, g \in \mathcal{M},\ h \leq g \quad \Longleftrightarrow \quad h(x) \leq_2 g(x)$$

for all $x \in \mathcal{E}$.

Now, we consider the function $J : \mathcal{M}_{\widetilde{\rho}} \to \mathcal{M}_{\widetilde{\rho}}$ defined by

$$J(g)(x) := \frac{1}{2} g(2x).$$

Now, by several steps, we show that conditions of Theorem 2.3 hold.

Step 1: $\widetilde{\rho}$ is a convex modular. It is sufficient to show that $\widetilde{\rho}$ satisfies the following condition:

$$\widetilde{\rho}(\alpha g + \beta h) \leq \alpha \widetilde{\rho}(g) + \beta \widetilde{\rho}(h)$$

if $\alpha + \beta = 1$ for any $\alpha, \beta \geq 0$. For any $\varepsilon > 0$, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq \widetilde{\rho}(g) + \varepsilon, \quad \rho(g(x)) \leq c_1 \phi(x,x)$$

and

$$c_2 \leq \widetilde{\rho}(h) + \varepsilon, \quad \rho(h(x)) \leq c_2 \phi(x,x).$$

If $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, then we get

$$\rho(\alpha g(x) + \beta h(x)) \leq \alpha \rho(g(x)) + \beta \rho(h(x)) \leq (\alpha c_1 + \beta c_2) \phi(x,x),$$

whence

$$\widetilde{\rho}(\alpha g + \beta h) \leq \alpha \widetilde{\rho}(g) + \beta \widetilde{\rho}(h) + (\alpha + \beta)\varepsilon$$

and so

$$\widetilde{\rho}(\alpha g + \beta h) \leq \alpha\widetilde{\rho}(g) + \beta\widetilde{\rho}(h).$$

Step 2: $\mathcal{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$–complete. Let $\{g_n\}$ be a $\widetilde{\rho}$–Cauchy sequence in $\mathcal{M}_{\widetilde{\rho}}$ and let $\varepsilon > 0$ be given. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $\widetilde{\rho}(g_n - g_m) \leq \varepsilon$ for all $n, m \geq n_0$. Now, by considering the definition of the modular $\widetilde{\rho}$, we see that

$$\rho(g_n(x) - g_m(x)) \leq \varepsilon\phi(x, x) \tag{3.6}$$

for all $x \in \mathcal{E}$ and $n, m \geq n_0$. If x is any given point of $\mathcal{E}$, (3.6) implies that $\{g_n(x)\}$ is a ρ–Cauchy sequence in $\mathcal{X}_\rho$. Since $\mathcal{X}_\rho$ is ρ–complete, it follows that $\{g_n(x)\}$ is ρ–convergent to a point in $\mathcal{X}_\rho$ for each $x \in \mathcal{E}$. Hence we can define a function $g : \mathcal{E} \to \mathcal{X}_\rho$ by

$$g(x) := \lim_{n\to\infty} g_n(x)$$

for all $x \in \mathcal{E}$. Letting $m \to \infty$, (3.6) implies that

$$\widetilde{\rho}(g_n - g) \leq \varepsilon$$

for all $n \geq n_0$. On the other hand ρ has the Fatou property. Then $\{g_n\}$ is a $\widetilde{\rho}$–convergent sequence in $\mathcal{M}_{\widetilde{\rho}}$. Therefore, $\mathcal{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$–complete.

Step 3: J is a quasi-contraction. Let $g, h \in \mathcal{M}_{\widetilde{\rho}}$, $g \leq h$ and let $c \in [0, \infty]$ be a constant with $\widetilde{\rho}(g - h) \leq c$. From the definition of $\widetilde{\rho}$, we have

$$\rho(g(x) - h(x)) \leq c\phi(x, x)$$

for all $x \in \mathcal{E}$. By the assumption and the last inequality, we get

$$\rho\left(\frac{g(2x)}{2} - \frac{h(2x)}{2}\right) \leq \frac{1}{2}\rho(g(2x) - h(2x)) \leq \frac{1}{2}c\phi(2x, 2x) \leq Lc\phi(x, x)$$

for all $x \in \mathcal{E}$. Hence we have $\widetilde{\rho}(J(g) - J(h)) \leq L\widetilde{\rho}(g - h)$, that is, J is a $\widetilde{\rho}$–strict contraction.

Step 4: $\delta_{\widetilde{\rho}}(f) < \infty$. It is clear that $f(0) = 0$. Putting $z := x$ and $y = w := 0$ in (3.3), we get

$$\rho(f(2\ x) - 2\ f(x)) \leq \phi(x, x) \tag{3.7}$$

for all $x \in \mathcal{E}$. Since ρ is convex, we have

$$\rho\Big(\frac{f(2\ x)}{2} - f(x)\Big) \leq \frac{1}{2}\rho(f(2\ x) - 2\ f(x)) \leq \frac{1}{2}\ \phi(x, x) \leq \phi(x, x)$$

for all $x \in \mathcal{E}$. Replacing $2^n x$ in x in (3.2), it follows from (3.7) that

$$\begin{aligned}\rho(2^k f(2^{n+1}x) - 2^{k+1} f(2^n x)) &\leq 2^k \rho(f(2^{n+1}x) - 2f(2^n x)) \\ &\leq 2^k \phi(2^n x, 2^n x)\end{aligned}$$

for all $x \in \mathcal{E}$ and $n, k \in \mathbb{N}$. Since ρ is convex, we have

$$\begin{aligned}\rho\Big(2^k \frac{f(2^{n+1}x)}{2^{n+1}} - 2^k \frac{f(2^n x)}{2^n}\Big) &\leq \frac{1}{2^{n+1}} \rho(2^k f(2^{n+1}x) - 2^{k+1} f(2^n x)) \\ &\leq \frac{2^k}{2^{n+1}} \phi(2^n x, 2^n x) \\ &\leq 2^k\ L^n \phi(x, x)\end{aligned}$$

for all $x \in \mathcal{E}$ and $n, k \in \mathbb{N}$. This shows that

$$\rho(2^k J^{n+1}(f)(x) - 2^k J^n(f)(x)) \leq 2^k \ L^n \phi(x,x)$$

for all $x \in \mathcal{E}$ and $n, k \in \mathbb{N}$. This implies that

$$\widetilde{\rho}(2^k J^{n+1}(f) - 2^k J^n(f)) \leq 2^k \ L^n \phi(x,x) \tag{3.8}$$

for all $n, k \in \mathbb{N}$. From (3.8), we obtain

$$\begin{aligned}
\widetilde{\rho}(J^m(f) - J^n(f)) &= \widetilde{\rho}(\sum_{i=n}^{m-1}(J^{i+1}(f) - J^i(f)) \\
&\leq \sum_{i=1}^{m-n} \frac{1}{2^i}\widetilde{\rho}(2^i(J^{m-i+1}(f) - J^{m-i}(f))) \\
&\leq \sum_{i=1}^{m-n} \frac{1}{2^i} 2^i \ L^{m-i} \\
&\leq \sum_{i=1}^{\infty} L^i = \frac{L}{1-L} < \infty
\end{aligned}$$

for all $m, n \in \mathbb{N}$ with $m \geq n$. This shows that $\delta_{\widetilde{\rho}}(f) < \infty$.

Step 5: $\widetilde{\rho}$ satisfies the Fatou property. Let $\{g_n\}$ be $\widetilde{\rho}$–convergent to a point $g \in \mathcal{M}_{\widetilde{\rho}}$. Suppose that $\liminf_{n\to\infty} \widetilde{\rho}(g_n) < \widetilde{\rho}(g)$ and $\alpha = \liminf_{n\to\infty} \widetilde{\rho}(g_n) < \infty$. Then there exists a subsequence $\{g_{ni}\}$ in $\{g_n\}$ such that $\lim_{i\to\infty} \widetilde{\rho}(g_{ni}) = \alpha$. Let $\epsilon > 0$ be given. Then there exists $i_0 \in \mathbb{N}$ such that $\widetilde{\rho}(g_{ni}) \leq \epsilon + \alpha$ for all $i \geq i_0$. This shows that

$$\rho(g_{ni}(x)) \leq (\epsilon + \alpha)\phi(x,x)$$

for all $i \geq i_0$ and $x \in \mathcal{E}$. Thus it follows that

$$\liminf_{i\to\infty} \rho(g_{ni}(x)) \leq \alpha\phi(x,x)$$

for all $x \in \mathcal{E}$. On the other hands, for all $x \in \mathcal{E}$, we have $\rho(g_{ni}(x) - g(x)) \to 0$ as $i \to \infty$. Since ρ satisfies the Fatou property, we have

$$\rho(g(x)) \leq \liminf_{i\to\infty} \rho(g_{ni}(x)) \leq \alpha\phi(x,x)$$

for all $x \in \mathcal{E}$. This shows that

$$\widetilde{\rho}(g) \leq \alpha,$$

which is a contradiction. Therefore, it follows that $\widetilde{\rho}(g) \leq \liminf_{n\to\infty} \widetilde{\rho}(g_n)$.

Step 6: J is nondecreasing. By the definition of J and the condition (a), it is clear that J is a nondecreasing mapping.

Step 7: J is $\widetilde{\rho}$-continuous. Let $\{h_n\}$ be $\widetilde{\rho}$–converges to a point $h \in \mathcal{M}_{\widetilde{\rho}}$ and $\epsilon > 0$ be given. Then there exist $C < \epsilon$ and $N \in \mathbb{N}$ such that

$$\widetilde{\rho}(h_n - h) \leq c < \epsilon$$

for all $n \geq N$. This shows that

$$\rho(h_n(x) - h(x)) \leq c\phi(x,x)$$

for all $x \in \mathcal{E}$ and $n \geq N$. Since ρ is convex, we have

$$\rho\Big(\frac{h_n(2x)}{2} - \frac{h(2x)}{2}\Big) \leq \frac{1}{2}\rho(h_n(2x) - h(2x)) \leq \frac{C}{2}\phi(2x, 2x) \leq LC\phi(x, x)$$

for all $x \in \mathcal{E}$ and $n \geq N$. This shows that $\widetilde{\rho}(J(h_n) - J(h)) < \epsilon$ for all $n \geq N$ and so $J(h_n)$ is $\widetilde{\rho}$-convergent to $J(h)$. It is obvious that $f \leq J(f)$. Now, by applying Theorem 3.4, J has a fixed point. Let T be a fixed point of J. Then $\lim_{n\to\infty} \widetilde{\rho}(J^n(f) - T) = 0$. This shows that

$$T(x) = \lim_{n\to\infty} \frac{f(2^n\ x)}{2^n} \tag{3.9}$$

for all $x \in \mathcal{E}$. Since J is a nondecreasing mapping, for any $x \in \mathcal{E}$, the sequence $\{\frac{f(2^n\ x)}{2^n}\}$ is a nondecreasing sequence in $\mathcal{X}_\rho$. Thus, by using the condition (d), we see that $\frac{f(2^n\ x)}{2^n} \leq_2 T(x)$ for all $n \geq 0$. In particular, $f(x) \leq_2 T(x)$. This shows that $f \leq T$. Let $x \in \mathcal{E}$ be an arbitrary element. By using (3.9), we have $T(2x) = \lim_{n\to\infty} \frac{f(2^{n+1}\ x)}{2^n}$ and, by Fatou property of ρ, we get

$$\begin{aligned}
\rho(f(2x) - T(2x)) &\leq \liminf_{k\to\infty} \rho(f(2x) - \frac{f(2^{k+1}x)}{2^k}) \\
&= \liminf_{k\to\infty} \rho(2\frac{f(2^{k+1}x)}{2^{k+1}} - 2\frac{f(2x)}{2}) \\
&\leq \liminf_{k\to\infty} \rho(2(\sum_{i=1}^{k} \frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i})) \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^{k} \frac{1}{2^i}\rho(2^{i+1}(\frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i})) \\
&\leq \sum_{i=1}^{\infty} \frac{1}{2^{2i}}\rho(2^i(f(2^{i+1}x) - 2f(2^i x))) \\
&\leq \sum_{i=1}^{\infty} \frac{1}{2^i}\phi(2^i x, 2^i x) \\
&\leq \sum_{i=1}^{\infty} L^i\phi(x, x).
\end{aligned}$$

This shows that $\widetilde{\rho}(2J(f) - 2J(T)) \leq \frac{L}{1-L}$. Therefore, By (3.7) and the last inequality, we have

$$\widetilde{\rho}(f - T) \leq \frac{1}{2}\widetilde{\rho}(2f - 2J(f)) + \frac{1}{2}\widetilde{\rho}(2J(f) - 2J(T)) \leq \frac{1}{2} + \frac{1}{2}\ \frac{L}{1-L} = \frac{1}{2(1-L)}.$$

This shows that the inequality (3.5) holds.

Now, we show that T is a additive mapping. To this end, let $x, y \in \mathcal{E}$. By the condition (b), there exists $z \in \mathcal{E}$ such that z is comparable to x and y. Thus $2^n z$ is comparable to $2^n x$ and $2^n y$ for all $n \in \mathbb{N}$. By using (3.3), we have

$$\begin{aligned}&\rho(f(2^n x+2^n y)-f(2^n x)-f(2^n y))\\&=\rho(f(2^n x+2^n z+2^n y-2^n z)-f(2^n x)-f(2^n z)-f(2^n y)+f(2^n z))\\&\le \phi(2^n x,2^n z)+\phi(2^n y,2^n z)\end{aligned}$$

for all $n \in \mathbb{N}$. Since ρ is convex, we have

$$\begin{aligned}\rho\Big(\frac{f(2^n x+2^n y)}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}\Big)&\le \frac{1}{2^n}\phi(2^n x,2^n z)+\frac{1}{2^n}\phi(2^n y,2^n z)\\&\le L^n\phi(x,z)+L^n\phi(y,z)\end{aligned}$$

for all $n \in \mathbb{N}$. On the other hands, the convexity of ρ shows that

$$\begin{aligned}&\rho\Big(\frac{1}{2^4}T(x+y)-\frac{1}{2^4}T(x)-\frac{1}{2^4}T(y)\Big)\\&\le \frac{1}{2^4}\rho(T(x+y)-\frac{1}{2^n}f(2^n\ (x+y)))+\frac{1}{2^4}\rho(T(x)-\frac{1}{2^n}f(2^n\ x))\\&\quad-\frac{1}{2^4}\rho(T(y)-\frac{1}{2^n}f(2^n\ y))+\frac{1}{2^4}\rho(\frac{f(2^n x+2^n y)}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n})\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that T is additive.

For the uniqueness property of T, let $T_1 : \mathcal{E} \to \mathcal{X}_\rho$ be another additive mapping satisfying (3.5). It is easy to show that $J(T_1) = T_1$. For any $x \in \mathcal{E}$, there exists $g(x) \in \mathcal{X}_\rho$ such that $g(x)$ is an upper bound of $\{T(x), T_1(x)\}$. This shows that $g \in \mathcal{M}$ and g is comparable to T and T_1. Since $\widetilde{\rho}$ is convex, we have $g \in \mathcal{M}_{\widetilde{\rho}}$ and so

$$\begin{aligned}\widetilde{\rho}\Big(\frac{T-T_1}{2}\Big)&\le \frac{1}{2}\widetilde{\rho}(J^n(T)-J^n(g))+\frac{1}{2}\widetilde{\rho}(J^n(T_1)-J^n(g))\\&\le \frac{1}{2}L^n\ \widetilde{\rho}(T-g)+\frac{1}{2}L^n\ \widetilde{\rho}(T_1-g).\end{aligned}$$

Since $L \in (0,1)$, we have $\widetilde{\rho}(\frac{T-T_1}{2}) = 0$ and hence $T = T_1$. This completes the proof. □

Corollary 3.2. *Let $(\mathcal{E}, \le_1)$ be a partially ordered normed space and let $(\mathfrak{F}, \le_2)$ be a partially ordered Banach space. Suppose that $f : \mathcal{E} \to \mathfrak{F}$ is a mapping with $f(0) = 0$ and there exist the constants $\varepsilon, \theta \ge 0$ and $p \in [0, 1)$ such that*

$$\|f(x+z+y-w)-f(x)-f(z)-f(y)+f(w)\| \le \varepsilon+\theta(\|x\|^p+\|y\|^p)$$

for all $x, y, z, w \in \mathcal{E}$ which x is comparable to z and y is comparable to w. Then there exists a unique additive mapping $j : \mathcal{E} \to \mathfrak{F}$ such that

$$\|f(x)-j(x)\| \le \frac{\varepsilon}{2-2^p}+\frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in \mathcal{E}$.

Proof. It is known that every normed space is a modular space with the modular $\rho(x) = \|x\|$. If we define $\phi(x, y) = \varepsilon + \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 3.1, then the conclusion follows. □

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