SOME NEW RESULTS ON WEIGHTED GRAND LORENTZ SPACES

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The concept of Lebesgue space has been generalized to the grand Lebesgue space with non-weight and weight, and the classical Lorentz space concept has been generalized to grand Lorentz spaces with a similar logic in literature. In this study, instead of using rearrangement function of a measurable function, weighted grand Lorentz spaces are defined by using the maximal function for $1 \leq p, q \leq \infty$. In addition, being Banach function space and some inclusion properties in weighted grand Lorentz spaces are investigated.

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1. **Introduction**

The concept of grand Lebesgue spaces, introduced by Iwaniec and Sbordone in [10] and in somewhat more general form in [8], provides a natural generalization of classical Lebesgue spaces. These spaces, denoted by $L^{p)}$, are defined for $1 < p < \infty$ on a finite measure space (X, Σ, μ). Unlike standard Lebesgue spaces, grand Lebesgue spaces encompass measurable functions whose integrability varies over a range of exponents. For any $f \in L^{p)}$, the functional

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\int_X |f(x)|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}}$$

defines a norm, making $L^{p)}$ a Banach function space. These spaces satisfy the inclusion $L^p \subset L^{p)} \subset L^{p-\varepsilon}$ for $0 < \varepsilon \leq p - 1$, highlighting their flexibility in handling functions that exhibit varying integrability.

Recent researchs [5, 7, 11, 12, 13, 14, 15] and [18] have shown that grand Lebesgue spaces are particularly useful in applications involving the Jacobian integrability problem, partial differential equations, variational problems and harmonic analysis. These spaces have been employed in studying maximal functions, extrapolation theory and other areas where classical Lebesgue spaces may be insufficient. Moreover, the harmonic analysis of grand Lebesgue spaces has been extensively developed, with applications ranging from weighted inequalities to operator theory.

A significant generalization of grand Lebesgue spaces, denoted by

$$\bigcap_{0<\varepsilon\leq p-1} L^{p-\varepsilon} := L^{p),\theta}(X),$$

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was introduced in [8]. These spaces extend $L^{p)}(X)$ by incorporating an additional parameter $\theta \geq 0$, which provides further flexibility. The norm for any $f \in L^{p),\theta}(X)$ is given by

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X |f(x)|^{p-\varepsilon} d\mu \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon\leq p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon},$$

where $L^{p),\theta}(X)$ reduces to classical Lebesgue spaces when $\theta = 0$ and to grand Lebesgue spaces when $\theta = 1$. These spaces have applications in weighted inequalities and their structural properties, such as being rearrangement invariant Banach function space, have been studied in depth [1, 8]. Also we have $L^p(X) \subset L^{p),\theta}(X) \subset L^{p-\varepsilon}(X)$ for $0 < \varepsilon \leq p-1$ and $L^p(X) \subset L^{p),\theta_1}(X) \subset L^{p),\theta_2}(X)$ for $0 \leq \theta_1 < \theta_2$ [1, 9].

It is important to remember that the subspace of test functions $C_0^\infty(X)$ is not dense in $L^{p),\theta}(X)$. If one calls the closure of $C_0^\infty(X)$ in $L^{p),\theta}(X)$ as $E^{p),\theta}(X)$, then we get that

$$E^{p),\theta}(X) = \left\{ f \in L^{p),\theta}(X) : \lim_{\varepsilon\to 0^+} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{p-\varepsilon} = 0 \right\}$$

is a closed subspace of $L^{p),\theta}(X)$ and $L^{p),\theta_1}(X) \subset E^{p),\theta_2}(X)$ for $0 \leq \theta_1 < \theta_2$.

The small Lebesgue spaces are introduced in [5]. According to that, the small Lebesgue space $L^{(p}$ consists of all measurable functions g on a finite measure space (X, μ) which can be represented in the form $g = \sum_{k=1}^{\infty} g_k$ (convergence a.e.) and such that the following norm is finite:

$$\|g\|_{(p} := \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k\|_{(p'-\varepsilon)'}$$

where $\|g\|_p$ stands for the normalised norm in L^p space:

$$\|g\|_p = \left(\frac{1}{|X|} \int_\Omega |g(x)|^p dx \right)^{\frac{1}{p}}$$

and $1 < p < \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$. In [7], the authors characterised these spaces as $\sum -$extrapolation and interpolation spaces. They showed that

$$\|g\|_{(p} \approx \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p}} \|g_k\|_{L^{p+\varepsilon,p}},$$

holds for $1 < p < \infty$. Here $L^{p,r}, 0 < p < \infty, 0 < r \leq \infty$ are the usual Lorentz spaces with the quasi-norm

$$\|g\|_{L^{p,r}} := \left\{ \int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^r dt/t \right\}^{\frac{1}{r}}.$$

Let $1 < p < \infty$ and w be a weight function, i.e. measurable, positive almost everywhere and locally integrable on X. Weighted grand Lebesgue spaces denoted by $L_w^{p)}$ are the space of $\Sigma-$measurable functions defined on finite measure space (X, Σ, μ) such that

$$\|f\|_{p),w} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_X |f(x)|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}}$$

is finite for any $f \in L_w^{p)}$. Weighted grand Lebesgue spaces have been introduced to address problems involving weight functions in [6]. The boundedness of the Hardy-Littlewood maximal operator on the space $L_w^{p)}$ was also examined in the same paper. Further work has explored the Riesz potential and its boundedness on weighted grand Lebesgue spaces [13].

In addition to these, the classical weighted Lorentz and grand Lorentz spaces were compared and the boundedness of the maximal operator was examined in [4, 17].

2. **Preliminaries**

Throughout the paper $L(\mu)$ will denote the linear space of all equivalence classes of Σ-measurable functions on X and χ_A will be used for the characteristic function of a set A. For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A \prec B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \prec B$ and $B \prec A$, we write $A \approx B$ and say that A and B are equivalent.

Let $X = (X, \Sigma, \mu)$ be a $\sigma-$finite measure space and w be a weight function. Weighted Lorentz spaces or Lorentz spaces over weighted measure spaces $L(p, q, wd\mu)$ are studied and discussed in [3, 16] by taking the measure $wd\mu$ instead of the measure μ. Then the distribution function of f which is considered real-valued, measurable and defined on the measure space $(X, wd\mu)$

$$\lambda_{f,w}(y) = w\{x \in X : |f(x)| > y\} = \int\limits_{\{x\in X:\ |f(x)|>y\}} w(x)\, d\mu(x),\ y \geq 0$$

is found. The nonnegative rearrangement of f is given by

$$f_w^*(t) = \inf\{y > 0 : \lambda_{f,w}(y) \leq t\} = \sup\{y > 0 : \lambda_{f,w}(y) > t\},\ t \geq 0$$

where we assume that $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. Also the average (maximal) function of f on $(0, \infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t} \int_0^t f_w^*(s)\, ds.$$

Note that $\lambda_{f,w}(\cdot), f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are non-increasing and right continuous functions. The weighted Lorentz space $L(p, q, wd\mu)$ is the set of all classes of $\Sigma-$measurable functions f such that $\|f\|_{p,q,w}^* < \infty$, where

$$\|f\|_{p,q,w}^* = \begin{cases} \left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [f_w^*(t)]^q\, dt\right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f_w^*(t), & 0 < p \leq \infty, q = \infty \end{cases}. \tag{1}$$

In general, however, $\|\cdot\|_{p,q,w}^*$ is not a norm since the Minkowski inequality may fail. But by replacing f_w^* with f_w^{**} in (1), we get that $L(p, q, wd\mu)$ is a normed space, with the norm $\|\cdot\|_{p,q,w}$ under certain restrictions on p and q.

If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q,w}^* \leq \|f\|_{p,q,w} \leq \frac{p}{p-1} \|f\|_{p,q,w}^*$$

where the first inequality is an immediate consequence of the fact that $f_w^* \leq f_w^{**}$ and the second follows from the Hardy inequality.

For more on weighted Lorentz spaces one can refer to [3, 16] and references therein.

Remark 2.1. *The case $p = \infty$ and $0 < q < \infty$ is not important for $L(p, q, wd\mu)$ because of $L(p, q, wd\mu) = \{0\}$.*

3. **Main Results**

Let $X = (X, \Sigma, \mu)$ be a finite measure space and w be a weight function, i.e. $\Sigma-$measurable, positive, locally integrable function on X. Using average function $f_w^{**}(\cdot)$ instead of nonnegative rearrangement $f_w^*(\cdot)$ used in the definition of grand Lorentz space in [17], we defined the weighted grand Lorentz spaces as follows.

Definition 3.1. *The weighted grand Lorentz space* $L_w^{p,q)} = L_w^{p,q)}(X,\Sigma,\mu)$ *is the set of all classes of complex-valued,* $\Sigma-$*measurable functions which are defined on the measure space* $(X,\Sigma,wd\mu)$ *such that* $\|f\|_{p,q)}^w < \infty$ *for any* $f \in L_w^{p,q)}$ *where*

$$\|f\|_{p,q)}^w = \begin{cases} \sup\limits_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \left(\frac{q}{p}\int_0^{w(X)} t^{\frac{q}{p}-1} \left[f_w^{**}(t)\right]^{q-\varepsilon} dt\right)^{\frac{1}{q-\varepsilon}}, & 0<p,q<\infty \\ \sup\limits_{0<t<w(X)} t^{\frac{1}{p}} f_w^{**}(t), & 0<p\le\infty,\ q=\infty \end{cases}$$

and $w(X) = \int_X w(x)\,d\mu(x)$. *In particular, if* $1<p<\infty$, $1\le q\le\infty$; $p=q=1$ *or* $p=q=\infty$, *then the functional* $\|\cdot\|_{p,q)}^w$ *is a norm and so the normed space* $L_w^{p,q)}$ *is a Banach space.*

Remark 3.1. $L_w^{p,q)}$ *space is not normable if* $0<p<\infty$, $0<q<1$ *or* $0<p<1$, $1\le q\le\infty$ *or* $p=1$, $1<q\le\infty$.

Lemma 3.1. *Let* $f \in L(p,q,wd\mu)$. *Then* $t^{\frac{1}{p}} f_w^{**}(t) \le \|f\|_{p,q,w}$.

Proof. For any $f \in L(p,q,wd\mu)$, we have

$$\begin{aligned} \|f\|_{p,q,w}^q &= \frac{q}{p}\int_0^\infty \left[t^{\frac{1}{p}} f_w^{**}(t)\right]^q \frac{dt}{t} \\ &= \frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}\left[f_w^{**}(t)\right]^q dt \\ &\ge \frac{q}{p}\int_0^t \left[f_w^{**}(s)\right]^q s^{\frac{q}{p}-1} ds \\ &\ge \frac{q}{p}\left[f_w^{**}(t)\right]^q \int_0^t s^{\frac{q}{p}-1} ds \\ &= \left[f_w^{**}(t)\right]^q t^{\frac{q}{p}} \end{aligned} \tag{2}$$

by using the non-increasing property of $f_w^{**}(\cdot)$. □

The following theorem gives the inclusion property of weighted grand Lorentz spaces and regular weighted Lorentz spaces.

Theorem 3.1. *If* $1<p<\infty$ *and* $1<q\le\infty$, *then* $L(p,q,wd\mu) \subset L_w^{p,q)}$.

Proof. Firstly, let $1 < q < \infty$. By using (2) in the preceding Lemma, for any $f \in L(p,q,wd\mu)$ we have

$$\begin{aligned}
\|f\|^{w}_{p,q)} &= \sup_{0<\varepsilon<q-1} \varepsilon^{\frac{1}{q-\varepsilon}} \left(\frac{q}{p} \int_0^{w(X)} t^{\frac{q}{p}-1} \left[f^{**}_w(t)\right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} \\
&= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^{w(X)} t^{\frac{\varepsilon}{p}-1} \left[t^{\frac{1}{p}} f^{**}_w(t)\right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} \\
&\le \sup_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^{w(X)} t^{\frac{\varepsilon}{p}-1} \|f\|^{q-\varepsilon}_{p,q,w} dt \right)^{\frac{1}{q-\varepsilon}} \\
&= \|f\|_{p,q,w} \sup_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^{w(X)} t^{\frac{\varepsilon}{p}-1} dt \right)^{\frac{1}{q-\varepsilon}} \\
&= \|f\|_{p,q,w} \sup_{0<\varepsilon<q-1} \left(q w(X)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{q-\varepsilon}} \\
&= q w(X)^{\frac{q-1}{p}} \|f\|_{p,q,w}.
\end{aligned}$$

Similarly for $q = \infty$,

$$\|f\|^{w}_{p,q)} = \sup_{0<t<w(X)} t^{\frac{1}{p}} f^{**}_w(t) \le \sup_{0<t} t^{\frac{1}{p}} f^{**}_w(t) = \|f\|_{p,q,w}$$

can be written. Therefore we have showed an embedding result in the framework of grand and classical weighted Lorentz spaces. □

Example 3.1. *Let $E \in \Sigma$ with $w(E) < \infty$. The non-increasing rearrangement of χ_E is found as*

$$(\chi_E)^*_w(t) = \chi_{[0,w(E))}(t).$$

Following this, we get

$$(\chi_E)^{**}_w(t) = \frac{1}{t}\int_0^t (\chi_E)^*_w(s)\,ds = \begin{cases} 1, & \text{if } 0 \le t < w(E) \\ \frac{1}{t} w(E), & \text{if } t \ge w(E) \end{cases}.$$

Therefore $\|\chi_E\|_{p,\infty,w} = (w(E))^{\frac{1}{p}}$ and

$$\begin{aligned}
\|\chi_E\|^q_{p,q,w} &= \frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1} \left[(\chi_E)^{**}_w(t)\right]^q dt \\
&= \frac{q}{p}\int_0^{w(E)} t^{\frac{q}{p}-1} dt + \frac{q}{p}\int_{w(E)}^{w(X)} t^{\frac{q}{p}-1} \left[\frac{1}{t} w(E)\right]^q dt \qquad (3)\\
&= (w(E))^{\frac{q}{p}} + \frac{p'}{p}\left[(w(E))^q w(X)^{\frac{-q}{p'}} - (w(E))^{\frac{q}{p}}\right],
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Consequently, we see that $\chi_E \in L(p,q,wd\mu)$ and so $\chi_E \in L^{p,q)}_w$ by Theorem 3.1. Moreover, we have

$$\begin{aligned}
\|\chi_E\|^w_{p,\,q)} &= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p}\varepsilon \int_0^{w(X)} t^{\frac{q}{p}-1} \left((\chi_E)^{**}_w(t)\right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} \\
&= (q-1)(w(E))^{\frac{q}{p}} + \frac{(q-1)p'}{p}\left[(w(E))^q w(X)^{\frac{-q}{p'}} - (w(E))^{\frac{q}{p}}\right]
\end{aligned}$$

if $1<p,q<\infty$, *and*

$$\|\chi_E\|^w_{p,\,q)} = \sup_{0<t<w(X)} t^{\frac{1}{p}} \left(\chi_E\right)^{**}_w(t) = \left(w\left(E\right)\right)^{\frac{1}{p}}$$

if $1<p<\infty, q=\infty$.

Lemma 3.2. *Let h be a decreasing function on $(0,1)$. If $0<\alpha\leq 1$ and $\beta\geq 0$, then*

$$\left(\int_0^1 u^{\beta-1}h\left(u\right)du\right)^{\alpha} \leq \alpha\beta^{1-\alpha}\int_0^1 u^{\alpha\beta-1}\left(h\left(u\right)\right)^{\alpha}du$$

and the constant $\alpha\beta^{1-\alpha}$ is sharp [2].

Lemma 3.3. *Let $1<p<\infty,\ 1<q<r<\infty$ and $f\in L^{p,q)}_w\cap L^{p,r)}_w$. Then*

$$\left(\int_0^{w(X)} t^{\frac{r}{p}-1}(f^{**}_w(t))^{r-\varepsilon}dt\right)^{\frac{1}{r-\varepsilon}} \leq \left(\int_0^{w(X)} t^{\frac{q}{p}-1}(f^{**}_w(t))^{q-\varepsilon}\,dt\right)^{\frac{1}{q-\varepsilon}} \left(\frac{p}{\varepsilon}w\left(X\right)^{\frac{\varepsilon}{p}}\right)^{\frac{q-r}{(q-\varepsilon)(r-\varepsilon)}}.$$

for any $0<\varepsilon<q-1$.

Proof. Let $1<p<\infty,\ 1<q<r<\infty$. For any $f\in L^{p,q)}_w\cap L^{p,r)}_w$, we have

$$\left(\int_0^{w(X)} t^{\frac{r}{p}-1}(f^{**}_w(t))^{r-\varepsilon}dt\right)^{\frac{1}{r-\varepsilon}} = \left(\int_0^{w(X)} t^{\frac{r-\varepsilon}{p}}(f^{**}_w(t))^{r-\varepsilon}t^{\frac{\varepsilon-p}{p}}\,dt\right)^{\frac{1}{r-\varepsilon}}.$$

If we apply Hölder's inequality with parameters $a=\frac{q-\varepsilon}{r-\varepsilon}$, $b=\frac{q-\varepsilon}{q-r}$ and the measure $t^{\frac{(\varepsilon-p)}{p}}dt$, then we get

$$\begin{aligned}
\left(\int_0^{w(X)} t^{\frac{r}{p}-1}(f^{**}_w(t))^{r-\varepsilon}dt\right)^{\frac{1}{r-\varepsilon}} &= \left(\int_0^{w(X)} t^{\frac{r-\varepsilon}{p}}\left(f^{**}_w(t)\right)^{r-\varepsilon}t^{\frac{\varepsilon-p}{p}}\,dt\right)^{\frac{1}{r-\varepsilon}} \\
&\leq \left[\left(\int_0^{w(X)}\left(t^{\frac{r-\varepsilon}{p}}\left(f^{**}_w(t)\right)^{r-\varepsilon}\right)^{a}t^{\frac{\varepsilon-p}{p}}\,dt\right)^{\frac{1}{a(r-\varepsilon)}}\right. \\
&\qquad \left.\left(\int_0^{w(X)} t^{\frac{\varepsilon-p}{p}}\,dt\right)^{\frac{1}{b(r-\varepsilon)}}\right] \\
&= \left[\left(\int_0^{w(X)} t^{\frac{q-p}{p}}\left(f^{**}_w(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}\right. \\
&\qquad \left.\left(\frac{p}{\varepsilon}w\left(X\right)^{\frac{\varepsilon}{p}}\right)^{\frac{q-r}{(q-\varepsilon)(r-\varepsilon)}}\right].
\end{aligned}$$

In other words,

$$\left(\int_0^{w(X)} t^{\frac{r}{p}-1}(f_w^{**}(t))^{r-\varepsilon}dt\right)^{\frac{1}{r-\varepsilon}} \leq \left[\left(\int_0^{w(X)} t^{\frac{q-p}{p}}(f_w^{**}(t))^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \left(\frac{p}{\varepsilon}w(X)^{\frac{\varepsilon}{p}}\right)^{\frac{q-r}{(q-\varepsilon)(r-\varepsilon)}}\right] \tag{4}$$

can be written. □

Theorem 3.2. *Let* $1 < p < \infty$ *and* $1 < q_1 \leq q_2 < \infty$*. Then*

$$\|\cdot\|_{p,q_2)}^{w} \leq C\|\cdot\|_{p,q_1)}^{w}$$

where the sharp constant is

$$C = \frac{q_2}{q_1}q_1^{q_1-q_2}w(X)^{\frac{(q_2-1)(q_1-q_2)}{p}}.$$

Therefore $L_w^{p,q_1)} \hookrightarrow L_w^{p,q_2)}$.

Proof. Let's take any $f \in L_w^{p,q_1)}$. If we use the inequality (4), then the following

$$\begin{aligned}
\|f\|_{p,q_2)}^{w} &= \sup_{0<\varepsilon<q_2-1}\left(\frac{q_2}{p}\varepsilon\int_0^{w(X)} t^{\frac{q_2}{p}-1}\left[f_w^{**}(t)\right]^{q_2-\varepsilon}dt\right)^{\frac{1}{q_2-\varepsilon}}\\
&= \sup_{0<\varepsilon<q_2-1}\left(\frac{q_2}{q_1}\right)^{\frac{1}{q_2-\varepsilon}}\left(\frac{q_1}{p}\varepsilon\right)^{\frac{1}{q_2-\varepsilon}}\left(\int_0^{w(X)} t^{\frac{q_2}{p}-1}\left[f_w^{**}(t)\right]^{q_2-\varepsilon}dt\right)^{\frac{1}{q_2-\varepsilon}}\\
&\leq \sup_{0<\varepsilon<q_2-1}\left[\left(\frac{q_2}{q_1}\right)^{\frac{1}{q_2-\varepsilon}}\left(\frac{q_1}{p}\varepsilon\right)^{\frac{1}{q_2-\varepsilon}}\left(\int_0^{w(X)} t^{\frac{q_1-p}{p}}(f_w^{**}(t))^{q_1-\varepsilon}dt\right)^{\frac{1}{q_1-\varepsilon}}\right.\\
&\qquad\left.\left(\frac{p}{\varepsilon}\right)^{\frac{(q_1-q_2)}{(q_1-\varepsilon)(q_2-\varepsilon)}}w(X)^{\frac{\varepsilon(q_1-q_2)}{p(q_1-\varepsilon)(q_2-\varepsilon)}}\right]\\
&= \frac{q_2}{q_1}\sup_{0<\varepsilon<q_2-1}\left[\left(\frac{q_1}{p}\varepsilon\right)^{\frac{q_1-q_2}{(q_1-\varepsilon)(q_2-\varepsilon)}}\left(\frac{q_1}{p}\varepsilon\int_0^{w(X)} t^{\frac{q_1-p}{p}}(f_w^{**}(t))^{q_1-\varepsilon}dt\right)^{\frac{1}{q_1-\varepsilon}}\right.\\
&\qquad\left.\left(\frac{p}{\varepsilon}\right)^{\frac{(q_1-q_2)}{(q_1-\varepsilon)(q_2-\varepsilon)}}w(X)^{\frac{\varepsilon(q_1-q_2)}{p(q_1-\varepsilon)(q_2-\varepsilon)}}\right]\\
&\leq \frac{q_2}{q_1}q_1^{q_1-q_2}w(X)^{\frac{(q_2-1)(q_1-q_2)}{p}}\|f\|_{p,q_1)}^{w}
\end{aligned}$$

can be written and $C(p, q_1, q_2, w(X))$ is sharp. □

Remark 3.2. *An alternative proof can be given by using Lemma 3.2 in which* $\beta = \frac{q_2}{p}$, $[f_w^{**}(t)]^{q_2-\varepsilon} = h(t)$ *and* $\alpha = \frac{(q_1-\varepsilon)}{(q_2-\varepsilon)}$.

Corollary 3.1. *Since weighted grand Lorentz spaces are defined on a finite measure space, the embedding* $L_w^{p,q)} \hookrightarrow L_w^{r,s)}$ *can be proved easily for all* $0 < r < p < \infty$ *and* $0 < q, s \leq \infty$.

Theorem 3.3. *Let* $1 < p < \infty$ *and* $1 < q \leq \infty$*. If* $w_2 \prec w_1$*, then* $L_{w_1}^{p,q)} \subset L_{w_2}^{p,q)}$.

Proof. Let $1 < p < \infty$ and $1 < q \leq \infty$ and w_1, w_2 be weights satisfying $w_2 \prec w_1$. Since $w_2 \prec w_1$, then there exists $C > 0$ such that $w_2(x) \leq Cw_1(x)$ for all $x \in X$. By using the definitions of distribution, rearrangement and maximal(average) functions, we get

$$\begin{aligned}
\lambda_{f,w_2}(y) &= w_2\{x \in X : |f(x)| > y\} \\
&= \int\limits_{\{x\in X:|f(x)|>y\}} w_2(x)\, d\mu(x) \\
&\leq \int\limits_{\{x\in X:|f(x)|>y\}} Cw_1(x)\, d\mu(x) \\
&= Cw_1\{x \in X : |f(x)| > y\} \\
&= C\lambda_{f,w_1}(y)
\end{aligned}$$

for any $y \geq 0$. Moreover

$$\begin{aligned}
f^*_{w_2}(t) &= \inf\{y > 0 : \lambda_{f,w_2}(y) \leq t\} \\
&\leq \inf\{y > 0 : C\lambda_{f,w_2}(y) \leq t\} \\
&= f^*_{w_1}\left(\frac{t}{C}\right)
\end{aligned}$$

and so

$$\begin{aligned}
f^{**}_{w_2}(t) &= \frac{1}{t}\int\limits_0^t f^*_{w_2}(s)\, ds \leq \frac{1}{t}\int\limits_0^t f^*_{w_1}\left(\frac{s}{C}\right) ds \\
&= \frac{C}{t}\int\limits_0^{\frac{t}{C}} f^*_{w_1}(u)\, du = f^{**}_{w_1}\left(\frac{t}{C}\right)
\end{aligned}$$

can be written by change of variable. If we take the norm of $f \in L^{p,q)}_{w_1}$ according to $\|\cdot\|^{w_2}_{p,q)}$, then

$$\begin{aligned}
\|f\|^{w_2}_{p,q)} &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon \int\limits_0^{w_2(X)} t^{\frac{q}{p}-1}\left[f^{**}_{w_2}(t)\right]^{q-\varepsilon} dt\right)^{\frac{1}{q-\varepsilon}} \\
&\leq \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon \int\limits_0^{Cw_1(X)} t^{\frac{q}{p}-1}\left[f^{**}_{w_1}\left(\frac{t}{C}\right)\right]^{q-\varepsilon} dt\right)^{\frac{1}{q-\varepsilon}} \\
&= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon \int\limits_0^{w_1(X)} u^{\frac{q}{p}-1}C^{\frac{q}{p}-1}\left[f^{**}_{w_1}(u)\right]^{q-\varepsilon} Cdu\right)^{\frac{1}{q-\varepsilon}} \\
&\leq \sup_{0<\varepsilon<q-1}\left(C^{\frac{q}{p}}\frac{q}{p}\varepsilon \int\limits_0^{w_1(X)} u^{\frac{q}{p}-1}\left[f^{**}_{w_1}(u)\right]^{q-\varepsilon} du\right)^{\frac{1}{q-\varepsilon}} \\
&= C^{\frac{q}{p}}\|f\|^{w_1}_{p,q)}
\end{aligned}$$

is found. Therefore $L^{p,q)}_{w_1} \subset L^{p,q)}_{w_2}$ when $w_2 \prec w_1$. □

Discussion

This study highlights the complexity and significance of weighted and grand Lorentz spaces in functional analysis. Weighted spaces offer flexibility through weight functions, distributing function spaces to various mathematical models and applications. Grand Lorentz spaces extend Lorentz spaces, enriching function space theory and revealing intricate geometric properties.

Our exploration uncovered key results, from fundamental properties to operator behavior, with applications in signal processing, image reconstruction, and mathematical physics. As this field continues to evolve, the interplay between weight functions, function space geometry, and operator theory presents new research opportunities.

Ultimately, weighted grand Lorentz spaces remain fundamental in functional analysis, paving the way for future discoveries and advancements in this dynamic field.

Conclusion

After reading this paper, one can

(i) consider further investigating the properties and characterizations of weighted grand Lorentz spaces. This could include analyzing specific function classes, exploring continuity properties, and examining their connections with other function spaces.

(ii) study the behavior of various operators on weighted grand Lorentz spaces, including maximal, integral, differential operators, and the Hilbert transform. This exploration may involve analyzing their boundedness, compactness, and spectral properties within the framework of weighted spaces.

(iii) study the approximation properties of functions in weighted grand Lorentz spaces, investigating like interpolation methods or wavelet decompositions. This includes exploring optimal approximation rates and the role of weight functions in approximation accuracy.

(iv) apply harmonic analysis and PDE techniques to study phenomena in these spaces. This includes investigating existence, uniqueness and regularity of PDE solutions on weighted domains and exploring connections with harmonic analysis, such as Fourier analysis on weighted spaces.

(v) examine the interplay between weight functions and geometric properties such as smoothness, convexity, and the Radon-Nikodym property and deal with embeddings of weighted spaces into other function spaces.

(vi) explore applications of weighted grand Lorentz spaces in fields like image processing, data analysis and mathematical physics, leveraging their flexibility to model and analyze real-world phenomena more effectively.

(vii) extend weighted grand Lorentz spaces to broader frameworks like variable exponent Lebesgue spaces, Karamata spaces or Orlicz spaces, exploring their properties and applications

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