U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 3, 2021 ISSN 1223-7027

IMPLICIT CHAOS IN COMPLEX SYSTEMS IN THE FORM OF PERIOD DOUBLING THROUGH HARMONIC MAPPINGS

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In the Multifractal Theory of Motion, in the form of Schrödinger – type "regimes", non – linear behaviors of a complex system are analysed. Then, in the non - stationary case, symmetries of SL(2R)-type for structural units of any complex system can be highlighted. These become functional, for example in the form of period doubling "synchronization mode". This "mode" can mime a possible scenario toward chaos (period doubling scenario), without concluding in chaos (non – manifest chaos).

Keywords: Chaos, Complex Systems, Fractal Analysis, Multifractal, Group invariances

1. Introduction

Theoretical modelling of complex systems is one of the best tools in understanding the formation, evolution and extinction of a wide range of natural phenomena, which transcend the border of various scientific areas: cosmology, physics, biology, economy, finances, computer science, human societies, the Internet etc. [1]. Analysing non-linear complex systems is not a simple task, as the nature of non-linear complex systems is understood as containing many interacting structural units whose individual interactions lead to a collective behavior [2]. The large number of interactions and the nonlinear properties of most natural systems often make the fractal approach to their description the best option in describing their evolution [3]. The evolution of non-linear complex systems usually leads to chaotic or turbulent dynamics. There are already a few well-established scenarios that can describe such systems, through: intermittences [4], period doubling [5], quasi-periodicity [6], sub-harmonic bifurcation avalanche [7]. In most cases, the

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route to chaos is governed by well-defined control parameters which can transition the complex system to various stages of the chaotic dynamics.

In recent years, the analysis of non-linear complex dynamics has also led to the proposal of novel scenarios of chaos, through mapping with discontinuities [8, 9]. Moreover, for understanding the non-chaotic-chaotic transitions for coupled non-linear dynamic complex systems, non-chaos base states must be explained [10]. This reveals that the spectrum for the chaos analysis is vast and new and also that "outside the box" approaches are always needed to cover the multitude of possibilities offered by nature.

In the present paper, considering the multifractal paradigm as being functional (in the form of Multifractal Theory of Motion), we analyse the implicit chaos in the form of period doubling, in complex systems through harmonic mappings from the usual space to the hyperbolic one. In such a conjecture, a possible scenario toward chaos in the form of period doubling, without concluding in chaos (non-manifest chaos) can be mimed.

2. Mathematical Model

The fundamental hypothesis postulates that the structural units' dynamics of any complex system are described by continuous but non – differentiable curves (multifractal curves). Indeed, such an assumption is sustained by the following example, related to the interaction processes in a complex fluid: between two successive collisions, the trajectory of the complex fluid structural unit is a straight line. This trajectory becomes non – differentiable in the impact point. Considering that all the collision impact points form an uncountable set of points, it results that the trajectories of the complex fluid structural units become continuous and non – differentiable curves, i.e. fractal curves. Obviously, the reality is much more complicated, taking into account both the diversity of the particles which compose a complex fluid and the various interactions between them, in the form of double/triple collisions etc. Then, the complex fluid becomes multifractal. Extrapolating the previous reasoning for any complex system, it results that it can be assimilated to a multifractal.

In such context, the dynamics of the non-linear complex system's structural units can become operational in the multifractal paradigm through the Multifractal Theory of Motion, in the form of a Schrödinger equation of a multifractal type [2, 11-14]:

$$\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\partial^l\partial_l\Psi + i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial_t\Psi = 0 \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial_t}, \partial_l = \frac{\partial}{\partial X^l}, \partial_l\partial_p = \frac{\partial}{\partial X^l}\frac{\partial}{\partial X^p}, i = \sqrt{-1}, \qquad l, p = 1, 2, 3 \tag{2}$$

In the previous relations, the significations of the terms are the following:

a) t is the non-multifractal time coordinate and plays the role of the affine parameter of the motion curves;
b) dt is the scale resolution;
c) X^l are the multifractal spatial coordinates;
d) λ is a constant coefficient imposed by the differential-non-differential scale transition;
e) $f[\alpha(D_F)]$ is the singularity spectrum of order α;
f) α is the singularity index;
g) D_F is the fractal dimension of the motion.

There are various modes for defining the fractal dimension: the Hausdorff-Besikovitch fractal dimension, Kolmogorov fractal dimension [14] etc. (these are the most frequently employed). Choosing one such definition and operating it in the framework of complex system dynamics, the value of the fractal dimension shall be arbitrary and constant for the entirety of the dynamic analysis pertaining to the complex system. In such a context, for correlative processes in complex system dynamics it is commonly established that $D_F < 2$ while, for non – correlative processes in complex system dynamics, $D_F > 2$.

It is noted that, when complex system dynamics are described through the singularity spectrum of $f[\alpha(D_F)]$, then the Scale Relativity Theory will become operational on multifractal manifolds through the Multifractal Theory of Motion. Conversely, when working with a singular fractal dimension D_F, then the Scale Relativity Theory will become operational on a monofractal manifold, through the model of Nottale [11]. Thus, the benefits of the Multifractal Theory of Motion are evident [15, 16]:

a) the areas of complex systems dynamics that are portrayed by a specific fractal dimension;
b) the number of areas in the complex systems dynamics, for whom the fractal dimensions are found in a span of values;
c) the types of universality in the complex systems dynamics, even in the cases when regular or strange attractors exhibit diverse aspects.

New data regarding the non-linear behaviors of complex systems, complementary to the class of solutions associated to equation (1) (generated through initial and boundary conditions), can also be given on the base of the transformation groups (which leave invariant the equation (1)) [17, 18]. These transformation groups constitute, in the most general case of the one – dimensional non-linear complex system dynamics, a realization of the Lie group $SL(2R)$[19], through the action [17, 18]:

$$t' \to \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad X' \to \frac{X}{\gamma t + \delta} \tag{3}$$

where α, β, γ and δ are real elements.

Let it be considered that, in accordance with general mathematical procedures from [17, 18], the complex system dynamics may be generally described with the help of a 2 X 2 matrix with real elements. In any complex system, it is obvious that the problem revolves around a family of such matrices, each of them describing the dynamics of any complex system entity (structural unit). The interactions between the complex system entities can then be expressed through relations between the representative matrices. These relations must contain certain parameters which characterize the structure of the complex system, adequate to the description of the complex system dynamics.

Therefore, the matrix which generates an anharmonic curve [17, 18] is a 2 X 2 matrix with real elements. This matrix has the form:

$$\widehat{\mathrm{M}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R} \tag{4}$$

The elements of matrix $\widehat{\mathrm{M}}$ contain both the physical parameters and the possible initial conditions of the complex system dynamics, but in an unspecified form. More precisely, the elements of matrix (4) depend on the scale resolution in the sense of the Multifractal Theory of Motion [12, 13].

A set of such matrices, with variable elements, may be admitted as relevant for any complex system dynamics. In a particular case, a fundamental spinor set can be used, given by 2 X 2 matrices which describe the complex system dynamics [20].

Now, any 2 X 2 matrix of form (4) can be written as a linear combination with real coefficients, which implies two special matrices: the unity matrix $\widehat{U}$ and a null – trace matrix $\hat{I}$, such that:

$$\widehat{M} = \lambda \widehat{U} + \mu \hat{I} \tag{5}$$

The involution $\hat{I}$ has some important properties, such as the ones that its squared form is a multiple of $\widehat{U}$ and the fixed points of its homographic action are the ones of matrix $\widehat{M}$.

In (5), we have the liberty to choose a parameterization in which, up to a sign, the squared form of $\widehat{U}$ can be the unity matrix. In this case, the elements of $\hat{I}$ may be expressed with only two parameters. These parameters represent the asymptotic directions of matrix $\widehat{M}$. If the asymptotic directions are complex, in the form $u \pm iv$, the representation of the matrix $\hat{I}$ through asymptotic directions is of a spherical type. Satisfying the above properties implies, for the matrix $\hat{I}$, the form:

$$\hat{I} = \frac{1}{v}\begin{pmatrix} -u & -u^2 - v^2 \\ 1 & u \end{pmatrix}, \qquad \hat{I} = -\widehat{U} \tag{6}$$

When analyzing this problem, the proposed model allows an explicit differential description of the complex system dynamics, through matrix geometry, identic to the metric geometry of space at a certain moment - the hyperbolic geometry of second type [17, 18].

The representation of complex system dynamics through 2 X 2 matrices leads to a natural matrix of the matrices' space. For example, the Killing – Cartan metric of $SL(2R)$ – type algebra of these matrices may be chosen [17, 18]. The basic co – vectors of such geometry are, in the general case of matrix (4), given by the 1-differential forms:

$$\omega^1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}, \qquad \omega^2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}, \qquad \omega^3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta}$$
$$\Delta = \alpha\gamma - \beta^2 \tag{7}$$

In the parameterization given through (5) and (6), (7) becomes:

$$\omega^1 = \frac{1}{v} d\Phi + \sin^2\Phi \frac{du}{v^2} - \sin\Phi\cos\Phi\frac{dv}{v^2}$$
$$\omega^2 = 2\frac{u}{v} d\Phi + 2\sin^2\Phi \frac{udu + vdv}{v^2} + 2\sin\Phi\cos\Phi \frac{vdu - udv}{v^2}$$
$$\omega^3 = \frac{u^2 + v^2}{v} d\Phi + \sin^2\Phi \frac{(u^2 - v^2)du + 2uvdv}{v^2} + \sin\Phi\cos\Phi \frac{2uvdu - (u^2 - v^2)dv}{v^2} \tag{8}$$

where:

$$\tan\Phi = \frac{\mu}{\lambda} \tag{9}$$

Related to these co – vectors, the metric is given by the squared form:

$$ds^2 = \omega^1\omega^3 - \left(\frac{\omega^2}{2}\right)^2 = d\Phi^2 - \sin^2\Phi \frac{du^2 + dv^2}{v^2} \tag{10}$$

As such, for as long as the complex system is defined by the core property that the proposed model admits as being essential – which is the complex system dynamic – its description mode is a metric geometry. Then, the metric is given through (10), where Φ is an arbitrary "phase", and u and v are "coordinates" obtained from the (local) dynamic of the complex system, in a previously described method.

Such a metric approach for the complex system dynamics can be certainly delegated to harmonic maps, from the complex system to the space. As soon as the mapping mode of the complex system on the space available its disposal is solved, the quantities Φ, u and v – and, the elements of the matrix family which represent the complex system – are obtained. In principle, a "position" function will be sufficient to correctly define a specific quantity of the complex system.

The difficulty of representing the complex system in this form is overcome though the harmonic map $\boldsymbol{X} = (X_1, X_2, X_3) \to \boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$ which can provide a set of quantities as functions of spatial coordinates. Let it be considered the functional corresponding to the harmonic mapping principle (for details see [21]):

$$J = \frac{1}{2} \iiint d^3 \boldsymbol{X} \sqrt{|h|}\, h^{il}(\boldsymbol{X}) \frac{\partial \xi^\mu \partial \xi^\nu}{\partial X^i \partial X^l} g_{\mu\nu}(\boldsymbol{\xi}) \tag{11}$$

In (11), h is the space metric and g is the associated metric of the complex system. Cancelling the first-degree variation of this functional, in relation to the spatial coordinates, gives the sought harmonic map. Taking into account the fact that the space is Euclidean and using (9) for the metric tensor associated to the complex system, for the integrand of (11) the expression will be:

$$\sqrt{|h|}\, h^{il}(\boldsymbol{X}) \frac{\partial \xi^\mu \partial \xi^\nu}{\partial X^i \partial X^l} g_{\mu\nu}(\boldsymbol{\xi}) \equiv \partial_l \Phi \partial^l \Phi - \left(\frac{\sin \Phi}{v}\right)^2 [\partial_l u \partial^l u + \partial_l v \partial^l v] \tag{12}$$

Introducing the complex variables

$$h = u + iv, \quad \bar{h} = u - iv, \tag{13}$$

the relation (12) becomes:

$$\sqrt{|h|}\, h^{il}(\boldsymbol{X}) \frac{\partial \xi^\mu \partial \xi^\nu}{\partial X^i \partial X^l} g_{\mu\nu}(\xi) \equiv \partial_l \Phi \partial^l \Phi - (\sin \Phi)^2 \left[\frac{\partial_l h \partial^l \bar{h}}{\left(h - \bar{h}\right)^2}\right] \tag{14}$$

where the usual notation ∂_l denotes the gradient.

In the case of the dynamics' synchronization of complex system's structural units, i.e. $\Phi = const. \neq n\pi$, with $n = 0,1,2 ...$, the Euler - equations corresponding to the functional (14) are:

$$\left(h - \bar{h}\right) \nabla(\nabla h) = 2(\nabla h)^2 \tag{15}$$

which admits

$$h = i \frac{\cosh \chi - \sinh \chi e^{-i\bar{\Omega}}}{\cosh \chi + \sinh \chi e^{-i\bar{\Omega}}}, \qquad \chi = \frac{\psi}{2} \tag{16}$$

as a solution.

Relation (16) represents harmonic mappings from the usual space to the Lobacevsky plane, having the metric

$$ds^2 = -\frac{dh d\bar{h}}{\left(h - \bar{h}\right)^2} = -\frac{du^2 + dv^2}{v^2} \tag{17}$$

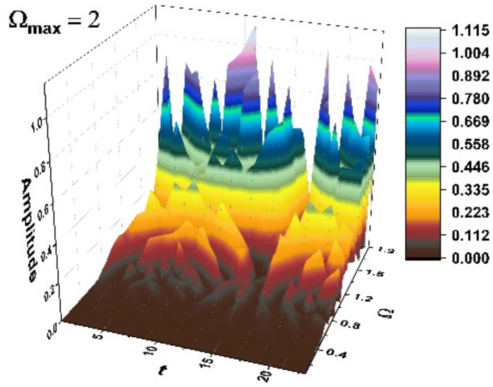
as long as χ (and thus ψ) are solutions of a Laplace – type equation for the free space.

3. Results and Discussion

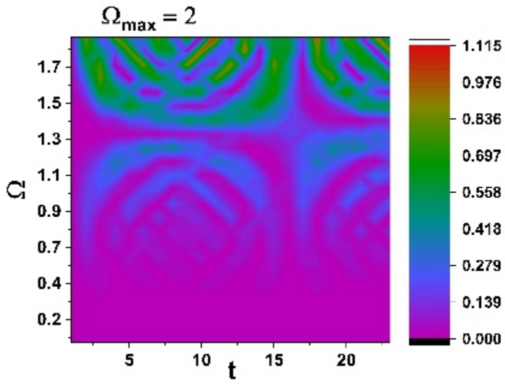
Therefore, space-time "synchronization modes" in phase and amplitude of the complex system structural units imply group invariances of a $SL(2R)$ type.

Then, period doubling emerges as a natural behavior in the complex systems dynamics (see Figures 1 a-d where $r = tanh\chi$, $|h| \equiv Amplitude$ and $\overline{\Omega} = \Omega t$ at various scale resolutions, given by means of the maximum value of Ω , i.e., Ω_{max}).

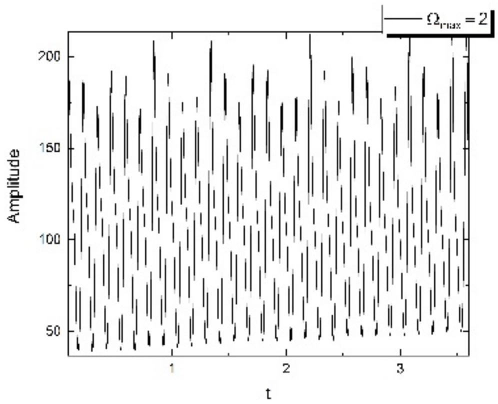
a



b



c



d

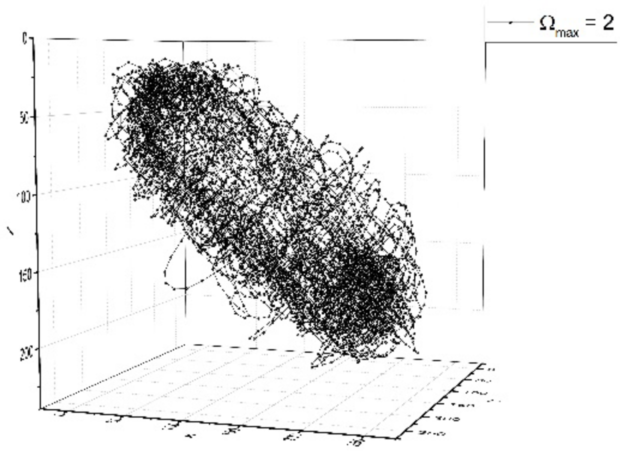


Fig. 1 a-d. A period doubling (a, b, c, d) "synchronization mode" of complex structural units (3D, contour plot, time – series and reconstructed attractors) for the scale resolution given by $\Omega_{max} = 2$.

As it can be observed in Figures 1 a-d, the natural transition of a complex system is to evolve from a normal period doubling state towards damped oscillating and a strong modulated dynamics. The complex system never reaches a chaotic state but it permanently evolves towards that state. There is a periodicity to the whole series of transitions, the system evolves through period doubling, damped oscillations even reaching in some cases an internment state but it never reaches a pure chaotic state. The evolution of the systems sees a "jump" into a period doubling oscillation state and the transition resumes towards a quasi-chaotic state.

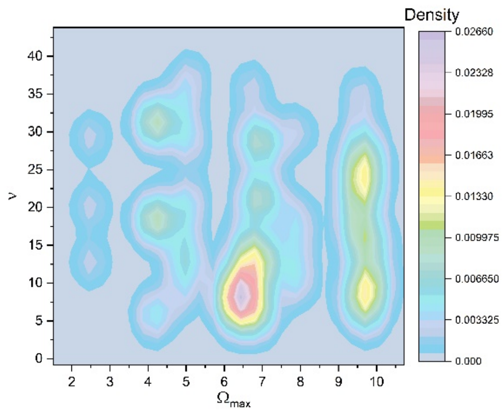


Fig. 2: Oscillation frequency of the complex system as a function of a scale resolution chosen by Ω_{max} Bifurcation Map

The Bifurcation Map is presented (Fig. 2) where again it is observed that the complex system starts from a steady state (double period state) and evolves towards a chaotic one ($\Omega_{max} = 2$) but it never reaches that state. For each periodic transition scenario, it is possible to observe the system swiping through all the previously mentioned dynamic states. Therefore, there is an overall-periodicity with a continuous increase in oscillation amplitude.

Let us note that the mathematical formalism of the Multifractal Theory of Motion implies various operational procedures (invariance groups, harmonic mappings, groups isomorphism, embedding manifolds etc.) with quite a number of applications in complex systems dynamics [22-33].

4. Conclusions

In the non - stationary case of Schrödinger equation of a multifractal type, a symmetry of $SL(2R)$ type is highlighted, situation in which the period doubling "synchronization mode" among the structural units of a complex system becomes functional. In such a manner, a possible scenario toward chaos (period doubling scenario), without concluding in chaos (nonmanifest chaos) can be mimed.

Also, for each periodic transition scenario, it is observed that the system is swiping through all the previously mentioned dynamic states, in paper.

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