FIXED POINT RESULTS FOR A PAIR OF α_*- DOMINATED MULTIVALUED MAPPINGS WITH APPLICATIONS

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The purpose of this paper is to find out fixed point results for a pair of semi α_*-dominated multivalued mappings fulfilling a generalized locally Ciric type rational $F-$dominated multivalued contractive condition on a closed ball in complete dislocated b-metric space. Examples and applications have been given. Our results extend several comparable results in the existing literature.

Keywords: fixed point; closed ball; semi α_*-dominated multivalued mapping; graphic contractions; order, application to integral equations.

MSC2010: 54H25; 47H10.

1. Introduction and Preliminaries

Fixed point theory plays a foundational role in functional analysis. Recently, many fixed point results have been proved satisfying contraction on a closed ball instead of a whole space (see [16, 17, 18, 19, 20, 21, 22]). In the present paper, we have achieved fixed point results for generalized F-contraction on a sequence contained in a closed ball for a general class of semi α_*-dominated mappings rather than α_*-admissible mappings and for a weaker class of strictly increasing mappings F rather than class of mappings F introduced by Wardowski [23]. The notion of multi graph dominated mapping is introduced. Fixed point results with graphic contractions on a closed ball for such mappings are established. We establish the existence of common fixed of multi $\preceq$-dominated mappings in an ordered space. Examples are given to demonstrate the variety of our results. An application is given to approximate the unique common solution of nonlinear integral equations. Moreover, we investigate our results in a better framework of dislocated b-metric space. New results in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space and metric space can be obtained as corollaries of our results. We give the following definitions and results which will be needed in the sequel.

Definition 1.1. [12] Let X be a nonempty set and let $d_b : X \times X \to [0, \infty)$ be a function, called a dislocated b-metric (or simply d_b-metric), if there exists $b \geq 1$ such that for any $x, y, z \in X$, the following conditions hold:

(i) If $d_b(x, y) = 0$, then $x = y$;

(ii) $d_b(x, y) = d_b(y, x)$;

(iii) $d_b(x, y) \leq b[d_b(x, z) + d_b(z, y)]$.

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The pair (X, d_b) is called a dislocated b-metric space. It should be noted that every dislocated metric is a dislocated b-metric with $b = 1$.

It is clear that if $d_b(x, y) = 0$, then from (i), $x = y$. But if $x = y$, $d_b(x, y)$ may not be 0. For $x \in X$ and $\varepsilon > 0$, $\overline{B(x, \varepsilon)} = \{y \in X : d_b(x, y) \leq \varepsilon\}$ is a closed ball in (X, d_b). We use $D.B.M$ space instead dislocated b-metric space.

Definition 1.2. [12] Let (X, d_b) be a $D.B.M$ space .

(i) A sequence $\{x_n\}$ in (X, d_b) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $n, m \geq n_0$ we have $d_b(x_m, x_n) < \varepsilon$ or $\lim\limits_{n,m\to\infty} d_b(x_n, x_m) = 0$.

(ii) A sequence $\{x_n\}$ dislocated b-converges (for short d_b -converges) to x if $\lim\limits_{n\to\infty} d_b(x_n, x) = 0$. In this case x is called a d_b-limit of $\{x_n\}$.

(iii) (X, d_b) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d_b(x, x) = 0$.

Definition 1.3. Let K be a nonempty subset of $D.B.M$ space of X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d_b(x, K) = d_b(x, y_0), \text{ where } d_b(x, K) = \inf_{y\in K} d_b(x, y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

Let Ψ_b, where $b \geq 1$, denote the family of all nondecreasing functions $\psi_b : [0, +\infty) \to [0, +\infty)$ such that $\sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty$ and $b\psi_b(t) < t$ for all $t > 0$, where ψ_b^k is the k^{th} iterate of ψ_b. Also $b^{n+1}\psi_b^{n+1}(t) = b^n b\psi_b(\psi_b^n(t)) < b^n\psi_b^n(t)$.
We denote $P(X)$ be the set of all closed proximinal subsets of X.

Definition 1.4. [20] The function $H_{d_b} : P(X) \times P(X) \to R^+$, defined by

$$H_{d_b}(N, M) = \max\{\sup_{n\in N} d_b(n, M), \sup_{m\in M} d_b(N, m)\}$$

is called dislocated Hausdorff $b-$metric on $P(X)$.

Definition 1.5. Let (X, d_b) be a $D.B.M$ space. Let $S : X \to P(X)$ be multivalued mapping and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$, we say that the S is semi α_*-dominated on H, whenever $\alpha_*(i, Si) \geq 1$ for all $i \in H$, where $\alpha_*(i, Si) = \inf\{\alpha(i, l) : l \in Si\}$. If $H = X$, then we say that the S is α_*-dominated. If $S : X \to X$ be a self mapping, then S is semi α-dominated on H, whenever $\alpha(i, Si) \geq 1$ for all $i \in H$.

Lemma 1.1. Let (X, d_b) be a $D.B.M$ space. Let $(P(X), H_{d_b})$ be a dislocated Hausdorff $b-$metric space on $P(X)$. Then, for all $G, H \in P(X)$ and for each $g \in G$ such that $d_b(g, H) = d_b(g, h_g)$, where $h_g \in H$. Then the following holds:

$$H_{d_b}(G, H) \geq d_b(g, h_g).$$

2. **Main Result**

Let (X, d_b) be a $D.B.M$ space, $x_0 \in X$ and $S, T : X \to P(X)$ be the multifunctions on X. Let $x_1 \in Sx_0$ be an element such that $d_b(x_0, Sx_0) = d_b(x_0, x_1)$. Let $x_2 \in Tx_1$ be such that $d_b(x_1, Tx_1) = d_b(x_1, x_2)$. Let $x_3 \in Sx_2$ be such that $d_b(x_2, Sx_2) = d_b(x_2, x_3)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ...$ Also $d_b(x_{2n}, Sx_{2n}) = d_b(x_{2n}, x_{2n+1})$, $d_b(x_{2n+1}, Tx_{2n+1}) = d_b(x_{2n+1}, x_{2n+2})$. We denote this iterative sequence by $\{TS(x_n)\}$. We say that $\{TS(x_n)\}$ is a sequence in X generated by x_0. For $x, y \in X$, $a > 0$, we define $D_b(x, y)$ as

$$D_b(x, y) = \max\{d_b(x, y), \frac{d_b(x, Sx) .d_b(y, Ty)}{a + d_b(x, y)}, d_b(x, Sx), d_b(y, Ty)\}.$$

Theorem 2.1. Let (X,d_b) be a complete $D.B.M$ space. Suppose there exist a function $\alpha : X\times X \to [0,\infty)$. Let, $r>0$, $x_0 \in \overline{B_{d_b}(x_0,r)}$, F be a strictly increasing function and $S,T : X \to P(X)$ be two α_*-dominated mappings on $\overline{B_{d_b}(x_0,r)}$. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$\tau + F(H_{d_b}(Sx,Ty)) \leq F(\psi_b(D_b(x,y))) \tag{1}$$

for all $x,y \in \overline{B_{d_b}(x_0,r)} \cap \{TS(x_n)\}$, $\alpha(x,y)\geq 1$ and $H_{d_b}(Sx,Ty)>0$. Also if, $\overline{B_{d_b}(x_0,r)}$ is closed and

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_b(x_0,Sx_0))\} \leq r \text{ for all } n\in\mathbb{N}\cup\{0\} \text{ and } b\geq 1. \tag{2}$$

Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_b}(x_0,r)}$, $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $\{TS(x_n)\}\to x^*\in\overline{B_{d_b}(x_0,r)}$. Also if the inequality (1) holds for x^* and either $\alpha(x_n,x^*)\geq 1$ or $\alpha(x^*,x_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then S and T have common fixed point x^* in $\overline{B_{d_b}(x_0,r)}$.

Proof. Consider a sequence $\{TS(x_n)\}$. From (2), we get

$$d_b(x_0,x_1) \leq \sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_b(x_0,Sx_0))\} \leq r.$$

Let $x_2,\cdots,x_j \in \overline{B_{d_b}(x_0,r)}$ for some $j\in\mathbb{N}$. If $j=2i+1$, where $i=1,2,\ldots,\frac{j-1}{2}$. Since $S,T: X\to P(X)$ be a α_*-dominated mappings on $\overline{B_{d_b}(x_0,r)}$, so $\alpha_*(x_{2i},Sx_{2i})\geq 1$ and $\alpha_*(x_{2i+1},Tx_{2i+1})\geq 1$. As $\alpha_*(x_{2i},Sx_{2i})\geq 1$, this implies $\inf\{\alpha(x_{2i},b) : b\in Sx_{2i}\}\geq 1$. Also $x_{2i+1}\in Sx_{2i}$, so $\alpha(x_{2i},x_{2i+1})\geq 1$. Now by using Lemma 1.1, we obtain,

$$\begin{aligned}\tau + F(d_b(x_{2i+1},x_{2i+2})) &\leq \tau + F(H_{d_b}(Sx_{2i},Tx_{2i+1})) \leq F(\psi_b(D_b(x_{2i},x_{2i+1})))\\ &\leq F\left(\psi_b(\max\{d_b(x_{2i},x_{2i+1}),d_b(x_{2i+1},x_{2i+2})\}\right)\end{aligned}$$

If $\max\{d_b(x_{2i},x_{2i+1}),d_b(x_{2i+1},x_{2i+2})\} = d_b(x_{2i+1},x_{2i+2})$, then

$$d_b(x_{2i+1},x_{2i+2}) < b\psi_b(d_b(x_{2i+1},x_{2i+2})).$$

This is the contradiction to the fact that $b\psi_b(t)<t$ for all $t>0$. So

$$\max\{d_b(x_{2i},x_{2i+1}),d_b(x_{2i+1},x_{2i+2})\} = d_b(x_{2i},x_{2i+1}).$$

Hence, we obtain

$$d_b(x_{2i+1},x_{2i+2}) < \psi_b(d_b(x_{2i},x_{2i+1})). \tag{3}$$

As $\alpha_*(x_{2i-1},Tx_{2i-1})\geq 1$ and $x_{2i}\in Tx_{2i-1}$, so $\alpha(x_{2i-1},x_{2i})\geq 1$. Now, by using Lemma 1.1, we have

$$\begin{aligned}\tau + F(d_b(x_{2i},x_{2i+1})) &\leq \tau + F(H_{d_b}(Tx_{2i-1},Sx_{2i})) \leq F(\psi_b(D_b(x_{2i},x_{2i-1})))\\ &\leq F(\psi_b(\max\{d_b(x_{2i},x_{2i-1}),d_b(x_{2i},x_{2i+1})\})).\end{aligned}$$

Since F is strictly increasing function then we have

$$d_b(x_{2i},x_{2i+1}) < \psi_b(\max\{d_b(x_{2i},x_{2i-1}),d_b(x_{2i},x_{2i+1})\})$$

If $\max\{d_b(x_{2i},x_{2i-1}),d_b(x_{2i},x_{2i+1})\} = d_b(x_{2i},x_{2i+1})$, then

$$d_b(x_{2i},x_{2i+1}) < \psi_b(d_b(x_{2i},x_{2i+1})) < b\psi_b(d_b(x_{2i},x_{2i+1})).$$

This is the contradiction to the fact that $b\psi_b(t)<t$ for all $t>0$. Hence, we obtain

$$d_b(x_{2i},x_{2i+1}) < \psi_b(d_b(x_{2i-1},x_{2i})). \tag{4}$$

As ψ_b is nondecreasing, so

$$\psi_b(d_b(x_{2i},x_{2i+1})) < \psi_b(\psi_b(d_b(x_{2i-1},x_{2i}))).$$

By using above inequality in (3), we have

$$d_b(x_{2i+1}, x_{2i+2}) < \psi_b^2(d_b(x_{2i-1}, x_{2i}))).$$

Continuing in this way, we obtain

$$d_b(x_{2i+1}, x_{2i+2}) < \psi_b^{2i+1}(d_b(x_0, x_1)). \tag{5}$$

Now, if $j = 2i$, where $i = 1, 2, \ldots \frac{j}{2}$. By using (4) and similar procedure as above, we have

$$d_b(x_{2i}, x_{2i+1}) < \psi_b^{2i}(d_b(x_0, x_1)). \tag{6}$$

Now, by combining (5) and (6)

$$d_b(x_j, x_{j+1}) < \psi_b^j(d_b(x_0, x_1)) \text{ for all } j \in N. \tag{7}$$

Now, by using triangle inequality and by (7), we have

$$\begin{aligned} d_b(x_0, x_{j+1}) &\leq bd_b(x_0, x_1) + b^2 d_b(x_1, x_2) + \ldots + b^{j+1} d_b(x_j, x_{j+1}) \\ &< bd_b(x_0, x_1) + b^2 \psi_b(d_b(x_0, x_1)) + \ldots + b^{j+1} \psi_b^j(d_b(x_0, x_1)) \\ &< \sum_{i=0}^{j} b^{i+1} \{\psi_b^i(d_b(x_0, x_1))\} < r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B_{d_b}(x_0, r)}$. Hence $x_n \in \overline{B_{d_b}(x_0, r)}$ for all $n \in N$, therefore $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_b}(x_0, r)}$. As S, T are α_*−dominated on $\overline{B_{d_b}(x_0, r)}$, so $\alpha_*(x_{2n}, Sx_{2n}) \geq 1$ and $\alpha_*(x_{2n+1}, Tx_{2n+1}) \geq 1$. This implies $\alpha(x_n, x_{n+1}) \geq 1$. Also inequality (7) can be written as

$$d_b(x_n, x_{n+1}) < \psi_b^n(d_b(x_0, x_1)), \text{ for all } n \in N. \tag{8}$$

As $\sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty$, then for some $p \in N$, then the series

$$\sum_{k=1}^{+\infty} b^k \psi_b^k(\psi_b^{p-1}(d_b(x_0, x_1))),$$

converges. As $b\psi_b(t) < t$, so

$$b^{n+1}\psi_b^{n+1}(\psi_b^{p-1}(d_b(x_0, x_1))) < b^n \psi_b^n(\psi_b^{p-1}(d_b(x_0, x_1))) \text{ for all } n \in N.$$

Fix $\varepsilon > 0$, then there exists $p(\varepsilon) \in N$, such that

$$b\psi_b(\psi_b^{p(\varepsilon)-1}(d_b(x_0, x_1))) + b^2\psi_b^2(\psi_b^{p(\varepsilon)-1}(d_b(x_0, x_1))) + \cdots < \varepsilon$$

Let $n, m \in N$ with $m > n > p(\varepsilon)$, then, we obtain

$$\begin{aligned} d_b(x_n, x_m) &\leq bd_b(x_n, x_{n+1}) + b^2 d_b(x_{n+1}, x_{n+2}) + \cdots + b^{m-n} d_b(x_{m-1}, x_m) \\ &< b\psi_b(\psi_b^{n-1}(d_b(x_0, x_1))) + \cdots + b^{m-n}\psi_b^{m-n}(\psi_b^{n-1}(d_b(x_0, x_1))) \\ &< b\psi_b(\psi_b^{p(\varepsilon)-1}(d_b(x_0, x_1))) + b^2\psi_b^2(\psi_b^{p(\varepsilon)-1}(d_b(x_0, x_1))) + \cdots < \varepsilon. \end{aligned}$$

Thus we proved that $\{TS(x_n)\}$ is a Cauchy sequence in $(\overline{B_{d_b}(x_0, r)}, d_b)$. As every closed set in a complete $D.B.M$ space is complete, so there exists $x^* \in \overline{B_{d_b}(x_0, r)}$ such that $\{TS(x_n)\} \to x^*$, that is

$$\lim_{n \to \infty} d_b(x_n, x^*) = 0. \tag{9}$$

Now,

$$\begin{aligned} d_b(x^*, Sx^*) &\leq bd_b(x^*, x_{2n+2}) + bd_b(x_{2n+2}, Sx^*) \\ &\leq bd_b(x^*, x_{2n+2}) + bH_{d_b}(Tx_{2n+1}, Sx^*). \quad \text{(by Lemma 1.1)} \end{aligned}$$

By assumption, $\alpha(x_n, x^*) \geq 1$. Suppose that $d_b(x^*, Sx^*) > 0$, then there exist positive integer k such that $d_b(x_n, Sx^*) > 0$ for all $n \geq k$. For $n \geq k$, we have

$$\begin{aligned} d_b(x^*, Sx^*) \quad < \quad & bd_b(x^*, x_{2n+2}) + b\psi_b(\max\{d_b(x^*, x_{2n+1}), d_b(x^*, Sx^*), \\ & \frac{d_b\left(x^*, Sx^*\right).d_b\left(x_{2n+1}, x_{2n+2}\right)}{a + d_b\left(x^*, x_{2n+1}\right)}, d_b(x_{2n+1}, x_{2n+2})\}). \end{aligned}$$

Letting $n \to \infty$, and using the inequality (9), we obtain a contradiction. So our supposition is wrong. Hence $d_b(x^*, Sx^*) = 0$ or $x^* \in Sx^*$. Similarly, by using the Lemma 1.1 and (9) we can show that $x^* = Tx^*$ or $x^* \in Tx^*$. Hence the S and T have a common fixed point x^* in $\overline{B_{d_l}(c_0, r)}$. Now,

$$d_l(x^*, x^*) \leq bd_l(x^*, Tx^*) + bd_l(Tx^*, x^*) \leq 0.$$

This implies that $d_b(x^*, x^*) = 0$. □

Theorem 2.2. Let (X, d_b) be a complete $D.B.M$ space. Suppose $S : X \to P(X)$ be a multivalued mappings and F be a strictly increasing function. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$\tau + F(H_{d_b}(Sx, Sy)) \leq F(\psi_b(D_b(x, y)))$$

for all $x, y \in \{SS(x_n)\}$. Then $\{SS(x_n)\} \to x^* \in X$ and S has a fixed point x^* in X and $d_b(x^*, x^*) = 0$.

Definition 2.1. Let X be a nonempty set, $\preceq$ is a partial order on X and $A \subseteq X$. We say that $a \preceq B$ whenever for all $b \in B$, we have $a \preceq b$. A mapping $S : X \to P(X)$ is said to be multi $\preceq$-dominated on A if $a \preceq Sa$ for each $a \in A \subseteq X$. If $A = X$, then $S : X \to P(X)$ is said to be multi $\preceq$-dominated.

Theorem 2.3. Let $(X, \preceq, d_b)$ be an ordered complete $D.B.M$ space. Let, $r > 0$, $x_0 \in \overline{B_{d_b}(x_0, r)}$, F be a strictly increasing function and $S, T : X \to P(X)$ be two multi $\preceq$-dominated mappings on $\overline{B_{d_b}(x_0, r)}$. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$\tau + F(H_{d_b}(Sx, Ty)) \leq F(\psi_b(D_b(x, y))) \tag{10}$$

for all $x, y \in \overline{B_{d_b}(x_0, r)} \cap \{TS(x_n)\}$, $x \preceq y$ and $H_{d_b}(Sx, Ty) > 0$. Also if, $\overline{B_{d_b}(x_0, r)}$ is closed and

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_b(x_0, x_1))\} \leq r \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } b \geq 1. \tag{11}$$

Then $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_b}(x_0, r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_b}(x_0, r)}$. Also if the inequality (10) holds for x^* and either $x_n \preceq x^*$ or $x^* \preceq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then S and T have a common fixed point x^* in $\overline{B_{d_b}(x_0, r)}$ and $d_b(x^*, x^*) = 0$.

Proof. Let $\alpha : X \times X \to [0, +\infty)$ be a mapping defined by $\alpha(x, y) = 1$ for all $x \in \overline{B_{d_b}(x_0, r)}$, $x \preceq y$, and $\alpha(x, y) = 0$ for all other elements $x, y \in X$. As S and T are the dominated mappings on $\overline{B_{d_b}(x_0, r)}$, so $x \preceq Sx$ and $x \preceq Tx$ for all $x \in \overline{B_{d_b}(x_0, r)}$. This implies that $x \preceq b$ for all $b \in Sx$ and $x \preceq c$ for all $c \in Tx$. So, $\alpha(x, b) = 1$ for all $b \in Sx$ and $\alpha(x, c) = 1$ for all $c \in Tx$. This implies that $\inf\{\alpha(x, y) : y \in Sx\} = 1$ and $\inf\{\alpha(x, y) : y \in Tx\} = 1$. Hence $\alpha_*(x, Sx) = 1$, $\alpha_*(x, Tx) = 1$ for all $x \in \overline{B_{d_b}(x_0, r)}$. So, $S, T : X \to P(X)$ are the α_*-dominated mapping on $\overline{B_{d_b}(x_0, r)}$. Moreover, inequality (10) can be written as

$$\tau + F(H_{d_b}(Sx, Ty)) \leq F(\psi_b(D_b(x, y)))$$

for all elements x, y in $\overline{B_{d_b}(x_0, r)} \cap \{TS(x_n)\}$, $\alpha(x, y) \geq 1$. Also, inequality (11) holds. Then, by Theorem 2.1, we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_b}(x_0, r)}$ and $\{TS(x_n)\} \to x^* \in \overline{B_{d_b}(x_0, r)}$. Now, $x_n, x^* \in \overline{B_{d_b}(x_0, r)}$ and either $x_n \preceq x^*$ or $x^* \preceq x_n$ implies that either

$\alpha(x_n, x^*) \geq 1$ or $\alpha(x^*, x_n) \geq 1$. So, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, S and T have a common fixed point x^* in $\overline{B_{d_b}(x_0, r)}$ and $d_b(x^*, x^*) = 0$. □

We have the following result without closed ball in an ordered complete $D.B.M$ space. Also we write the result only for one multivalued mapping.

Theorem 2.4. Let $(X, \preceq, d_b)$ be an ordered complete $D.B.M$ space. Let $S : X \to P(X)$ be a multi $\preceq$-dominated mappings on X and F be a strictly increasing function. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$\tau + F(H_{d_b}(Sx, Sy)) \leq F(\psi_b(D_b(x, y))) \tag{12}$$

for all $x, y \in \{SS(x_n)\}$ with $x \preceq y$. Then $\{SS(x_n)\} \to x^* \in X$. Also if the inequality (12) holds for x^* and either $x_n \preceq x^*$ or $x^* \preceq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then S has a fixed point x^* and $d_b(x^*, x^*) = 0$.

Example 2.1. Let $X = Q^+ \cup \{0\}$ and let $d_b : X \times X \to X$ be the complete dislocated b-metric on X defined by

$$d_b(x, y) = (x + y)^2 \text{ for all } x, y \in X$$

with parameter $b = 2$. Define the multivalued mappings, $S, T : X \times X \to P(X)$ by,

$$Sx = \begin{cases} [\frac{x}{3}, \frac{2}{3}x] \text{ if } x \in [0, 19] \cap X \\ [x, x+1] \text{ if } x \in (19, \infty) \cap X, \end{cases}$$

and

$$Tx = \begin{cases} [\frac{x}{4}, \frac{3}{4}x] \text{ if } x \in [0, 19] \cap X \\ [x+1, x+3] \text{ if } x \in (19, \infty) \cap X. \end{cases}$$

Considering, $x_0 = 1, r = 400$, then $\overline{B_{d_b}(x_0, r)} = [0, 19] \cap X$. Now $d_b(x_0, Sx_0) = d_b(1, S1) = d_b(1, \frac{1}{3}) = \frac{16}{9}$. So we obtain a sequence $\{TS(x_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, ...\}$ in X generated by x_0. Let $\psi_b(t) = \frac{4t}{10}$, then $b\psi_b(t) < t$. Define

$$\alpha(x, y) = \left\{ \begin{array}{cc} 1 & \text{if } x > y \\ \frac{1}{2} & \text{otherwise} \end{array} \right\}.$$

Now, if $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{TS(x_n)\}$ with $\alpha(x, y) \geq 1$, we have

$$\begin{aligned} H_{d_l}(Sx, Ty) &= \max\{\sup_{a \in Sx} d_l(a, [\frac{y}{4}, \frac{3y}{4}]), \sup_{b \in Ty} d_l([\frac{x}{3}, \frac{2x}{3}], b)\} \\ &= \max\left\{\left(\frac{2x}{3} + \frac{y}{4}\right)^2, \left(\frac{x}{3} + \frac{3y}{4}\right)^2\right\} \\ &< \psi_b\left\{\max\left((x+y)^2, \frac{25x^2y^2}{9(1+(x+y)^2)}, \left(\frac{4x}{3}\right)^2, \left(\frac{5y}{4}\right)^2\right)\right\}. \end{aligned}$$

Thus,

$$H_{d_b}(Sx, Ty)) < \psi_b(D_b(x, y))$$

which implies that, for any $\tau \in (0, \frac{12}{95}]$ and for a strictly increasing mapping $F(s) = \ln s$, we have

$$\tau + F(H_{d_l}(Sx, Ty)) \leq F(\psi_b(D_b(x, y))).$$

Note that, for 20, 21 $\in X$, then $\alpha(20, 21) \geq 1$. But, we have

$$\tau + F(H_{d_l}(S20, T21)) > F(\psi_b(D_b(20, 21))).$$

So condition (1) does not hold on X. Also, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_b(x_0,x_1))\} = \frac{16}{9} \times 2\sum_{i=0}^{n}(\frac{4}{5})^i < 400 = r.$$

Thus the mappings S and T are satisfying all the conditions of Theorem 2.1 only for $x, y \in \overline{B_{d_l}(x_0,r)} \cap \{TS(x_n)\}$ with $\alpha(x,y) \geq 1$. Hence S and T have a common fixed point.

3. **Fixed Point Results For Graphic Contractions**

In this section we presents an application of Theorem 2.1 in graph theory. Jachymski, [14], proved the result concerning for contraction mappings on metric space with a graph. A graph G is connected graph if there must be exist a path among any two different vertices (see for detail [8]).

Definition 3.1. Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$, $A \subseteq X$. A mapping $S : X \to P(X)$ is said to be multi graph dominated on A if $(x,y) \in E(G)$, for all $y \in Sx$ and $x \in A$.

Theorem 3.1. Let (X, d_l) be a complete $D.B.M$ space endowed with a graph G with constant $b \geq 1$. Let $r > 0$, $x_0 \in \overline{B_{d_l}(x_0,r)}$ and $S,T : X \to P(X)$. Assume that for some $\psi_b \in \Psi_b$, the following hold:

(i) S and T are multi graph dominated on $\overline{B_{d_l}(x_0,r)} \cap \{TS(x_n)\}$.

(ii) There exist $\tau > 0$ satisfying and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Sx,Ty)) \leq F(\psi_b(D_b(x,y))), \tag{13}$$

whenever $x, y \in \overline{B_{d_l}(x_0,r)} \cap \{TS(x_n)\}$, $(x,y) \in E(G)$ and $H_{d_l}(Sx,Ty) > 0$.

(iii) $\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_b(x_0,x_1))\} \leq r$.

Also if, $\overline{B_{d_b}(x_0,r)}$ is closed then, $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$, $(x_n,x_{n+1}) \in E(G)$ and $\{TS(x_n)\} \to m^*$. Also, if the inequality (13) holds for m^* and $(x_n,m^*) \in E(G)$ or $(m^*,x_n) \in E(G)$ for all $n \in N \cup \{0\}$, then S and T have common fixed point m^* in $\overline{B_{d_l}(x_0,r)}$.

Proof. Define, $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \overline{B_{d_l}(x_0,r)},\ (x,y) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

As S and T are semi graph dominated on $\overline{B_{d_l}(x_0,r)}$, then for $x \in \overline{B_{d_l}(x_0,r)}$, $(x,y) \in E(G)$ for all $y \in Sx$ and $(x,y) \in E(G)$ for all $y \in Tx$. So, $\alpha(x,y) = 1$ for all $y \in Sx$ and $\alpha(x,y) = 1$ for all $y \in Tx$. This implies that $\inf\{\alpha(x,y) : y \in Sx\} = 1$ and $\inf\{\alpha(x,y) : y \in Tx\} = 1$. Hence $\alpha_*(x,Sx) = 1$, $\alpha_*(x,Tx) = 1$ for all $x \in \overline{B_{d_l}(x_0,r)}$. So, $S,T : Z \to P(Z)$ are the semi α_*-dominated mapping on $\overline{B_{d_l}(x_0,r)}$. Moreover, inequality (13) can be written as

$$\tau + F(H_{d_l}(Sx,Ty)) \leq F(\psi_b(D_b(x,y)))$$

whenever $x, y \in \overline{B_{d_l}(x_0,r)} \cap \{TS(x_n)\}$, $\alpha(x,y) \geq 1$ and $H_{d_l}(Sx,Ty) > 0$. Also, (iii) holds. Then, by Theorem 2.1, we have $\{TS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$ and $\{TS(x_n)\} \to m^* \in \overline{B_{d_l}(x_0,r)}$. Now, $x_n, m^* \in \overline{B_{d_l}(x_0,r)}$ and either $(x_n,m^*) \in E(G)$ or $(m^*,x_n) \in E(G)$ implies that either $\alpha(x_n,m^*) \geq 1$ or $\alpha(m^*,x_n) \geq 1$. So, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, S and T have a common fixed point m^* in $\overline{B_{d_l}(x_0,r)}$ and $d_l(m^*,m^*) = 0$. □

4. **Application to the systems of integral equations**

Theorem 4.1. Let (X, d_b) be a complete $D.B.M$ space with constant $b \geq 1$. Let $c_0 \in X$ and $S, T : X \to X$. Assume that, there exist $\tau > 0$ and a strictly increasing mapping F such that the for some $\psi_b \in \Psi_b$, the following holds:

$$\tau + F(d_l(Sx, Ty)) \leq F(\psi_b(D_b(x, y))), \tag{14}$$

whenever $x, y \in \{TS(x_n)\}$ and $d(Sx, Ty) > 0$. Then $\{TS(x_n)\} \to u \in X$. Also if the inequality (14) holds for u, then S and T have unique common fixed point u in X.

Proof. The proof of this Theorem is similar as Theorem 2.1. We have to prove the uniqueness only. Let v be another common fixed point of S and T. Suppose $d_l(Su, Tv) > 0$. Then, we have

$$\tau + F(d_l(Su, Tv)) \leq F(\psi_b(D_b(x, y)))$$

This implies that

$$d_l(u, v) < \psi_b(d_l(u, v) < d_l(u, v),$$

which is a contradiction. So $d_l(Su, Tv) = 0$. Hence $u = v$. □

In this section, we discuss the application of Theorem 4.1 in the form of Volterra type integral equations.

$$u(k) = \int_0^k H_1(k, h, u(h))dh, \tag{15}$$

$$c(k) = \int_0^k H_2(k, h, c(h))dh, \tag{16}$$

for all $k \in [0, 1]$. We find the solution of (15) and (16). Let $X = C([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete dislocated $b-$metric. For $u \in C([0, 1], \mathbb{R}_+)$, define supremum norm as: $\|u\|_\tau = \sup_{k\in[0,1]} \{|u(k)| e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(u, c) = \left[\sup_{k\in[0,1]} \{|u(k) + c(k)| e^{-\tau k}\} \right]^{\frac{1}{2}} = \|u + c\|_\tau^2$$

for all $u, c \in C([0, 1], \mathbb{R}_+)$, with these settings, $(C([0, 1], \mathbb{R}_+), d_\tau)$ becomes a complete $D.B.M$ space.

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 4.2. Assume the following conditions are satisfied:

(i) $H_1, H_2 : [0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$;

(ii) Define

$$\begin{aligned} Su(k) &= \int_0^k H_1(k, h, u(h))dh, \\ Tc(k) &= \int_0^k H_2(k, h, c(h))dh. \end{aligned}$$

Suppose there exist $\tau > 0$, such that

$$|H_1(k, h, u) + H_2(k, h, c)| \leq \frac{\tau N(u, c)e^{\tau h}}{\tau N(u, c) + 1}$$

for all $k, h \in [0,1]$ and $u, c \in C([0,1], \mathbb{R})$, where

$$N(u,c) = \psi_b \left(\max \left\{ \begin{array}{c} \|u(k)+c(k)\|_\tau, \frac{\|u(k)+Su(k)\|_\tau \cdot \|c(k)+Tc(k)\|_\tau}{1+\|u(k)+c(k)\|_\tau}, \\ \|u(k)+Su(k)\|_\tau, \|u(k)+Tc(k)\|_\tau \end{array} \right\} \right).$$

Then integral equations (15) and (16) has a unique solution.

Proof. By assumption (ii)

$$\begin{aligned} |Su(k)+Tc(k)| &= \int_0^k |H_1(k,h,u(h)+H_2(k,h,c(h)))|\, dh, \\ &\leq \int_0^k \frac{\tau N(u,c)e^{\tau h}}{\tau N(u,c)+1} dh \\ &\leq \frac{N(u,c)}{\tau N(u,c)+1} e^{\tau k}. \end{aligned}$$

This implies

$$|Su(k)+Tc(k)|\, e^{-\tau k} \leq \frac{N(u,c)}{\tau N(u,c)+1}.$$

$$\tau + \frac{1}{N(u,c)} \leq \frac{1}{\|Su(k)+Tc(k)\|_\tau}.$$

So all the conditions of Theorem 4.1 are satisfied for $F(c) = -\frac{-1}{\sqrt{c}}$, $c > 0$ and $d_\tau(u,c) = \|u+c\|_\tau^2$. Hence integral equations given in (15) and (16) has a unique common solution. □

5. **Conclusions**

In the present paper, we have obtained fixed point results for a pair of generalized $F-$ dominated contractive multivalued mappings on an intersection of a closed ball and a sequence. We have used α_*-dominated mappings which are and showed that fixed point exists even if the contractive condition holds on subspaces rather than on whole spaces. Many fixed point results for multivalued and singlevalued contractive mappings can also be obtained as corollaries of our results.

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