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THREE-STEP ITERATIVE ALGORITHM FOR MULTIVALUED NONEPANSIVE MAPPINGS IN CAT(κ) SPACES

G. S. Saluja[1]

We study a three-step iteration process for multivalued nonexpansive mappings and establish some strong convergence theorems and a Δ-convergence theorem for the iteration scheme and mappings in the setting of a CAT(κ) space with $\kappa > 0$. Our results continue in a natural way the study of Abbas and Nazir in [Mat. Vesnik 66(2) (2014), 223-234].

Keywords: Multivalued nonexpansive mapping, modified three-step iteration scheme, fixed point, strong convergence, Δ-convergence, CAT(κ) space.

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1. Introduction

A CAT(κ) space is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space with curvature κ for $\kappa \in \mathbb{R}$. The terminology CAT(κ) spaces was introduced by Gromov in [12]. In recent years, CAT(κ) spaces are studied by many researchers, these one playing an important role in different aspects of geometry. A very wide discussion on these spaces and the role they play in geometry can be found in the monograph by. Bridson and Haefliger [5]

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [24]. Fixed point theory in CAT(κ) space was initiated by Kirk (see [16, 17]). His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces (see, e.g., [4, 7, 8, 9, 11, 20, 21]). We underline that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$, see [5].

In 2011, Piatek in [28] proved that the sequence generated by the Halpern scheme converges to a fixed point in the complete CAT(κ) spaces. In 2012, He *et al.* [13] proved that the sequence defined by Mann's algorithm Δ-converges to a fixed point in complete CAT(κ) spaces. Recently, Rashwan and Altwqi [30], studied the sequence defined by SP-iterative scheme for multivalued nonexpansive mappings and proved that the above said scheme strong and Δ-converges to common fixed points in the setting of CAT(κ) spaces.

The concept of Δ-convergence in a general metric space was introduced by Lim [22]. In 2008, Kirk and Panyanak [18] used the notion of Δ-convergence introduced by Lim [22] to prove in the CAT(0) space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [10] obtained Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space. Since then, the existence problem and the Δ-convergence problem of iterative sequences to a fixed point and common fixed points for different classes of mappings have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction: see, e.g., [1, 7, 10, 15, 19, 25, 31, 32, 33, 34, 36, 37, 38, 41].

Department of Mathematics, Govt. K.P.G. College, Jagdalpur - 494001 (C.G.), India, e-mail: saluja1963@gmail.com

2. **Preliminaries**

Let $(\mathcal{X}, d)$ be a metric space. A *geodesic path* joining $x \in \mathcal{X}$ to $y \in \mathcal{X}$ (or a geodesic from x_1 to x_2) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to ρ such that $c(0) = x_1$, $c(l) = x_2$ and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x_1, x_2) = l$. The image α of c is called a geodesic (or metric) *segment* joining x_1 and x_2. We say that $\mathcal{X}$ is (i) a *geodesic space* if any two points of $\mathcal{X}$ are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x_1 and x_2 for each $x_1, x_2 \in \mathcal{X}$, which we will denote by $[x_1, x_2]$, called the segment joining x_1 to x_2. This means that $z \in [x_1,\, x_2]$ if and only if $d(x_1, z) = (1-\alpha)d(x_1, x_2)$ and $d(x_2, z) = \alpha\, d(x_1, x_2)$.

In this case, we write $z = \alpha\, x_1 \oplus (1-\alpha)x_2$. The space $(\mathcal{X}, d)$ is said to be a geodesic space (D-geodesic space) if every two points of ρ (every two points of distance smaller than D) are joined by a geodesic, and $\mathcal{X}$ is said to be uniquely geodesic (D-uniquely geodesic) if there is exactly one geodesic joining x_1 and x_2 for each $x_1, x_2 \in \mathcal{X}$ (for $x_1, x_2 \in \mathcal{X}$ with $d(x_1, x_2) < D$). A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be convex if $\mathcal{K}$ includes every geodesic segment joining any two of its points. The set $\mathcal{K}$ is said to be bounded if $diam(\mathcal{K}) := \sup\{d(x_1, x_2) : x_1, x_2 \in \mathcal{K}\} < \infty$.

The model spaces $\mathcal{M}_k^2$ are defined as follows.

Definition 2.1. Given a real number κ, we denote by $\mathcal{M}_\kappa^2$ the following metric spaces:

(i) if $\kappa = 0$ then $\mathcal{M}_\kappa^2$ is Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$ then $\mathcal{M}_\kappa^2$ is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $\frac{1}{\sqrt{\kappa}}$;

(iii) if $\kappa < 0$ then $\mathcal{M}_\kappa^2$ is obtained from hyperbolic space $\mathbb{H}^n$ by by multiplying the distance function by $\frac{1}{\sqrt{-\kappa}}$.

The metric space $(\mathcal{X}, d)$ is called a CAT(κ) space if it is D_κ-geodesic and any geodesic triangle in $\mathcal{X}$ of perimeter less than $2D_\kappa$ satisfies the CAT(κ) inequality.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(\mathcal{X}, d)$ consists of three points in $\mathcal{X}$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(\mathcal{X}, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathcal{M}_\kappa^2$ such that $d(x_1, x_2) = d_{\mathcal{M}_\kappa^2}(\overline{x_1}, \overline{x_2})$, $d(x_2, x_3) = d_{\mathcal{M}_\kappa^2}(\overline{x_2}, \overline{x_3})$ and $d(x_3, x_1) = d_{\mathcal{M}_\kappa^2}(\overline{x_3}, \overline{x_1})$. If $k \le 0$, then such a comparison triangle always exists in $\mathcal{M}_\kappa^2$. If $\kappa > 0$, then such a triangle exists whenever $d(x_1, x_2)+d(x_2, x_3)+d(x_3, x_1) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{\mathcal{M}_\kappa^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in $\mathcal{X}$ is said to satisfy the CAT(κ) inequality if for any $p, q \in \triangle(x_1, x_2, x_3)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x_1}, \bar{x_2}, \bar{x_3})$, one has $d(p, q) = d_{\mathcal{M}_\kappa^2}(\overline{p}, \overline{q})$.

Definition 2.2. If $\kappa \le 0$, then $\mathcal{X}$ is called a CAT(κ) space if and only if $\mathcal{X}$ is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.

If $\kappa > 0$, then $\mathcal{X}$ is called a CAT(κ) space if and only if $\mathcal{X}$ is D_κ-geodesic and any geodesic triangle $\triangle(x_1, x_2, x_3)$ in $\mathcal{X}$ with $d(x_1, x_2)+d(x_2, x_3)+d(x_3, x_1) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space $(\mathcal{X}, d)$ if $x, y, z \in \mathcal{X}$, then the CAT(0) inequality implies

$$d^2\Big(x, \frac{y \oplus z}{2}\Big) \quad \le \quad \frac{1}{2}\, d^2(x, y) + \frac{1}{2}\, d^2(x, z) - \frac{1}{4}\, d^2(y, z). \quad \text{(CN)}$$

The above introduced (CN) is inequality of Bruhat and Tits [6]. This inequality is extended by Dhompongsa and Panyanak in [10] as

$$d^2(z, \alpha x \oplus (1-\alpha)y) \le \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y) \quad (\text{CN}^*)$$

for all $\alpha \in [0,1]$ and $x, y, z \in \mathcal{X}$. In fact, if $\mathcal{X}$ is a geodesic space, then the following statements are equivalent:

(i) $\mathcal{X}$ is a CAT(0);

(ii) $\mathcal{X}$ satisfies (CN) inequality;

(iii) $\mathcal{X}$ satisfies (CN*) inequality.

Let $R \in (0,2]$. Recall that a geodesic space $(\mathcal{X}, d)$ is said to be *R-convex* for R (see [27]) if for any three points $x, y, z \in \mathcal{X}$, we have

$$d^2(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d^2(z,x) + (1-\alpha)d^2(z,y) - \frac{R}{2}\alpha(1-\alpha)d^2(x,y). \tag{1}$$

It follows from (CN^*) that a geodesic space $(\mathcal{X}, d)$ is a CAT(0) space if and only if $(\mathcal{X}, d)$ is R-convex for R=2.

$\mathbb{R}$-trees are a particular class of CAT(κ) spaces for any real κ which will be named at certain points of our exposition (see [5], pg. 167 for more details).

Definition 2.3. An $\mathbb{R}$-tree is a metric space $\mathcal{X}$ such that:

(i) it is uniquely geodesic metric space,

(ii) if x, y and $z \in \mathcal{X}$ are such that $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

Therefore the family of all closed convex subset of a CAT(κ) space has uniform normal structure in the usual metric (or Banach space) sense.

We now recall the following elementary facts about CAT(κ) spaces. Most of them are proved in the framework of CAT(1) spaces. For completeness, we state the results in CAT(κ) space with $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space $(\mathcal{X}, d)$. For $x \in \mathcal{X}$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{X}\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in \mathcal{X} : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

It is known from Proposition 4.1 of [11] that in a CAT(κ) space with $diam(\mathcal{X}) = \frac{\pi}{2\sqrt{\kappa}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of Δ-convergence and collect some of its basic properties.

Definition 2.4 ([18, 22])**.** A sequence $\{x_n\}$ in $\mathcal{X}$ is said to Δ-converge to $x \in \mathcal{X}$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Definition 2.5. A mapping $\mathcal{T}\colon \mathcal{X} \to CC(\mathcal{X})$ is called hemi-compact if for any sequence $\{x_n\}$ in $\mathcal{X}$ such that $d(x_n, \mathcal{T}x_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in \mathcal{X}$.

Definition 2.6. Let φ be a nondecreasing self-map on $[0,\infty)$ with $\varphi(0) = 0$ and $\varphi(r) > 0$ for all $r \in (0,\infty)$ and let $d(x, \mathcal{F}) = \inf\{d(x,y) : y \in \mathcal{F}\}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued maps with $\mathcal{F} = F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3) \neq \emptyset$. Then the three maps are said to satisfy condition (GI) if

$$d(x, \mathcal{T}_1 x) \geq \varphi(d(x,\mathcal{F})) \text{ or } d(x, \mathcal{T}_2 x) \geq \varphi(d(x,\mathcal{F})) \text{ or } d(x, \mathcal{T}_3 x) \geq \varphi(d(x,\mathcal{F}))$$

Recall that a subset $\mathcal{K}$ in a metric space $\mathcal{X}$ is said to be Δ-compact [22] if every sequence in $\mathcal{K}$ has a Δ-convergent subsequence. A mapping $\mathcal{T}$ from a metric space $\mathcal{X}$ to a metric space $\mathcal{Y}$ is said to be completely continuous if $\mathcal{T}(\mathcal{K})$ is a compact subset of $\mathcal{Y}$ whenever $\mathcal{K}$ is a Δ-compact subset of $\mathcal{X}$.

A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be convex if $\mathcal{K}$ includes every geodesic segment joining any two of its points. A subset $\mathcal{K}$ is called proximal if for each $x \in \mathcal{X}$, there exists an element $\kappa \in \mathcal{K}$ such that $d(x, \kappa) = \inf\{\|x - y\| : y \in \mathcal{K}\} = d(x, \mathcal{K})$. We know that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are proximal.

We shall denote $CC(\mathcal{K})$, $C(\mathcal{K})$ and $P(\mathcal{K})$ by the families of all nonempty closed and convex subsets, nonempty compact subsets and nonempty proximal subsets of $\mathcal{K}$, respectively. Let H denote the Hausdorff metric induced by the metric d of $\mathcal{X}$, that is,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for every $A, B \in CC(\mathcal{X})$, where $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

A multivalued mapping $\mathcal{T}\colon \mathcal{K} \to CC(\mathcal{K})$ is said to be a *contraction* if there exists a constant $t \in [0, 1)$ such that for any $x, y \in \mathcal{K}$,

$$H(\mathcal{T}x, \mathcal{T}y) \le t\, d(x, y),$$

and $\mathcal{T}$ is said to be *nonexpansive* if

$$H(\mathcal{T}x, \mathcal{T}y) \le d(x, y),$$

for all $x, y \in \mathcal{K}$. A point $x \in \mathcal{K}$ is called a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}x$. Denote the set of all fixed points of $\mathcal{T}$ by $F(\mathcal{T})$.

Throughout this paper, we suppose that $\kappa > 0$ and $(\mathcal{X}, d)$ is a CAT(κ) space with the property:

$$\operatorname{diam}(\mathcal{X}) = \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}, \quad \text{for some } \varepsilon \in (0, \pi/2). \qquad \text{(p)}$$

In the sequel we need the following lemmas.

Lemma 2.1 ([5], p.176)**.** *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p)*. Then*

$$d((1 - \alpha)x \oplus \alpha\, y, z) \le (1 - \alpha)d(x, z) + \alpha\, d(y, z)$$

for all $x, y, z \in \mathcal{X}$ and $\alpha \in [0, 1]$.

Lemma 2.2. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p)*. Then the following statements hold:*

(i) ([11]*, Corollary 4.4) Every sequence in $\mathcal{X}$ has a Δ-convergent subsequence.*

(ii) ([11]*, Proposition 4.5) If $\{x_n\} \subseteq \mathcal{X}$ and Δ-$\lim_{n\to\infty} x_n = x$, then $x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \dots\}$, where $\overline{conv}(A) = \bigcap\{B : B \supseteq A$ and B is closed and convex$\}$.*

By the uniqueness of asymptotic center, we can obtain the following lemma in ([10]).

Lemma 2.3 ([10], Lemma 2.8)**.** *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p)*. If $\{x_n\}$ is a bounded sequence in $\mathcal{X}$ with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4 ([30], Theorem 4.5)**.** *Let $\mathcal{K}$ be a nonempty closed convex subset of a complete CAT(κ) space $\mathcal{X}$ with $\kappa > 0$ and rad($\mathcal{K}$) $< \frac{\pi}{2\sqrt{\kappa}}$ and let $\mathcal{T}\colon \mathcal{K} \to CC(\mathcal{K})$ satisfying condition (I). If $\{x_n\}$ be the sequence in $\mathcal{K}$ defined by*

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n u_n, \ n \ge 1, \end{cases}$$

where $u_n \in \mathcal{T}x_n$ and $\{\alpha_n\} \in [0, 1]$ such that $\lim_{n\to\infty} d(x_n, \mathcal{T}x_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = v$, then $v \in \mathcal{T}v$.

Lemma 2.5 ([40]). *Let $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$p_{n+1} \leq (1+q_n)p_n + r_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

Proposition 2.1 ([25], Proposition 3.12). *Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space $\mathcal{X}$, and let $\mathcal{K}$ be a closed convex subset of $\mathcal{X}$ which contains $\{x_n\}$. Then*
(i) Δ-$\lim_{n\to\infty} x_n = x$ implies that $x_n \rightharpoonup x$,
(ii) the converse is true if $\{x_n\}$ is regular.

For single valued mappings, we list some relevant iterative processes.

In 1953, Mann [23] introduced the following iteration process for single valued mapping.

Algorithm 1. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

In 1974, Ishikawa [14] introduced a new iteration process for single valued nonexpansive mappings in Banach space.

Algorithm 2. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. This iteration scheme reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal *et al.* [3] introduced and studied the following iteration process for single valued mappings.

Algorithm 3. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)\mathcal{T}x_n + \alpha_n \mathcal{T}y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

In 2000, Noor [26] introduced a three-step iteration process for single valued nonexpansive mappings in Banach space.

Algorithm 4. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

In 2011, Phuengrattana and Suantai [29] defined the following three-step iteration process for single valued mappings.

Algorithm 5. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n \mathcal{T} y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T} z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T} x_n, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

Recently, Abbas and Nazir [2] introduced and studied the following iteration scheme: let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{X}$ and $\mathcal{T}$ be a nonlinear mapping of $\mathcal{K}$ into itself.

Algorithm 6. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)\mathcal{T} y_n + \alpha_n \mathcal{T} z_n, \\ y_n = (1-\beta_n)\mathcal{T} x_n + \beta_n \mathcal{T} z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T} x_n, \ n \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. They showed that this process converges faster than both Picard and Agarwal *et al.* for the subclass of contractive mappings ([3]) and in support gave analytic proof by a numerical example (for more details, see ([2])). For a very recent development in this direction, please see [39], [42], [43].

Now, we modify the above scheme (2) for three mappings as follows.

Algorithm 7. The sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)\mathcal{T}_2 y_n + \alpha_n \mathcal{T}_3 z_n, \\ y_n = (1-\beta_n)\mathcal{T}_1 x_n + \beta_n \mathcal{T}_3 z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}_1 x_n, \ n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

Our purpose in this paper is to extend the iteration (3) to the case of three multivalued nonexpansive mappings on closed and convex subset in the setting of CAT(κ) spaces and establish some strong and a Δ-convergence theorems.

We modify iterative scheme (3) as follows.

Definition 2.7. Let $\mathcal{X}$ be a CAT(κ) space, $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued nonexpansive mappings. The sequence $\{x_n\}$ of the modified AN (Abbas and Nazir)-iteration is defined by:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)v_n \oplus \alpha_n w_n, \\ y_n = (1-\beta_n)u_n \oplus \beta_n w_n, \\ z_n = (1-\gamma_n)x_n \oplus \gamma_n u_n, \ n \geq 1, \end{cases} \tag{4}$$

where $u_n \in \mathcal{T}_1 x_n$, $v_n \in \mathcal{T}_2 y_n$, $w_n \in \mathcal{T}_3 z_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$.

3. Main Results

First of all we prove the following lemmas which will play a key role in our investigation. Assume that $\mathcal{F} = F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$ denotes the set of all common fixed points of the multivalued mappings $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Lemma 3.1. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p). *Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by* (4). *If $\mathcal{F} \neq \emptyset$ and $\mathcal{T}_1(p) = \mathcal{T}_2(p) = \mathcal{T}_3(p) = \{p\}$ for any $p \in \mathcal{F}$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$.*

Proof. Assume that $\mathcal{F} \neq \emptyset$. Let $p \in \mathcal{F}$. Then from (4), we have

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n u_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(u_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n H(\mathcal{T}_1(x_n), \mathcal{T}_1(p)) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{5}$$

Again, using (4) and (5), we get

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)u_n \oplus \beta_n w_n, p) \\ &\leq (1-\beta_n)d(u_n, p) + \beta_n d(w_n, p) \\ &\leq (1-\beta_n)H(\mathcal{T}_1(x_n), \mathcal{T}_1(p)) + \beta_n H(\mathcal{T}_3(z_n), \mathcal{T}_3(p)) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{6}$$

Finally, using (4), (5) and (6), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)v_n \oplus \alpha_n w_n, p) \\ &\leq (1-\alpha_n)d(v_n, p) + \alpha_n d(w_n, p) \\ &\leq (1-\alpha_n)H(\mathcal{T}_2(y_n), \mathcal{T}_2(p)) + \alpha_n H(\mathcal{T}_3(z_n), \mathcal{T}_3(p)) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(z_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus by Lemma 2.5, we get that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$ and hence $\{x_n\}$ is bounded. This completes the proof. □

Lemma 3.2. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p). *Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by* (4) *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. If $\mathcal{F} \neq \emptyset$ and $\mathcal{T}_1(p) = \mathcal{T}_2(p) = \mathcal{T}_3(p) = \{p\}$ for any $p \in \mathcal{F}$, then $\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = 0$, $\lim_{n\to\infty} d(x_n, \mathcal{T}_2 y_n) = 0$ and $\lim_{n\to\infty} d(x_n, \mathcal{T}_3 z_n) = 0$.*

Proof. Let $p \in \mathcal{F} \neq \emptyset$. From Lemma 3.1, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$. Since $\{x_n\}$ is bounded, there exists $R_1 > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\} \subset B_{R_1}(p)$ for all

$n \geq 1$ with $R_1 < D_{\kappa/2}$. In view of (1), we have

$$\begin{aligned} d^2(z_n, p) &= d^2((1-\gamma_n)x_n \oplus \gamma_n u_n, p) \\ &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(u_n, p) \\ &\quad -\frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n, u_n) \\ &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n H^2(\mathcal{T}_1(x_n), \mathcal{T}_1(p)) \\ &\quad -\frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n, u_n) \\ &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(x_n, p) \\ &\quad -\frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n, u_n) \\ &= d^2(x_n, p) - \frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n, u_n). \end{aligned} \tag{7}$$

This implies that

$$d^2(z_n, p) \leq d^2(x_n, p). \tag{8}$$

Again using (1) and (8), we obtain

$$\begin{aligned} d^2(y_n, p) &= d^2((1-\beta_n)u_n \oplus \beta_n w_n, p) \\ &\leq (1-\beta_n)d^2(u_n, p) + \beta_n d^2(w_n, p) \\ &\quad -\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n, w_n) \\ &\leq (1-\beta_n)H^2(\mathcal{T}_1(x_n), \mathcal{T}_1(p)) + \beta_n H^2(\mathcal{T}_3(z_n), \mathcal{T}_3(p)) \\ &\quad -\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n, w_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) \\ &\quad -\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n, w_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) \\ &\quad -\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n, w_n) \\ &= d^2(x_n, p) - \frac{R}{2}\beta_n(1-\beta_n)d^2(u_n, w_n). \end{aligned} \tag{9}$$

This implies that

$$d^2(y_n, p) \leq d^2(x_n, p). \tag{10}$$

Now using (1), (8) and (10), we get

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\alpha_n)v_n \oplus \alpha_n w_n, p) \\ &\leq (1-\alpha_n)d^2(v_n,p)+\alpha_n d^2(w_n,p) \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n) \\ &\leq (1-\alpha_n)H^2(\mathcal{T}_2(y_n),\mathcal{T}_2(p))+\alpha_n H^2(\mathcal{T}_3(z_n),\mathcal{T}_3(p)) \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n) \\ &\leq (1-\alpha_n)d^2(y_n,p)+\alpha_n d^2(z_n,p) \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n) \\ &\leq (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(x_n,p) \\ &\quad -\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n) \\ &= d^2(x_n,p)-\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n). \end{aligned}$$

This implies that

$$\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n) \quad \leq \quad d^2(x_n,p)-d^2(x_{n+1},p).$$

Since $d(x_n,p)<R_1$, we have

$$\frac{R}{2}\alpha_n(1-\alpha_n)d^2(v_n,w_n)<\infty.$$

Hence by the fact that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we have

$$\lim_{n\to\infty} d(v_n,w_n) \quad = \quad 0. \tag{11}$$

Now, equation (9) yields

$$\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n,w_n) \quad \leq \quad d^2(x_n,p)-d^2(y_n,p).$$

Since $d(x_n,p)<R_1$ and $d(y_n,p)<R_1$, we have

$$\frac{R}{2}\beta_n(1-\beta_n)d^2(u_n,w_n)<\infty.$$

Hence by the fact that $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, we have

$$\lim_{n\to\infty} d(u_n,w_n) \quad = \quad 0. \tag{12}$$

Again, equation (7) yields

$$\frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n,u_n) \quad \leq \quad d^2(x_n,p)-d^2(z_n,p).$$

Since $d(x_n,p)<R_1$ and $d(z_n,p)<R_1$, we have

$$\frac{R}{2}\gamma_n(1-\gamma_n)d^2(x_n,u_n)<\infty.$$

Hence by the fact that $\liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0$, we have

$$\lim_{n\to\infty} d(x_n,u_n) \quad = \quad 0. \tag{13}$$

Now note that

$$d(x_n,w_n) \quad \leq \quad d(x_n,u_n)+d(u_n,w_n).$$

Using equation (12) and (13), we get

$$\lim_{n\to\infty} d(x_n, w_n) \quad = \quad 0. \tag{14}$$

Also note that

$$d(x_n, v_n) \quad \leq \quad d(x_n, w_n) + d(w_n, v_n).$$

Using equation (11) and (14), we get

$$\lim_{n\to\infty} d(x_n, v_n) \quad = \quad 0. \tag{15}$$

Since

$$d(x_n, \mathcal{T}_1 x_n) \quad \leq \quad d(x_n, u_n).$$

Using equation (13), we obtain

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = 0.$$

Similarly

$$d(x_n, \mathcal{T}_2 y_n) \quad \leq \quad d(x_n, v_n).$$

Using equation (15), we get

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_2 y_n) = 0,$$

and

$$d(x_n, \mathcal{T}_3 z_n) \quad \leq \quad d(x_n, w_n).$$

Using equation (14), we obtain

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_3 z_n) = 0.$$

This completes the proof. □

Theorem 3.1. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete $CAT(\kappa)$ space with property* (p)*. Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued nonexpansive mappings satisfying condition (GI). Let $\{x_n\}$ be the sequence defined by* (4) *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in* $(0,1)$ *such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. If $\mathcal{F} \neq \emptyset$ and $\mathcal{T}_1(p) = \mathcal{T}_2(p) = \mathcal{T}_3(p) = \{p\}$ for any $p \in \mathcal{F}$, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof. Since $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ satisfies condition (GI), we have $\lim_{n\to\infty} \varphi(d(x_n, \mathcal{F})) = 0$. Thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset \mathcal{F}$ such that

$$d(x_{n_j}, p_j) < \frac{1}{2^j},$$

for all $j > 0$. By Lemma 3.1 we obtain that

$$d(x_{n_j+1}, p_j) \leq d(x_{n_j}, p_j) < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in $\mathcal{K}$. Observe that

$$\begin{aligned} d(p_{j+1}, p_j) &\leq d(p_{j+1}, x_{n_j+1}) + d(x_{n_j+1}, p_j) \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}}. \end{aligned}$$

This shows that $\{p_j\}$ is a Cauchy sequence in $\mathcal{K}$ and hence converges to $p \in \mathcal{K}$. Since

$$\begin{aligned} d(p_j, \mathcal{T}_1(p)) &\leq H(\mathcal{T}_1(p), \mathcal{T}_1(p_j)) \\ &\leq d(p, p_j), \end{aligned}$$

and $p_j \to p$ as $j \to \infty$, it follows that $d(p, \mathcal{T}_1(p)) = 0$, which implies that $p \in \mathcal{T}_1(p)$.

Similarly

$$\begin{aligned} d(p_j, \mathcal{T}_2(p)) &\leq H(\mathcal{T}_2(p), \mathcal{T}_2(p_j)) \\ &\leq d(p, p_j), \end{aligned}$$

and $p_j \to p$ as $j \to \infty$, it follows that $d(p, \mathcal{T}_2(p)) = 0$, which implies that $p \in \mathcal{T}_2(p)$.

Similarly

$$\begin{aligned} d(p_j, \mathcal{T}_3(p)) &\leq H(\mathcal{T}_3(p), \mathcal{T}_3(p_j)) \\ &\leq d(p, p_j), \end{aligned}$$

and $p_j \to p$ as $j \to \infty$, it follows that $d(p, \mathcal{T}_3(p)) = 0$, which implies that $p \in \mathcal{T}_3(p)$. Consequently, $p \in \mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, thus we conclude that $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. This completes the proof. □

Theorem 3.2. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p)*. Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three hemicompact and continuous multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by* (4) *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. If $\mathcal{F} \neq \emptyset$ and $\mathcal{T}_1(p) = \mathcal{T}_2(p) = \mathcal{T}_3(p) = \{p\}$ for any $p \in \mathcal{F}$, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof. By Lemma 3.2, we know that $d(x_n, \mathcal{T}_1 x_n) = d(x_n, \mathcal{T}_2 y_n) = d(x_n, \mathcal{T}_3 z_n) = 0$ and $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are hemicompact, so there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$ for some $p \in \mathcal{K}$. Since $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are continuous, we have

$$\begin{aligned} d(p, \mathcal{T}_1 p) &\leq d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_1 x_{n_k}) + H(\mathcal{T}_1 x_{n_k}, \mathcal{T}_1 p) \\ &\leq 2d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_1 x_{n_k}) \\ &\quad \to 0 \text{ as } k \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(p, \mathcal{T}_2 p) &\leq d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_2 y_{n_k}) + H(\mathcal{T}_2 y_{n_k}, \mathcal{T}_2 p) \\ &\leq 2d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_2 y_{n_k}) \\ &\quad \to 0 \text{ as } k \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(p, \mathcal{T}_3 p) &\leq d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_3 z_{n_k}) + H(\mathcal{T}_3 z_{n_k}, \mathcal{T}_3 p) \\ &\leq 2d(p, x_{n_k}) + d(x_{n_k}, \mathcal{T}_3 z_{n_k}) \\ &\quad \to 0 \text{ as } k \to \infty. \end{aligned}$$

This implies that $p \in \mathcal{T}_1 p$, $p \in \mathcal{T}_2 p$ and $p \in \mathcal{T}_3 p$. Since by Lemma 3.1 $\lim_{n\to\infty} d(x_n, p)$ exists, thus we conclude that $p \in \mathcal{F}$ is the strong limit of the sequence $\{x_n\}$ itself. This completes the proof. □

Now, we are in a position to prove the Δ-convergence theorems.

Theorem 3.3. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p). *Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ be three multivalued nonexpansive mappings satisfying condition (GI). Let $\{x_n\}$ be the sequence defined by* (4) *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. If $\mathcal{F} \neq \emptyset$ and $\mathcal{T}_1(p) = \mathcal{T}_2(p) = \mathcal{T}_3(p) = \{p\}$ for any $p \in \mathcal{F}$, then $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof. Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq \mathcal{F}$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in \mathcal{K}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_2 y_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_3 z_n) = 0$, so by Lemma 2.4, we have $v \in \mathcal{F}$ and the $\lim_{n\to\infty} d(x_n, v)$ exists by Lemma 3.1. Hence $u = v \in \mathcal{F}$ by Lemma 2.3, i.e., $\omega_w(x_n) \subseteq \mathcal{F}$.

To see that $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$, it is enough to prove that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq \mathcal{F}$ and by Lemma 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.3, we have $x = w \in \mathcal{F}$, therefore $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$ and the proof is complete. □

If we put $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}$ in Theorem 3.3, then we have the following result.

Corollary 3.1. *Let $\kappa > 0$ and $(\mathcal{X}, d)$ be a complete CAT(κ) space with property* (p). *Let $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$ and let $\mathcal{T} \colon \mathcal{K} \to CC(\mathcal{K})$ be a multivalued nonexpansive mapping satisfying condition (I). Let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n) v_n \oplus \alpha_n w_n, \\ y_n = (1-\beta_n) u_n \oplus \beta_n w_n, \\ z_n = (1-\gamma_n) x_n \oplus \gamma_n u_n, \ n \geq 1, \end{cases}$$

where $u_n \in \mathcal{T} x_n$, $v_n \in \mathcal{T} y_n$, $w_n \in \mathcal{T} z_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. If $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and $\mathcal{T}(p) = \{p\}$ for any $p \in \mathcal{F}(\mathcal{T})$, then $\{x_n\}$ Δ-converges to a fixed point of $\mathcal{T}$.

Example 3.1. Let us consider that $\mathcal{K} = [0,1]$ is equipped with the Euclidean metric. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to CC(\mathcal{K})$ (the family of closed and convex subset of $\mathcal{K}$) be defined by $\mathcal{T}_1(x) = [0, \frac{x}{2}]$, $\mathcal{T}_2(x) = [0, \frac{x}{4}]$ and $\mathcal{T}_3(x) = [0, \frac{x}{5}]$. It is easy to see that for any $x, y \in \mathcal{K}$, we have the inequality

$$\begin{aligned} H(\mathcal{T}_1(x), \mathcal{T}_1(y)) &= \max\Big\{\Big|\frac{x}{2} - \frac{y}{2}\Big|, 0\Big\} = \Big|\frac{x}{2} - \frac{y}{2}\Big| = \Big|\frac{x-y}{2}\Big| \\ &\leq |x-y|. \end{aligned}$$

By similar calculation, we obtain

$$\begin{aligned} H(\mathcal{T}_2(x), \mathcal{T}_2(y)) &= \max\Big\{\Big|\frac{x}{4} - \frac{y}{4}\Big|, 0\Big\} = \Big|\frac{x}{4} - \frac{y}{4}\Big| = \Big|\frac{x-y}{4}\Big| \\ &\leq |x-y|, \end{aligned}$$

and

$$\begin{aligned} H(\mathcal{T}_3(x), \mathcal{T}_3(y)) &= \max\Big\{\Big|\frac{x}{5} - \frac{y}{5}\Big|, 0\Big\} = \Big|\frac{x}{5} - \frac{y}{5}\Big| = \Big|\frac{x-y}{5}\Big| \\ &\leq |x-y|, \end{aligned}$$

showing that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are multivalued nonexpansive mappings. On other hand, it is clear that $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3) = \{0\}$. Hence, mappings $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ have a unique common fixed point.

Example 3.2 ([35]). Suppose that $\mathcal{X} = \mathcal{K} = [0, 1]$ is endowed with the usual metric, and $\{x_n\} = \{\frac{1}{n}\}$, $\{z_{n_k}\} = \{\frac{1}{k^n}\}$ for all $n, k \geq 1$ are sequences in $\mathcal{K}$. Then $A(\{x_n\}) = \{0\}$ and $A(\{z_{n_k}\}) = \{0\}$. This shows that the sequence $\{x_n\}$ is Δ-convergent to 0, that is, $\Delta\text{-}\lim_{n\to\infty} x_n = 0$. The sequence $\{x_n\}$ also converges strongly to 0, that is, $|x_n - 0| \to 0$ as $n \to \infty$. Moreover, it is weakly convergent to 0, that is, $x_n \rightharpoonup 0$ as $n \to \infty$, by Proposition 2.1. This analysis suggest us the following implications:

$$\text{strong convergence} \;\Rightarrow\; \Delta\text{-convergence} \;\Rightarrow\; \text{weak convergence.}$$

Note that in general the converse is not true.

The following example analyzes the case when a sequence $\{x_n\}$ is weakly convergent, and it is not Δ-convergent.

Example 3.3 ([25]). On $\mathcal{X} = \mathbb{R}$, with the usual metric, consider $\mathcal{K} = [-1, 1]$, and the sequences $\{x_n\} = \{1, -1, 1, -1, \dots\}$, $\{u_n\} = \{-1, -1, -1, \dots\}$ and $\{v_n\} = \{1, 1, 1, \dots\}$. Then $A(\{x_n\}) = A_{\mathcal{K}}(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$ and $A(\{v_n\}) = \{1\}$. This shows that the sequence $\{x_n\}$ is weakly convergent to 0 but it does not have a Δ-limit.

4. **Conclusion**

In this paper, we first generalize Abbas and Nazir [2] three-step iteration scheme for three mappings and then translate it to three multivalued nonexpansive mappings and establish a Δ-convergence and some strong convergence theorems in the setting of CAT(κ) spaces. The results in this paper extend and generalize several results from the current existing literature, and are thought as natural continuation of those in [2].

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