SPECTRAL DECOMPOSITION OF THE ELASTICITY MATRIX

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În Teoria Elasticității Anizotrope, matricea coeficienților elastici este, până la urmă, o transformare liniară, simetrică, pe spații vectoriale 6-dimensionale. Deci, orice asemenea matrice admite o descompunere spectrală. Structura unei descompuneri spectrale este determinată de mulțimile spațiilor invariante, specifice tipului de simetrie elastică a materialului studiat. Vectorii proprii, cel puțin parțial, nu depind de valorile constantelor elastice, în schimb, valorile proprii depind de aceste constante. Sunt prezentate matricile coeficienților elastici pentru categoriile semnificative de simetrie cristalină. Sunt calculate matricile coeficienților elastici pentru cazuri concrete de materiale reprezentative pentru fiecare categorie de simetrie cristalină.

In the anisotropic elasticity research domain, the elasticity matrix is a symmetric linear transformation on the six-dimensional vector spaces. So, the elasticity matrix can always have its own spectral decomposition. A spectral decomposition is determined by the sets of invariant subspaces that are consistent with the specific material symmetry. Eigenvectors, partially, do not depend on the values of the elastic constants, but the eigenvalues depend on them. For almost every symmetry group of crystallography, the structure of corresponding elasticity matrix spectral decomposition is presented. Also, for some representative materials belonging to each and every group, numerical results are presented.

Keywords: spectral decomposition, elasticity matrix, symmetry group, eigenvalues, eigenvectors

1. Introduction

A natural representation of the elasticity matrix in its spectral form is always possible, which further allows a simple geometrical interpretation of the relationship between stress and strain to take place, regardless the degree of anisotropy.

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The form of the elasticity matrix contains restrictions resulting from the symmetry theory of classical crystallography. These restrictions are reflected in the invariant structures of the spectral decompositions. The spectral forms are determined by the symmetry groups and do not depend on the values of the elastic constants.

Therefore, it becomes possible to compare different materials within the same symmetry group. In some cases, similarities between materials belonging to different symmetry groups are revealed. We have noticed striking similarities between the metals with cubic, hexagonal and tetragonal symmetry.

The eigenvalues and eigenvectors of the elasticity tensor were discussed, for the first time, by Kelvin (1856), and his results are summarized in Encyclopedia Britannica (1878). More recently, Cowin and Mehrabadi [1] and Mehrabadi and Cowin [2] have determined the eigenvalues and eigenvectors for anisotropic elasticity. Ting [3] has discussed the eigenvalues problem in connection with his study on the invariants of the elasticity tensor.

In these previous works, the elasticity tensor has been induced from a fourth-order symmetric linear transformation on the space of all 3×3 second-order tensors to an 6×6 second-order tensor.

Based on modern mathematical and elasticity theories, this paper, focusing some authors' results [4] on specific cases, presents a relatively simple method for the spectral decomposition of the elasticity matrix. There are numerical results presented for a selection of representative materials with measured (known) elastic constants, belonging to each and every symmetry group.

The first spectral decomposition of the elasticity tensor was made by Rychlewski and Zhang [5], using tensor products. Then, Sutcliffe [6] developed this method and he used it for different symmetry groups.

Due to the fact that composite materials are, generally, kind of non-homogenous and anisotropic, calculating their elastic characteristics is always a tremendous difficult task. In this respect, the elasticity matrix decomposition for each constituent proves to be extremely useful [7].

But, even in case of some categories of homogenous materials, the spectral decomposition of the elasticity matrix has, still, proven to be kind of essential [8-10].

Starting from solid mathematical fundamentals, this paper presents a new, original and comprehensive matrix-based method for the elasticity matrix spectral decomposition of such a homogenous constituent, whatsoever and no matter the kind of its anisotropy might be. The matrix-based aspect of this method could be rather useful, especially in Dynamics (vibrations) of certain composite structures (plates, bars) where the presentation of the mathematical model of those vibrations under a matrix-based form makes the model more comprehensive and engineering like. More else, when possible, this matrix-based aspect of the mathematical model could make the solving process of the model somehow easier by using matrix functions.

2. Generalities

Let be K an n dimensional Euclidian space, and B a general base of $e_1,...,e_n$ vectors.

We consider a symmetric operator A, having the space K as the domain and co-domain of definition, as well.

Depending on base B, we put in correspondence to the operator A, the symmetric matrix [M], with real elements $a_j^{(i)}, i,j=\overline{1,n}$, where

$$Ae_i = \sum_{j=1}^{n} a_j^{(i)} e_j, i = \overline{1,n}. \tag{1}$$

The space of linear operators A: $K \to K$ is isomorphic with the space of quadratic matrix with real elements, of n order.

Let λ be a real number. If there is $x \in K$, $x \neq 0_K$, such that

$$Ax = \lambda x, \tag{2}$$

then λ is called the eigenvalue of operator A and the vector (or vectors) x is called the eigenvector of operator A, corresponding to eigenvalue λ.

We note by $\sigma_p(A)$, the following set:

$$\sigma_p(A) = \{\lambda \in C | (\exists) x \in K, x \neq 0, Ax = \lambda x\}, \qquad p = \overline{1,r}, \tag{3}$$

and we'll call this set the point spectrum of operator A, C being the set of complex numbers. Particularly here, λ is a real number: $\lambda \in R$.

Since the operator A is kind of symmetric ($a_j^{(i)} = a_i^{(j)}, i,j=\overline{1,n}$), then A has only real eigenvalues and its corresponding matrix can be a diagonal one.

Let $\lambda_1,...,\lambda_r$ be the real and distinct eigenvalues of the operator A and their corresponding multiplicities: $m_1,...,m_r$, such that: $\sum_{p=1}^{r} m_p = n$. These eigenvalues are representing the solutions of the equation:

$$\det([M] - \lambda[I_n]) = 0, \tag{4}$$

where I_n is the unit matrix of n order.

The eigenvectors corresponding to eigenvalue $\lambda_p, p=\overline{1,r}$ will be the solutions of the following matrix equation:

$$\left([M]-\lambda_p[I_n]\right)\left[X_{\lambda_p}\right]=0, \tag{5}$$

where $\left[X_{\lambda_p}\right]=\left(x_1...x_n\right)^t$.

We shall note by $K^{\left(\lambda_p\right)}$ the set of all eigenvectors corresponding to eigenvalue λ_p, $p=\overline{1,r}$ (and null vector).

$K^{\left(\lambda_p\right)}$ is the subspace of *K*, called the eigenspace corresponding to the specific eigenvalue λ_p, $p=\overline{1,r}$ and, moreover,

$$\dim K^{\left(\lambda_p\right)}=m_p. \tag{6}$$

Since $\sum_{p=1}^{r} m_p=n$, the set $B_1=\left\{X_{\lambda_1}^1,...,X_{\lambda_1}^{m_1},...,X_{\lambda_r}^1,...,X_{\lambda_r}^{m_r}\right\}$, containing all the eigenvectors (corresponding to the eigenvalues $\lambda_1,...,\lambda_r$), generates a orthogonal base of the space *K*.

We shall normalize this base and we'll further note this new normalized base by $B^*=\left\{X_1,...,X_n\right\}$.

Depending on this base, the matrix [*M*] of the operator *A* is the diagonal matrix:

$$\left[M_{B^*}\right]=\begin{pmatrix}\lambda_1 & 0 & 0 & ... & 0 & 0 & 0\\ 0 & ... & 0 & ... & 0 & 0 & 0\\ 0 & 0 & \lambda_1 & ... & 0 & 0 & 0\\ ... & ... & ... & ... & ... & ... & ...\\ 0 & 0 & 0 & ... & \lambda_r & 0 & 0\\ 0 & 0 & 0 & ... & 0 & ... & 0\\ 0 & 0 & 0 & ... & 0 & 0 & \lambda_r\end{pmatrix}. \tag{7}$$

The matrix $\left[M_{B^*}\right]$ can be represented as follows:

$$\left[M_{B^*}\right]=\lambda_1\left[E_1\right]+...+\lambda_r\left[E_r\right], \tag{8}$$

where $\left[E_p\right]$, $p=\overline{1,r}$ are matrices fulfilling the following conditions:

$$\begin{cases}\left[E_p\right]\cdot\left[E_p\right]=\left[E_p\right],\ p=\overline{1,r}\,,\\ \left[E_i\right]\cdot\left[E_j\right]=\left[0_n\right],\ i,j=\overline{1,r};\ i\neq j,\\ \left[E_1\right]+\ldots+\left[E_r\right]=\left[I_n\right],\end{cases} \tag{9}$$

$\left[0_n\right]$ is the null matrix of n order.

If we note by $\left[X\right]$ the matrix of n order having as columns the coefficients of each and every eigenvector (the matrix $\left[X\right]$ is the transit matrix from base B to base B^*), then each and every matrix $\left[E_1\right],\ldots,\left[E_r\right]$ of (8) can be obtained from the relation:

$$\left[E_p\right]=\sum_{k=1}^{m_p}\left[X\right]\cdot\left[I_{(k)}\right]\cdot\left[X\right]^t,\quad p=\overline{1,r}, \tag{10}$$

where $\left[I_{(k)}\right]$ is the matrix having the only element equal with 1 on the k position of its principal diagonal, all the others elements being null.

The relation (8) is called the spectral decomposition of matrix [M] with respect to base $B^*=\left\{X_1,\ldots,X_n\right\}$ containing all eigenvectors ($\left\|X_i\right\|=1,\quad i=\overline{1,n}$).

3. Decomposition of the elasticity matrix

In case of the linear-elastic materials, the dependence between the deformation matrix components and the stress matrix components is a linear one:

$$\sigma_{ij}=\sum_{k=1}^{3}\sum_{l=1}^{3}S_{ijkl}\,\varepsilon_{kl}\ . \tag{11}$$

This dependence can be written as follows:

$$\overline{\sigma}=\left[S\right]\overline{\varepsilon}\,, \tag{12}$$

where:

$$\overline{\varepsilon}=\left(\varepsilon_{11};\varepsilon_{22};\varepsilon_{33};\sqrt{2}\ \varepsilon_{23};\sqrt{2}\ \varepsilon_{13};\sqrt{2}\ \varepsilon_{12}\right)^t;$$

$$\overline{\sigma}=\left(\sigma_{11};\sigma_{22};\sigma_{33};\sqrt{2}\ \sigma_{23};\sqrt{2}\ \sigma_{13};\sqrt{2}\ \sigma_{12}\right)^t; \tag{13}$$

$$[S]=\begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1113} & \sqrt{2}S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2213} & \sqrt{2}S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & \sqrt{2}S_{3323} & \sqrt{2}S_{3313} & \sqrt{2}S_{3312} \\ \sqrt{2}S_{2311} & \sqrt{2}S_{2322} & \sqrt{2}S_{2333} & 2S_{2323} & 2S_{2313} & 2S_{2312} \\ \sqrt{2}S_{1311} & \sqrt{2}S_{1322} & \sqrt{2}S_{1333} & 2S_{1323} & 2S_{1313} & 2S_{1312} \\ \sqrt{2}S_{1211} & \sqrt{2}S_{1222} & \sqrt{2}S_{1233} & 2S_{1223} & 2S_{1213} & 2S_{1212} \end{bmatrix}; \tag{14}$$

and:

$$S_{ijkl}=S_{jikl}=S_{ijlk}=S_{klij}\,. \tag{15}$$

We present the elasticity matrix for usual symmetry cases.
Triclinic:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}. \tag{16}$$

Monoclinic:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}. \tag{17}$$

Orthorhombic:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{18}$$

Trigonal:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & -\sqrt{2}C_{15} & 0 \\ C_{12} & C_{11} & C_{13} & 0 & \sqrt{2}C_{15} & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 2C_{15} \\ -\sqrt{2}C_{15} & \sqrt{2}C_{15} & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 2C_{15} & 0 & C_{11}-C_{12} \end{bmatrix}. \tag{19}$$

Tetragonal:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{20}$$

Hexagonal:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{11}-C_{12} \end{bmatrix}. \tag{21}$$

Cubic:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}. \tag{22}$$

Homogenous and isotropic:

$$[S]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11}-C_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{11}-C_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{11}-C_{12} \end{bmatrix}. \tag{23}$$

The triclinic symmetry does not require any restriction in terms of spectral decomposition of the elasticity matrix. Each and every one-dimension subspace of the symmetric tensors space remains invariant with respect to this kind of symmetry. So, the spectral decomposition of the elasticity matrix in case of triclinic systems consists in the existence of six distinct one-dimension eigenspaces and six distinct eigenvalues. In this kind of respect we have to add that the determining the eigenvalues formally leads the whole issue to the difficult task of solving an algebraic equation of 6^{th} degree in the unknown λ. Due to obvious reasons, in the most fortunate cases, the degree of the mentioned equation can't be ever smaller than three. Solving symbolically equations like these is really difficult (the Cardano formulae at least) and it's kind of ineffective due to the complicated form of the solutions. Solving numerically these equations, for specific cases, is kind of easy and effective.

Concerning the monoclinic symmetry systems, affirmations of here-above kind remain available, except the fact that the issue of finding the eigenvalues always leads to solve an algebraic equation of 4^{th} degree in the unknown λ. We have to add that, in this case, among the six subspaces, only four of them are eigenspaces, literally.

It is, also, important to notice that the specific case of orthorhombic symmetry can always be regarded as a particular case of the monoclinic symmetry.

In the case of the trigonal symmetry, the eigenvalues are:

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left[C_{11}+C_{12}+C_{33}+\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],\\
\lambda_2 &= \frac{1}{2}\left[C_{11}+C_{12}+C_{33}-\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],\\
\lambda_3 = \lambda_6 &= \frac{1}{2}\left[C_{11}-C_{12}+C_{44}+\sqrt{\left(C_{11}-C_{12}-C_{44}\right)^2+16C_{15}^2}\right],\\
\lambda_4 = \lambda_5 &= \frac{1}{2}\left[C_{11}-C_{12}+C_{44}-\sqrt{\left(C_{11}-C_{12}-C_{44}\right)^2+16C_{15}^2}\right].
\end{aligned} \tag{24}$$

and the matrix of eigenvectors is:

$$[X]=\begin{bmatrix} \frac{1}{\sqrt{2}}\sin\alpha & -\frac{1}{\sqrt{2}}\cos\alpha & \frac{1}{\sqrt{2}}\cos\beta & 0 & \frac{1}{\sqrt{2}}\sin\beta & 0 \\ \frac{1}{\sqrt{2}}\sin\alpha & -\frac{1}{\sqrt{2}}\cos\alpha & -\frac{1}{\sqrt{2}}\cos\beta & 0 & -\frac{1}{\sqrt{2}}\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & 0 & \sin\beta \\ 0 & 0 & -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & 0 & \cos\beta \end{bmatrix}, \tag{25}$$

where:

$$\cos\alpha=\frac{|\lambda_1-C_{11}-C_{12}|}{\sqrt{2C_{13}^2+(\lambda_1-C_{11}-C_{12})^2}};$$
$$\sin\alpha=\frac{|\lambda_2-C_{11}-C_{12}|}{\sqrt{2C_{13}^2+(\lambda_2-C_{11}-C_{12})^2}};$$
$$\sin\beta=\frac{|C_{11}-C_{12}-\lambda_3|}{\sqrt{(C_{11}-C_{12}-\lambda_3)^2+4C_{15}^2}};$$
$$\cos\beta=\frac{|C_{11}-C_{12}-\lambda_4|}{\sqrt{(C_{11}-C_{12}-\lambda_4)^2+4C_{15}^2}}. \tag{26}$$

The matrix of the spectral decomposition will be:

$$[E_1]=\begin{bmatrix} \frac{1}{2}\sin^2\alpha & \frac{1}{2}\sin^2\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{2}\sin^2\alpha & \frac{1}{2}\sin^2\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \cos^2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

$$[E_2]=\begin{bmatrix} \frac{1}{2}\cos^2\alpha & \frac{1}{2}\cos^2\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{2}\cos^2\alpha & \frac{1}{2}\cos^2\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \sin^2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

$$[E_3]=\begin{bmatrix} \frac{1}{2}\cos^2\beta & -\frac{1}{2}\cos^2\beta & 0 & 0 & -\frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 \\ -\frac{1}{2}\cos^2\beta & \frac{1}{2}\cos^2\beta & 0 & 0 & \frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^2\beta & 0 & \sin\beta\cos\beta \\ -\frac{1}{\sqrt{2}}\sin\beta\cos\beta & -\frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 & 0 & \sin^2\beta & 0 \\ 0 & 0 & 0 & \sin\beta\cos\beta & 0 & \cos^2\beta \end{bmatrix}, \tag{29}$$

$$[E_4]=\begin{bmatrix} \frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta & 0 & 0 & \frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}\sin^2\beta & 0 & 0 & -\frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2\beta & 0 & -\sin\beta\cos\beta \\ \frac{1}{\sqrt{2}}\sin\beta\cos\beta & -\frac{1}{\sqrt{2}}\sin\beta\cos\beta & 0 & 0 & \cos^2\beta & 0 \\ 0 & 0 & 0 & -\sin\beta\cos\beta & 0 & \sin^2\beta \end{bmatrix}. \tag{30}$$

In the case of the tetragonal symmetry, the eigenvalues are:

$$\lambda_1=\frac{1}{2}\left[C_{11}+C_{12}+C_{33}+\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],$$

$$\lambda_2=\frac{1}{2}\left[C_{11}+C_{12}+C_{33}-\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],$$

$$\lambda_3 = C_{11} - C_{12}\,,\ \ \lambda_4 = \lambda_5 = C_{44}\,,\ \ \lambda_6 = C_{66}\,. \tag{31}$$

and the matrix of eigenvectors is:

$$[X] = \begin{bmatrix} \frac{1}{\sqrt{2}}\sin\alpha & -\frac{1}{\sqrt{2}}\cos\alpha & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\sin\alpha & -\frac{1}{\sqrt{2}}\cos\alpha & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{32}$$

where:

$$\cos\alpha = \frac{\left|\lambda_1 - C_{11} - C_{12}\right|}{\sqrt{2C_{13}^2 + \left(\lambda_1 - C_{11} - C_{12}\right)^2}}\,, \sin\alpha = \frac{\left|\lambda_2 - C_{11} - C_{12}\right|}{\sqrt{2C_{13}^2 + \left(\lambda_2 - C_{11} - C_{12}\right)^2}}\,. \tag{33}$$

The matrices of the spectral decomposition will be:

$$[E_1] = \begin{bmatrix} \frac{1}{2}\sin^2\alpha & \frac{1}{2}\sin^2\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{2}\sin^2\alpha & \frac{1}{2}\sin^2\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \cos^2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{34}$$

$$[E_2] = \begin{bmatrix} \frac{1}{2}\cos^2\alpha & \frac{1}{2}\cos^2\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ \frac{1}{2}\cos^2\alpha & \frac{1}{2}\cos^2\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & -\frac{1}{\sqrt{2}}\sin\alpha\cos\alpha & \sin^2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$[E_3]=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$[E_4]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{37}$$

$$[E_6]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

In the case of the hexagonal symmetry, the eigenvalues are:

$$\lambda_1=\frac{1}{2}\left[C_{11}+C_{12}+C_{33}+\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],$$
$$\lambda_2=\frac{1}{2}\left[C_{11}+C_{12}+C_{33}-\sqrt{\left(C_{11}+C_{12}-C_{33}\right)^2+8C_{13}^2}\right],$$
$$\lambda_3=\lambda_6=C_{11}-C_{12}\,,\quad \lambda_4=\lambda_5=C_{44}\,. \tag{39}$$

The matrix of eigenvectors will be given by (32). The matrices $[E_1]$, $[E_2]$ and $[E_4]$ are given by relations (34), (35), (37) and the matrix $[E_3]$ is:

$$[E_3]=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

In case of the cubic symmetry, the eigenvalues are:

$$\lambda_1 = C_{11} + 2C_{12}\,,\;\; \lambda_2 = \lambda_3 = C_{11} - C_{12}\,,\;\; \lambda_4 = \lambda_5 = \lambda_6 = C_{44}\,. \tag{41}$$

The matrix of eigenvectors is:

$$[X]=\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

The matrices of the spectral decomposition will be:

$$[E_1]=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{43}$$

$$[E_2]=\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{44}$$

$$[E_4]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

For homogenous and isotropic materials, the eigenvalues are:

$$\lambda_1 = C_{11} + 2C_{12}\,, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = C_{11} - C_{12}\,. \tag{46}$$

The matrix of eigenvectors will be given by relation (42).
Finally, we obtain the values for the matrices of spectral decomposition:

$$[E_1]=\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

$$[E_2]=\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (48)$$

Finally, we have to add that using our new method of elasticity matrix spectral decomposition in some cases like:

- tourmaline and α-quartz concerning trigonal symmetry;
- tin and pentaerytritol concerning tetragonal symmetry;
- cobalt and beryl concerning hexagonal symmetry;
- topaz concerning orthorhombic symmetry (which is a particular case of monoclinic symmetry);
- copper concerning cubic symmetry;

and starting from the same values of elastic constants used by Sutcliffe in [6], we obtain the same numerical results he obtained in [6]. It's, also kind of appropriate to add that these results are experimentally validated and well-known in the Materials Research Domain.

4. Conclusions

Composite materials are, basically, kind of anisotropic and non-homogenous. In case that we have to deal with a non-homogenous material, the calculus of its elastic constants, based on elastic characteristics of its constituents, is kind of essential.

In order to study the elastic behavior of a certain composite material, one or other so-called homogenization theory is often used. When such a theory is, effectively, used it has to take into account the phenomena occurring on the separation surfaces between whatever two constituents (phases). Basically, whatever homogenization theory is used, it has to deal with existing reality of the continuity of displacements and stresses of both phases concerning the same separation surface. In this kind of respect the spectral decomposition-based homogenization theories were built.

The elasticity matrix can be regarded as a linear transformation and it can be expressed in terms of its spectral decomposition. The structures of the spectral decomposition are determined by the sets of invariant subspaces that are consistent with material symmetry. Eigenvalues depend on the values of the elastic constants, but eigenvectors are, partially, independent of the values of the elastic constants and that could be kind of important in terms of choosing constituents and arrangements of them in order to build-up new composite materials having required elastic characteristics.

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