U.P.B. Sci. Bull., Series C, Vol. 72, Iss. 4, 2010 ISSN 1454-234x

ON THE FEASIBILITY OF CONSTRAINED GENERALIZED PREDICTIVE CONTROL

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În acest articol se analizează legile de comandã predictivã în prezența constrângerilor asupra semnalelor de intrare, de ieşire şi a altor semnale interne care descriu starea sistemului. Constrângerile considerate sunt liniare, iar secvența de comandã este rezultatul unei probleme de optimizare pãtraticã. Problemele de fezabilitate sunt legate pe de o parte de consistența setului de constrângeri fizice, iar pe de altă parte, de adecvarea lor cu semnalul de referință care trebuie urmărit. Propagarea fezabilitații de-a lungul funcționarii sistemului este fundamentalã, prezentându-se, în acest sens, condițiile necesare.

This paper analyzes the generalized predictive control law under constraints on the input, output or other auxiliary signals that depend linearly on the system variables. These constraints are formulated as sets of linear equalities or inequalities; the control sequence is therefore elaborated based on parametric optimization problem. The feasibility issues are related on one hand to the well-posedness condition and on the other hand to the compatibility with the set-point constraints. The prediction of the feasibility is of great interest from this point of view and necessary feasibility conditions are presented.

Keywords: predictive control, feasibility, polyhedral representation

1. Introduction

The computer aided design of control laws must overcome important difficulties when dealing with constraints. These constraints may be forced by practical considerations as limitations on the input control signal amplitude or rate. Constraints may arise also from the desired closed-loop performances for the control law, a classical example being the output constraints. Generally, these types of constraints can be softened [1]. Other hidden constraints (from the end-user point of view) are to be reinforced, the classical example being the end-point stability constraints. In the present study all the constraints are expressed as linear equality or inequality constraints that have to be further considered in the control

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design procedure [2]. This set describes in fact a polyhedral domain for which a dual representation in terms of generators is available [3]. Analyzing the geometry and the evolutions of this polyhedron can characterize the viability of the control algorithm. An exhaustive analysis of the system of constraints may reveal useful properties such as the expression of the “switching surfaces” for the linear control laws and the corresponding affine formulations [4], [5].

This paper deals with another important aspect related to the constraints analysis, the feasibility of the related optimization problem [6], a crucial aspect for the validation of the predictive control law [7], [8]. This is equivalent with an off-line prediction of infeasibility. Results towards feasibility and their implications in the case of general types of set-points were presented in [9]. Theoretical aspects related to some classes of necessary feasibility conditions are recalled here and an algorithm to check these conditions is presented here, based on off-line information.

2. Generalized Predictive Control

Generalized predictive control (GPC) is part of the long-range predictive control (LRPC) or model predictive control (MPC) family [10]. GPC is characterized by two major characteristics. It uses first a CARIMA plant model :

$$A(q^{-1})y(t) = B(q^{-1})u(t-d) + C(q^{-1})\xi(t)\big/\Delta(q^{-1}) \tag{1}$$

where u, y are the system input and output respectively, $\xi(t)$ represents a centered Gaussian white noise, d the system time delay, A and B are polynomials in q^{-1} (the backward shift operator) of degree n_a and n_b, and $\Delta(q^{-1}) = 1 - q^{-1}$.

Then the cost function to be minimized is quadratic in the tracking error and control effort over a receding horizon

$$J = \sum_{j=N_1}^{N_2} \left[\hat{y}(t+j) - w(t+j)\right]^2 + \lambda \sum_{j=1}^{N_u} \left[\Delta u(t+j-1)\right]^2 \tag{2}$$

where $\hat{y}(t+j)$ is an optimum j-step-ahead predictor, N_1, N_2 are the minimum and maximum costing horizon, N_u the control horizon, λ a control weighting factor and w the setpoint.

Based on the model mentioned earlier and following the ideas of GPC [11] an optimal j-step ahead predictor can be constructed

$$\hat{y}(t+j) = \underbrace{F_j(q^{-1})y(t) + H_j(q^{-1})\Delta u(t-1)}_{l=\text{free response}} + \underbrace{G_j(q^{-1})\Delta u(t+j-1)}_{\text{forced response}} \tag{3}$$

where the F_j, G_j, H_j polynomials are solutions of the Diophantine equations

$$\Delta(q^{-1})A(q^{-1})J_j(q^{-1}) + q^{-j}F_j(q^{-1}) = 1$$
$$G_j(q^{-1}) + q^{-j}H_j(q^{-1}) = B(q^{-1})J_j(q^{-1}) \tag{4}$$

The index (2) is rewritten for optimization purpose

$$J = (\mathbf{G}\,\mathbf{k}_u + \mathbf{l} - \mathbf{w})^{\mathrm{T}}(\mathbf{G}\,\mathbf{k}_u + \mathbf{l} - \mathbf{w}) + \lambda\,\mathbf{k}_u{}^{\mathrm{T}}\mathbf{k}_u =$$
$$= 0.5\,\mathbf{k}_u{}^{\mathrm{T}}\mathbf{Q}\,\mathbf{k}_u + \mathbf{f}\,\mathbf{k}_u + J_0 \tag{5}$$

with the vector form of (3)

$$\hat{\mathbf{y}} = \mathbf{G}\,\mathbf{k}_u + \mathbf{l} = \mathbf{G}\,\mathbf{k}_u + \mathbf{if}\,\mathbf{y}_{past}(t) + \mathbf{ih}\,\Delta\,\mathbf{u}_{past}(t)$$

with:

$$\mathbf{k}_u = \begin{bmatrix} \Delta u(t) \\ \vdots \\ \Delta u(t+N_u-1) \end{bmatrix}; \mathbf{u}_{past}(t) = \begin{bmatrix} u(t-1) \\ \vdots \\ u(t-n_b) \end{bmatrix}$$

$$\mathbf{y}_{past}(t) = \begin{bmatrix} y(t) \\ \vdots \\ y(t-n_a) \end{bmatrix}; \hat{\mathbf{y}} = \begin{bmatrix} \hat{y}(t+N_1) \\ \vdots \\ \hat{y}(t+N_2) \end{bmatrix}; \mathbf{w} = \begin{bmatrix} w(t+N_1) \\ \vdots \\ w(t+N_2) \end{bmatrix}$$

$$\mathbf{ih} = \begin{bmatrix} H_{N_1}(1) & \cdots & H_{N_1}(n_b-1) \\ \vdots & & \vdots \\ H_{N_2}(1) & \cdots & H_{N_2}(n_b-1) \end{bmatrix}; \mathbf{if} = \begin{bmatrix} F_{N_1}(1) & \cdots & F_{N_1}(n_a) \\ \vdots & & \vdots \\ F_{N_2}(1) & \cdots & F_{N_2}(n_a) \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} g_{N_1} & g_{N_1-1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{N_2-1} & \cdots & \vdots & \vdots \\ g_{N_2} & \cdots & \cdots & g_{N_2-N_u+1} \end{bmatrix}$$

In the unconstrained case, the optimum of J derived through analytical minimization is given by the relation $\mathbf{k}_u = -\mathbf{Q}^{-1}\mathbf{f}$. By applying the first control action $\mathbf{k}_u(1)$ of this optimal sequence and restarting the procedure, a control law with improved performances under the RST form is performed [12].

3. Constrained GPC

All these properties have to be reanalyzed when constraints are taken into consideration [13]. The design procedures most often have to consider specific types of constraints originated by amplitude limits in the control signal, slew rate limits of the actuator, limits on the output signal or equality constraints at the end of the prediction horizon for stability purposes.

Generally the formal mathematical description is

$$\begin{cases} -\Delta u_{\min} \le u(t+k) - u(t+k-1) \le \Delta u_{\max} \\ -u_{\min} \le u(t+k) \le u_{\max}, \quad 0 \le k \le N_u - 1 \\ -y_{\min} \le \hat{y}(t+k) \le y_{\max}, \quad N_1 \le k \le N_2 \\ \hat{y}(t+N_2+k) = w(t+N_2), \quad k = 1...m \end{cases} \tag{6}$$

These constraints on the control action and outputs can be restated in a form depending only on control updates. Further, this description could be translated in a matrix form like in [2]

$$\begin{cases} -\mathbf{M}(N_u,1)\,\Delta u_{\min} \le \mathbf{I}\,\mathbf{k}_u \le \mathbf{M}(N_u,1)\,\Delta u_{\max} \\ -\mathbf{M}(N_u,1)\,\underline{u} \le \mathbf{L}\,\mathbf{k}_u \le \mathbf{M}(N_u,1)\,\overline{u}, \\ \qquad \underline{u} = -u_{\min} - u(t-1),\ \overline{u} = u_{\max} - u(t-1) \\ -\mathbf{M}(N,1)\,y_{\min} \le \mathbf{G}\,\mathbf{k}_u + \mathbf{l} \le \mathbf{M}(N,1)\,y_{\max} \\ \mathbf{G}_c\mathbf{k}_u + \mathbf{l}_c = \mathbf{M}(m,1)\,w(t+N_2) \end{cases} \tag{7}$$

where $N = N_2 - N_1 + 1$, $\mathbf{M}(q,r)$ is a matrix of dimension qxr whose entries are one on the first column and zero for the others, $\mathbf{L}$ is a N_uxN_u lower triangular matrix whose entries are one. $\mathbf{G}_c$ and $\mathbf{l}_c$ describe the dynamics and the free response of the constrained system, both found as in (3), (5).

When minimizing the index J in (2) with respect to the constraints, the methods presented in the relaxed case cannot be applied since they do not provide a solution when the global optimum violates the constraints

4. Constrained domain evolution

The description of the feasibility domain for a system under all types of constraints can be obtained in a compact form from (7)

$$\underbrace{\begin{bmatrix} \mathbf{F} \\ -\mathbf{F} \end{bmatrix}}_{\tilde{\mathbf{F}}} \boldsymbol{\theta}(t) \le \underbrace{\begin{bmatrix} \boldsymbol{\Gamma}_{\max} \\ \boldsymbol{\Gamma}_{\min} \end{bmatrix}}_{\boldsymbol{\Gamma}}; \quad \boldsymbol{\Gamma} > \mathbf{0} \tag{11}$$

with

$$\boldsymbol{\Gamma}_{\min} = \begin{bmatrix} \mathbf{M}(N_u,1)\,\Delta u_{\min} \\ \mathbf{M}(N_u,1)\,u_{\min} \\ \mathbf{M}(N,1)\,y_{\min} \\ \mathbf{M}(m,1)\,\varepsilon_{\min} \end{bmatrix} \quad \boldsymbol{\Gamma}_{\max} = \begin{bmatrix} \mathbf{M}(N_u,1)\,\Delta u_{\max} \\ \mathbf{M}(N_u,1)\,u_{\max} \\ \mathbf{M}(N,1)\,y_{\max} \\ \mathbf{M}(m,1)\,\varepsilon_{\max} \end{bmatrix}$$

$$\mathbf{F} = \left[\begin{array}{ccc|cc} 0 & 0 & \mathbf{I}_{N_u} & 0 & 0 \\ 0 & \mathbf{M}(N_u,n_b) & \mathbf{L} & 0 & 0 \\ \mathbf{if} & \boldsymbol{\Delta}\mathbf{ih} & \mathbf{G} & 0 & 0 \\ \hline \mathbf{if}_c & \boldsymbol{\Delta}\mathbf{ih}_c & \mathbf{G}_c & 0 & -\mathbf{M}(m,n_w) \end{array}\right]$$

$$\boldsymbol{\theta}(t) = \left[\mathbf{y}_{past}(t) \quad \mathbf{u}_{past}(t) \quad \mathbf{k}_u(t) \mid \mathbf{w}(t) \quad \mathbf{w}_c(t)\right]^{\mathrm{T}}$$

where the epsilon machine will represent the bounds for equality constraints, n_w is the required number of past known values that are necessary to properly evaluate the future setpoint evolution.

A possible way of modeling (11) considers the dual representation of the inequalities in (7)

$$P = conv.hull\{x_1, \ldots, x_v\} + cone\{y_1, \ldots, y_r\} + lin.spaceP \tag{12}$$

where conv.hullX denotes the set of all convex combinations of points in X, coneY denotes nonnegative combinations of unidirectional rays and lin.spaceP represents a linear combination of bi-directional rays. It can be rewritten as

$$\begin{gathered} P = \sum_{i=1}^{v} \lambda_i x_i + \sum_{i=1}^{r} \gamma_i y_i + \sum_{i=1}^{l} \mu_i z_i \\ 0 \le \lambda_i \le 1,\ \sum_{i=1}^{v} \lambda_i = 1,\ \gamma_i \ge 0\ ,\forall \mu_i \end{gathered} \tag{13}$$

Usually the polyhedral domain related with practical CGPC laws are in fact polytopes. These domains in a compact form can be analysed by their evolution, providing the dynamics of the constrained variables vector. This is the purpose of the next part.

From (5) one has

$$\begin{aligned} J &= \mathbf{k}_u^{\mathrm{T}}\left(\mathbf{G}^{\mathrm{T}}\mathbf{G} + \lambda\mathbf{I}\right)\mathbf{k}_u + 2\,\mathbf{k}_u^{\mathrm{T}}\mathbf{G}(\mathbf{l} - \mathbf{w}) = \\ &= \mathbf{k}_u^{\mathrm{T}}\left(\underbrace{\mathbf{G}^{\mathrm{T}}\mathbf{G} + \lambda\mathbf{I}}_{\mathbf{H}}\right)\mathbf{k}_u + 2\,\mathbf{k}_u^{\mathrm{T}}\mathbf{G}\,\mathbf{E}\,\boldsymbol{\theta}^*(t) \end{aligned} \tag{14}$$

where E is a matrix which allows the description of the vector $\mathbf{l} - \mathbf{w} = \mathbf{E}\boldsymbol{\theta}^*(t)$ when

$$\boldsymbol{\theta}^*(t+1) = \begin{bmatrix} \mathbf{y}_{past}(t+1) \\ \mathbf{u}_{past}(t+1) \\ \mathbf{w}(t+1) \\ \mathbf{w}_c(t+1) \end{bmatrix} = \boldsymbol{\Phi}^*\boldsymbol{\theta}(t) =$$

$$= \begin{bmatrix} \mathbf{D}_1(\mathbf{if}) & \mathbf{D}_2(\mathbf{ih}) & \mathbf{D}_3(\mathbf{G}) & 0 & 0 \\ 0 & \mathbf{I}_{dev} & \mathbf{D}_4 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_1 & 0 \\ 0 & 0 & 0 & \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{past}(t) \\ \mathbf{u}_{past}(t) \\ \mathbf{k}_u(t) \\ \mathbf{w}(t) \\ \mathbf{w}_c(t) \end{bmatrix} \tag{15}$$

One can find the description for the vector $\mathbf{k}_u$ by minimizing this index J under the constraints

$$\overline{\mathbf{A}}\mathbf{k}_u \leq \overline{\mathbf{b}} - \mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t) \tag{16}$$

where K is a matrix allowing the description of the affine part of the inequalities as a linear dependence on the context parameters $\boldsymbol{\theta}^*(t)$. The close form of the optimal control sequence for the CGPC is

$$\mathbf{k}_u{}^* = \left(\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{A}}_0\mathbf{H}^{-1} - \mathbf{H}^{-1}\right)\mathbf{GE}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t)$$

$$+\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\left(\overline{\mathbf{b}} - \mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t)\right)$$

$$= \Big[\left(\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{A}}_0\mathbf{H}^{-1} - \mathbf{H}^{-1}\right)\mathbf{GE}\boldsymbol{\Phi}^*$$

$$-\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\mathbf{K}\boldsymbol{\Phi}^*\Big]\boldsymbol{\theta}(t) + \mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}}(\overline{\mathbf{A}}_0\mathbf{H}^{-1}\overline{\mathbf{A}}_0^{\mathrm{T}})^{-1}\overline{\mathbf{b}}$$

with $\overline{\mathbf{A}}_0$ the matrix constructed by the subset of lines in $\overline{\mathbf{A}}$ for whom the inequality constraints are saturated.

As a conclusion, the elaborated control law is affine in the parameter vector $\boldsymbol{\theta}(t)$. However, the difficulties arise from the fact that the matrix $\overline{\mathbf{A}}_0 = \overline{\mathbf{A}}_0(\boldsymbol{\theta}(t))$ is not allowing an explicit dependence on the vector of parameters.

Remark: A parameterized polyhedron like the one in (16)

$$P = \left\{\mathbf{k}_u \middle| \overline{\mathbf{A}}\mathbf{k}_u \leq \overline{\mathbf{b}} - \mathbf{K}\boldsymbol{\Phi}^*\boldsymbol{\theta}(t)\right\}$$

has a dual representation where only the vertices are affected by the parameters

$$P = \left\{ \begin{array}{l} \mathbf{k}_u \middle| \mathbf{k}_u = \sum_{i=1}^{v} \lambda_i x_i(\boldsymbol{\theta}^*) + \sum_{i=1}^{r} \gamma_i y_i + \sum_{i=1}^{l} \mu_i z_i \\ \lambda_i \geq 0, \sum_{i=1}^{v} \lambda_i = 1, \gamma_i \geq 0, \forall \mu_i \end{array} \right\}$$

5. Necessary conditions by means of extreme point feasibility

Considering the polyhedral domain as described earlier, with the dual representation by the vertices, it can be interesting to look at the evolution of these vertices at each sampling time.

Proposition 1: The optimal control sequence corresponding to all extreme combinations of context parameters must lead to a point inside the projection of the initial polyhedral domain for a feasible CGPC law.

Sketch of proof: As explained earlier, the constraints on the CGPC law define a polyhedral domain

$$D = \left\{ \mathbf{k}_u \middle| \mathbf{k}_u = \sum_{i=1}^{v} \lambda_i k_{u\,i}(\boldsymbol{\theta}^*); \lambda_i \geq 0, \sum_{i=1}^{v} \lambda_i = 1 \right\} \tag{17}$$

By considering the involved system variables as parameters, this parameterized polyhedron can be extended to a fixed one of higher dimension.

$$P = \left\{ \boldsymbol{\theta} \middle| \boldsymbol{\theta} = \sum_{i=1}^{v} \lambda_i \boldsymbol{\theta}_i; \lambda_i \geq 0, \sum_{i=1}^{v} \lambda_i = 1 \right\} \tag{18}$$

The existence of these vertices does not guaranty the fact that the CGPC law will have the opportunity to reach each of them. Multiple vertices may correspond to the same context parameters. Thus, a useful manipulation may be the orthogonal projection of this domain on the subspace of the context parameters (as in Fig. 1).

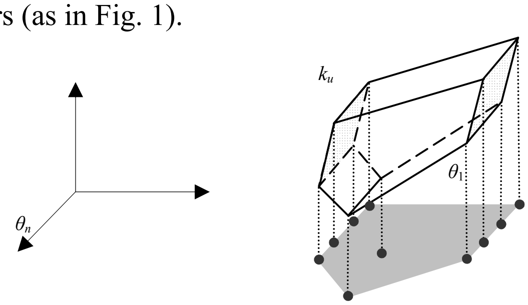

Fig. 1. The polyhedral domain and its projection on the context parameters subspace

This operation can be done explicitly by multiplying each vertex θ_i by a matrix $\lfloor e_j \rfloor$ where j are the indices of the context parameters in the vector $\boldsymbol{\theta}$. The resulting set is P^*, the convex combination of the points

$$\Im(P^*) = \left\{ \boldsymbol{\theta}^* \middle| \; \boldsymbol{\theta}^* = \left[e_j \right] \theta_i \right\} \tag{19}$$

Once the projection available, a redundancy check must be operated in order to obtain the minimal set of generators.

The resulting domain P^* can provide by its vertices the extremal points for the context parameters that can further be used for figuring the whole domain. Solving the parameterized quadratic problem related to CGPC, one can retrieve a hyper surface inside the original polyhedral domain D.

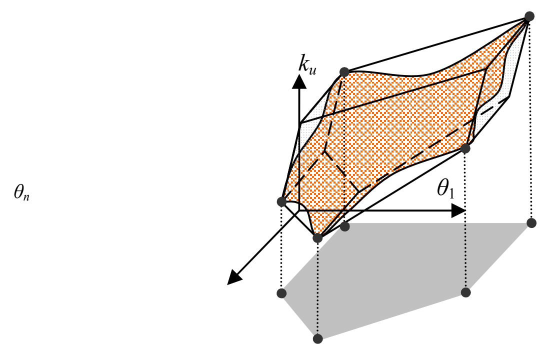

Fig. 2. The optimal solution of CGPC for each possible context parameters combination

The elaboration of this shape enables to solve all the analysis problems as it defines the whole behavior of the CGPC law. As far as the evolution of the context parameters domain is concerned, the image of the points on the CGPC shape must be found by the linear transformation (15). If this domain is denoted as P_+^*, the necessary and sufficient conditions for feasibility are resumed by

$$P^* \supset P_+^* \tag{20}$$

Due to limitations in the knowledge on the topology of the CGPC shape, this will resume on necessary conditions based on the extremal points. These necessary conditions may be expressed as in Fig. 3 by a set of inequalities

$$\boldsymbol{\theta}^*(t+1) \in P_+^* \tag{21}$$

which resumes the proposition.

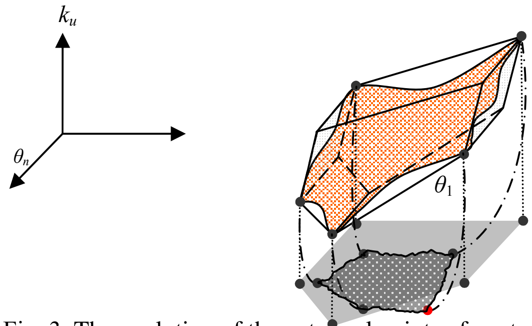

Fig. 3. The evolution of the extremal points of context parameters domain

For a complete analysis of the CGPC law, all the points inside the polyhedral domain P_+^* have to be checked in order to confirm the feasibility. This is not an obvious task as long as the optimal control sequence is affine in the context parameters, and the affine part even if linear in the parameter vector is changing the linear dependence in concordance with the active set of constraints. It is clear that the number of active constraints is maximal for the vertices and is subsequently decreasing for the points on the frontier where subsets of these sets of constraints are active. Following the same line as the proof, an algorithm based on tools of polyhedral computations and quadratic optimization can be designed in order to validate these necessary conditions. Such an algorithm can be resumed by the following steps.

Algorithm 1

- Compute the vertices of the polyhedral set by dual representation of the constraints
- Project the polytope on the parameters subspace
- Remove the redundant points
- Compute the close form of the control law in all the vertices of the constrained domain. (Compulsatory as it is not always equal with the value in the original polyhedron)
- For each such law, construct the evolution matrix and compute the corresponding next step parameters $\boldsymbol{\theta}(t+1)$
- Check if each such point $\boldsymbol{\theta}(t+1)$ is inside the projected polyhedron found at step 2. If it is not the case, that means that there exists at least one point in the constrained domain which, if reached, will lead to infeasibility.

6. Exemple

Consider in the following a second order linear system as the one reported in (Olaru and Dumur, 2003), with non-minimum phase characteristics

$$(1-q^{-1}+0.25q^{-2})y(t)=(-0.25-0.25q^{-1}+0.75q^{-2})u(t) \tag{22}$$

The step response of this system is given in Fig. 4.

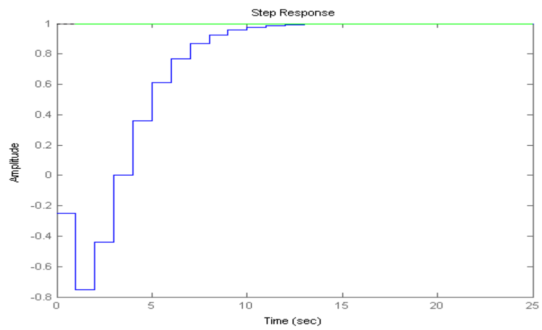

Fig. 4. Open loop step response

For CGPC law with $N_1=1$, $N_2=4$, $N_u=2$, the system proves to have an infeasible behavior for step setpoints and constraints on the output of magnitude $-1\leq y\leq 1$, based on snow-ball attitude [14] (Fig. 5).

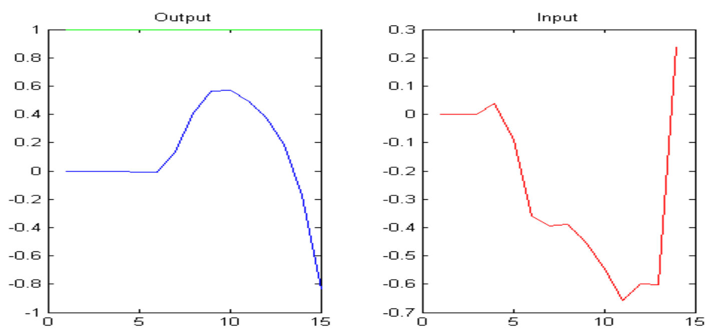

Fig. 5. CGPC closed loop behavior

Proceeding as explained in Algorithm 1, the constrained domain can be described as

$$D=\left\{\mathbf{k}_u \middle| \mathbf{1}\underline{y}\leq \mathbf{G}\,\mathbf{k}_u+\mathbf{l}\leq \mathbf{1}\overline{y}\right\} \tag{23}$$

where l is like in (3) and

$$\mathbf{G}=\begin{bmatrix}-0.25 & 0\\ -0.75 & -0.25\\ -0.40 & -0.80\\ 0 & -0.40\end{bmatrix} \tag{24}$$

The elaboration of the extended polyhedron requires the definition of

$$P = \{-\boldsymbol{\Gamma}_{\min} \le \mathbf{F}\,\boldsymbol{\theta}(t) \le \boldsymbol{\Gamma}_{\max} \quad \boldsymbol{\Gamma}_{\min}, \boldsymbol{\Gamma}_{\max} \ge \mathbf{0}\} \tag{25}$$

with

$$\boldsymbol{\Gamma}_{\min}^{\mathrm{T}} = \mathbf{1}^{\mathrm{T}}\underline{y} = \mathbf{1}^{\mathrm{T}};\boldsymbol{\Gamma}_{\max}^{\mathrm{T}} = \mathbf{1}^{\mathrm{T}}\overline{y} = \mathbf{1}^{\mathrm{T}}$$

$$\mathbf{F} = \begin{bmatrix} 2 & -1.25 & 0.25 & -0.25 & 0.75 & -0.25 & 0 \\ 2.75 & -2.25 & 0.5 & 0.25 & 1.5 & -0.75 & -0.25 \\ 3.2 & -3 & 0.7 & 0.8 & 2 & -0.4 & -0.8 \\ 3.6 & -3.4 & -0.8 & -1.2 & -2.4 & 0 & 0.4 \end{bmatrix}$$

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \mathbf{y}_{past}(t) & \Delta\mathbf{u}_{past}(t) & \mathbf{k}_u(t) \end{bmatrix}^{\mathrm{T}}$$

As the context parameters include the past outputs, three implicit constraints have been added as an upper part of F in order to avoid the analysis of non-reachable regions. The result is a square matrix of constraints describing a polytope with a dual representation containing 128 vertices. The projection on the subspace of the first five variables leads to a domain P^* that can be reduced by removing redundant pairs to the convex hull of 64 vertices like in Fig. 6.

θ1= [-148 -148 148 335 -193]/148
θ2= [148 148 -148 677 -303]/148
θ3=[148 -148 -148 793 -767]/148
…
θ64= [-148 148 148 -793 767]/148

Fig. 6. Convex hull for P^* computed by POLYLIB

The corresponding quadratic problems have to be solved in order to find the optimal control law in each such extreme context. The next step aims at computing the image of the resulting extended vectors $\boldsymbol{\theta}_{1..64}$ by the linear transformation

$$\boldsymbol{\theta}^*(t+1) = \begin{bmatrix} 2 & -1.25 & 0.25 & -0.25 & 0.75 & -0.25 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{past}(t) \\ \Delta\mathbf{u}_{past}(t) \\ \mathbf{k}_u(t) \end{bmatrix}$$

Checking their membership inside D ends the algorithm. In the studied case, there are 32 vertices which are positioned outside the feasible context polyhedron P^*. This means that there are at least 32 combinations of past inputs and outputs for which there is no feasible control sequence able to retain the system inside the constraints

$-1 \le y \le 1$

Thus as the necessary conditions are not fulfilled, the CGPC is infeasible.

6. Conclusion

This paper presented two possible approaches for off-line analysis of the feasibility of constrained generalized predictive control strategies. The advantages of this analysis consist in a set-point validation at the stage of parameter tuning for the predictive control laws. Subsequently, the adaptation of prediction horizon and/or the introduction of slack variables can be considered for feasibility reinforcement at the control design stage.

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