NEW APPROACH FOR THE DUFFING EQUATION INVOLVING BOTH INTEGRAL AND NON-INTEGRAL FORCING TERMS

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Duffing equations have a wide range of application in science and engineering. In this research, the sinc-collocation method is presented for solving nonlinear Duffing equation involving both integral and non-integral forcing terms. The properties of sinc functions required for our subsequent development are given. These properties are then used to reduce the computation of solution of Duffing equation to some algebraic equations. It is well known that the sinc procedure converges to the solution at an exponential rate. Numerical examples are included to demonstrate the validity and applicability of the new technique.

Keywords: Duffing equation, Sinc function, Collocation method, Integral equation, Boundary value problem.

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1. Introduction

In this work, we develop a framework to obtain the numerical solution of the following Duffing equation involving both integral and non-integral forcing terms

$$y''(x) + \sigma y'(x) + f(x, y(x), y'(x)) + \int_0^x k(x, t, y(t))\mathrm{d}t = 0, \qquad 0 < x < 1, \tag{1}$$

with separated boundary conditions

$$p_0 y(0) - q_0 y'(0) = a, \qquad p_1 y(1) + q_1 y'(1) = b, \tag{2}$$

where $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $k : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Also, $p_0, q_0, p_1, q_1, a, b \in \mathbb{R}$ and $\sigma \in \mathbb{R} - \{0\}$ are such that $p_0, q_0, p_1, q_1 > 0$.
The Duffing equation in one form or another frequently appear in many physical and engineering problems, e.g., chaotic phenomena, signal processing, fuzzy modelling, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, orbit extraction and brain modelling (see [16, 12, 5, 13, 10, 11, 15] and the references therein). Authors of [2] introduced the existence and uniqueness of the solution of (1)-(2) by a generalized quasilinearization (QSL) technique. Also the analytic approximation of the forced Duffing equation with continuous and discontinuous integral boundary conditions has been investigated in [4, 3] through (QSL) technique. Furthermore, authors of [5] developed an algorithm for the analytic solution of the forced Duffing type integro-differential equations with nonlinear three-point boundary conditions.

There are few references on the numerical solution of the problem (1)-(2). In [11] an improved variational iteration method is presented for solving this problem. As said in [11], the main advantage of this modification over the standard variational iteration method is that it can avoid unnecessary repeated computation in determining the unknown

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parameters in the initial solution. Yao [27] presented an iterative reproducing kernel method for solving problem (1)-(2). Also in [15] the hybrid functions of block-pulse and Bernoulli polynomials are used to approximate the solutions of (1)-(2). Furthermore, in [21] the Legendre pseudospectral method was developed for solving this problem.

In the last three decades a variety of numerical methods based on the sinc approximation have been developed. References [26, 25, 14] provide overviews of the methods based on the sinc function for solving ordinary differential equations, partial differential equations and integral equations. Sinc methods have also been employed as forward solvers in the solution of boundary value problems [22, 7, 24, 6, 9, 20], astrophysics equations [19], Blasius equation [18], elasto-plastic problem [1], heat distribution [8], integral and integro-differential equations [23, 28, 29, 17].

In this paper, the solution of Duffing equation (1) with separated boundary conditions (2) is presented by means of sinc-collocation method. Our method consists of reducing the solution of Eq. (1) to a set of algebraic equations. The properties of sinc function are then utilized to evaluate the unknown coefficients. It is known that the sinc collocation method with n collocation points converges at the rate of $\exp(-\kappa\sqrt{n})$ with some $\kappa > 0$ under certain condition [25].

The outline of the paper is as follows. First, in Section 2 we explain the formulation of sinc functions required for our subsequent development. In Section 3, we illustrate how the sinc method may be used to replace Duffing equation (1) with boundary conditions (2) by an explicit system of nonlinear algebraic equations, which is solved by Newton's method. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

2. **Sinc function preliminaries**

In what follows we outline some of the basic properties of the sinc functions. For more detailed overview of the sinc function properties readers can study [25, 14].
On the real line $\Re$ the sinc function is defined as

$$\operatorname{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{3}$$

For any step-size $h > 0$, the translated sinc functions with evenly spaced nodes are given by

$$S(j,h)(x) = \operatorname{Sinc}\left(\frac{x - jh}{h}\right) = \begin{cases} \frac{\sin[\frac{\pi}{h}(x-jh)]}{\frac{\pi}{h}(x-jh)}, & x \neq jh, \\ 1, & x = jh, \end{cases} \tag{4}$$

which are called the jth sinc functions. The sinc functions form an interpolatory set of functions, i.e.,

$$S(j,h)(kh) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{5}$$

If a function f is defined on $\Re$, then the cardinal function of f, denoted $C(f,h)(x)$, is defined by

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh) \operatorname{Sinc}\left(\frac{x - jh}{h}\right), \qquad h > 0, \tag{6}$$

whenever this series converges. The properties of this series are discussed thoroughly in [25, 14]. These properties are derived in the infinite strip D_S in the complex plane,

$$D_S = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}. \tag{7}$$

To construct an approximation on the interval $(0,1)$, which is used in this paper, we consider the conformal map

$$\phi(z) = \ln\left(\frac{z}{1-z}\right). \tag{8}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z \in \mathbb{C} : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \leq \frac{\pi}{2} \right\}, \tag{9}$$

onto the infinite strip D_S. The basis functions on the interval $(0,1)$ are taken to be the composite translated sinc functions:

$$S_j(x) \equiv S(j,h) \circ \phi(x) = \mathrm{Sinc}\left(\frac{\phi(x) - jh}{h}\right), \tag{10}$$

where $S(j,h) \circ \phi(x)$ is defined by $S(j,h)(\phi(x))$. The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{e^w}{1+e^w}. \tag{11}$$

We define the range of $\phi^{-1}(w)$ on the real line as

$$\Gamma = \left\{ \phi^{-1}(x) \in D_E : -\infty < x < \infty \right\} = (0,1). \tag{12}$$

Also, for the evenly spaced nodes $\{jh\}_{j=-\infty}^{\infty}$ on the real line, the image that corresponds to these nodes is

$$x_j = \phi^{-1}(jh) = \frac{e^{jh}}{1+e^{jh}}, \quad j = 0, \pm 1, \pm 2, ... \tag{13}$$

In the following, for our subsequent development, a required definition and some theorems related to functions of the class $L_\alpha(D_E)$ are presented (for more details, see [25, 14]).

Definition 2.1. Let $L_\alpha(D_E)$ be the set of all analytic functions u, for which there exists a constant C, such that

$$| u(z) | \leq C \frac{| \rho(z) |^\alpha}{[1+ | \rho(z) |]^{2\alpha}}, \quad z \in D_E, \quad 0 < \alpha \leqslant 1, \tag{14}$$

where $\rho(z) = e^{\phi(z)}$.

Theorem 2.1. Let $u \in L_\alpha(D_E)$, let N be a positive integer, and let h be selected by the formula

$$h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}, \tag{15}$$

then there exists a positive constant c_1, independent of N, such that

$$\sup_{x \in \Gamma} \left| u(x) - \sum_{j=-N}^{N} u(x_j) S(j,h) \circ \phi(x) \right| \leq c_1 e^{-(\pi d \alpha N)^{1/2}}. \tag{16}$$

The above expressions show that sinc interpolation on $L_\alpha(D_E)$ converges exponentially.

Theorem 2.2. Let $\frac{u}{\phi'} \in L_\alpha(D_E)$, let $\delta_{kj}^{(-1)}$ be defined as

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} \mathrm{d}t,$$

and take $h = \left(\frac{\pi d}{\alpha N}\right)^{1/2}$. Then there exists a constant c_2, which is independent of N, such that

$$\left| \int_0^{x_k} u(t)\,\mathrm{d}t - h \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(x_j)}{\phi'(x_j)} \right| \leqslant c_2 e^{-(\pi d \alpha N)^{1/2}}. \tag{17}$$

In addition, we require derivatives of composite sinc functions evaluated at the nodes x_j. The expressions required for the present discussion are [25]

$$\delta_{ij}^{(0)} = [S(i,h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{18}$$

$$\delta_{ij}^{(1)} = \frac{d}{d\phi}[S(i,h) \circ \phi(x)]|_{x=x_j} = \frac{1}{h} \begin{cases} 0, & i = j, \\ \frac{(-1)^{j-i}}{j-i}, & i \neq j, \end{cases} \tag{19}$$

and

$$\delta_{ij}^{(2)} = \frac{d^2}{d\phi^2}[S(i,h) \circ \phi(x)]|_{x=x_j} = \frac{1}{h^2} \begin{cases} \frac{-\pi^2}{3}, & i = j, \\ \frac{-2(-1)^{j-i}}{(j-i)^2}. & i \neq j. \end{cases} \tag{20}$$

3. **The sinc-collocation method**

For the boundary conditions in Eq. (2), the sinc basis functions $S_i(x)$ do not have a derivative when x tends to 0 or 1. Thus, we modify the sinc basis functions as $w(x)S_i(x)$, where $w(x) = x(1-x)$. Now the first derivative of the modified sinc basis functions are defined and tend to zero as x approaches 0 or 1. To solve problem (1)-(2), we first approximate $y(x)$ as

$$y_N(x) = Q(x) + U_N(x), \tag{21}$$

where

$$U_N(x) = w(x) \sum_{i=-N}^{N} u_i S_i(x), \tag{22}$$

and

$$Q(x) = c_1 q_1(x) + c_2 q_2(x) + c_3 q_3(x) + c_4 q_4(x). \tag{23}$$

Here, $q_1(x), q_2(x), q_3(x)$ and $q_4(x)$ consist of the cardinal functions for univariate cubic Hermite interpolation [6]:

$$\begin{aligned} q_1(x) &= x(1-x)^2, & q_2(x) &= (2x+1)(1-x)^2, \\ q_3(x) &= x^2(3-2x), & q_4(x) &= x^2(x-1). \end{aligned} \tag{24}$$

In Eq. (23), c_1, c_2, c_3 and c_4 are constants to be determined. The $2N+1$, coefficients $\{u_i\}_{i=-N}^{N}$, and c_1, c_2, c_3, c_4 are determined by substituting $y_N(x)$ into Eq. (1) and evaluating the result at the sinc points

$$x_j = \frac{e^{jh}}{1+e^{jh}}, \qquad j = -N-1, \ldots, N+1. \tag{25}$$

Note that

$$\frac{d}{dx}[S(i,h) \circ \phi(x)] = \phi'(x) \frac{d}{d\phi}[S(i,h) \circ \phi(x)], \tag{26}$$

and

$$\frac{d^2}{dx^2}[S(i,h) \circ \phi(x)] = \phi''(x) \frac{d}{d\phi}[S(i,h) \circ \phi(x)] + (\phi'(x))^2 \frac{d^2}{d\phi^2}[S(i,h) \circ \phi(x)]. \tag{27}$$

Using Eqs. (18)-(21), (26) and (27) gives

$$y_N(x_j) = Q(x_j) + w(x_j) u_j, \tag{28}$$

$$y'_N(x_j) = Q'(x_j) + w(x_j) \sum_{i=-N}^{N} u_i \phi'_j \delta_{ij}^{(1)} + w'(x_j) u_j, \tag{29}$$

and

$$\begin{aligned} y''_N(x_j) = Q''(x_j) + w(x_j) \sum_{i=-N}^{N} u_i \left(\phi''_j \delta_{ij}^{(1)} + (\phi'_j)^2 \delta_{ij}^{(2)} \right) + 2w'(x_j) \sum_{i=-N}^{N} u_i \phi'_j \delta_{ij}^{(1)} \\ + w''(x_j) u_j, \end{aligned} \tag{30}$$

where $\phi'_j = \phi'(x_j)$, $\phi''_j = \phi''(x_j)$. Application of Theorem 2.2 to the integral in Eq. (1) gives

$$\int_0^{x_j} k(x_j, t, y_N(t))\, \mathrm{d}t \simeq h \sum_{i=-N}^{N} \frac{k(x_j, x_i, y_N(x_i))}{\phi'(x_i)} \delta_{ji}^{(-1)}. \tag{31}$$

By replacing Eqs. (28)-(31) in Eq. (1), we obtain

$$\begin{aligned} y''_N(x_j) + \sigma y'_N(x_j) + f(x_j, y_N(x_j), y'_N(x_j)) + h \sum_{i=-N}^{N} \frac{k(x_j, x_i, y_N(x_i))}{\phi'(x_i)} \delta_{ji}^{(-1)} \\ = 0, \quad j = -N-1, \dots, N+1, \end{aligned} \tag{32}$$

where we used $u_{-N-1} = u_{N+1} = 0$. Also, substituting Eq. (21) in boundary conditions (2), we get

$$p_0 c_2 - q_0 c_1 = a, \quad p_1 c_3 + q_1 c_4 = b. \tag{33}$$

Eqs. (32) and (33) give $2N + 5$ nonlinear algebraic equations which can be solved for the unknown coefficients u_i and c_i by using Newton's method. Consequently $y_N(x)$ given in Eq. (21) can be calculated.

Remark 3.1. It is worth indicating that the Newton's method has a convergence rate of quadratic order, which directly depends on the equations of (32)-(33) and initial guesses for u_i and c_i. One way to discover the initial guesses is to solve the system analytically for the very small N by means of software programs, such as MATLAB or Maple. The solution can be calculated if the conditions for the existence of the solution and the convergence of the Newton process are fulfilled.

4. **Illustrative Examples**

To incorporate our discussion above, in this section, we will apply the sinc-collocation method to solve some examples. In all examples we choose $\alpha = \frac{1}{2}$ and $d = \frac{\pi}{2}$ which leads to $h = \frac{\pi}{\sqrt{N}}$. We use the absolute errors, $E_N(x) = |y_N(x) - y(x)|$, where $y(x)$ denotes the exact solution of the given example and $y_N(x)$ denotes the computed solution by our method. Note that we have computed the numerical results by MATLAB programming.

Example 1: Consider the Duffing equation [27],

$$\begin{cases} y''(x) + y'(x) + y(x)y'(x) + \displaystyle\int_0^x xty^2(t)\, \mathrm{d}t = f(x), \qquad 0 < x < 1, \\ y(0) - y'(0) = 0, \\ y(1) + y'(1) = 0, \end{cases} \tag{34}$$

where

$$f(x) = -3x - 3x^2 + \frac{5x^3}{2} + \frac{2x^4}{3} - \frac{x^5}{4} - \frac{2x^6}{5} + \frac{x^7}{6}.$$

The exact solution of this problem is $y(x) = 1 + x - x^2$. For the purpose of comparison in Table 1, we compare the absolute error $E_N(x)$ of our method with $N = 15$ and $N = 20$ at the same points as [27] together with the results given in [27]. From Table 1 we see that the sinc-collocation method is clearly reliable if compared with the method given in [27].

Table 1. Comparison of absolute error $E_N(x)$ for Example 1.

x	Exact solution	Method of [27]	Present method $N=15$	$N=20$
0.1	1.09	9.46600×10^{-6}	4.8910×10^{-7}	8.9651×10^{-9}
0.2	1.16	9.77708×10^{-6}	8.7097×10^{-8}	2.0289×10^{-8}
0.3	1.21	9.81938×10^{-6}	2.1538×10^{-7}	8.0284×10^{-8}
0.4	1.24	9.66575×10^{-6}	1.2703×10^{-6}	1.1747×10^{-7}
0.5	1.25	9.37204×10^{-6}	1.4226×10^{-6}	1.7725×10^{-7}
0.6	1.24	8.97794×10^{-6}	2.7771×10^{-7}	4.5333×10^{-8}
0.7	1.21	8.50917×10^{-6}	9.8545×10^{-7}	2.0085×10^{-7}
0.8	1.16	7.97984×10^{-6}	5.7859×10^{-7}	1.7829×10^{-8}
0.9	1.09	7.39469×10^{-6}	2.9480×10^{-7}	3.0525×10^{-8}
1	1	6.75105×10^{-6}	1.2745×10^{-7}	1.9952×10^{-8}

Example 2: As the second example, we consider the following problem

$$\begin{cases} y''(x) - 1.72y'(x) + e^{-y(x)} - \displaystyle\int_0^x (1-2t)y(t)\,\mathrm{d}t = f(x), \qquad 0 < x < 1, \\ y(0) - 3y'(0) = -1, \\ y(1) + 3y'(1) = -1, \end{cases} \tag{35}$$

where

$$\begin{aligned} f(x) = &\frac{43}{75}(2x-1)\cos(x-x^2) - \frac{2}{3}\cos(x-x^2) - \frac{1}{3}(2x-1)^2\sin(x-x^2) \\ &+ \exp\left(-\frac{1}{3}\sin(x-x^2)\right) - \frac{1}{3} + \frac{1}{3}\cos(x-x^2). \end{aligned} \tag{36}$$

The exact solution of this problem is $y(x) = \frac{1}{3}\sin(x-x^2)$. We applied the sinc-collocation method presented in this paper with $N = 25, 30$. Figure 1 shows the absolute error functions $E_{25}(x)$ and $E_{30}(x)$.

Example 3: We consider the following problem,

$$\begin{cases} y''(x) + y'(x) + y(x)(1 + y'(x)) + \displaystyle\int_0^x xty(t)\,\mathrm{d}t = f(x), \qquad 0 < x < 1, \\ y(0) - y'(0) = -1, \\ y(1) + y'(1) = -1, \end{cases} \tag{37}$$

where

$$f(x) = x(2x-1)(x-1) - x(x-1) - 2x - 1 - \frac{1}{12}x^4(3x-4).$$

The true solution is $y(x) = x(1-x)$. The absolute error functions $E_N(x)$ for $N = 15$ and $N = 20$ are plotted in Figure 2. For exploring the dependence of error of the solutions on the parameter N, and to have an overview of the rate of convergence, we apply the presented method on Examples 1, 2, and 3 for various values of N. The results are summarized in Figure 3, where we plot the maximum absolute errors defined as:

$$e_N = \max\{|y_N(x) - y(x)|, \quad 0 < x < 1\}.$$

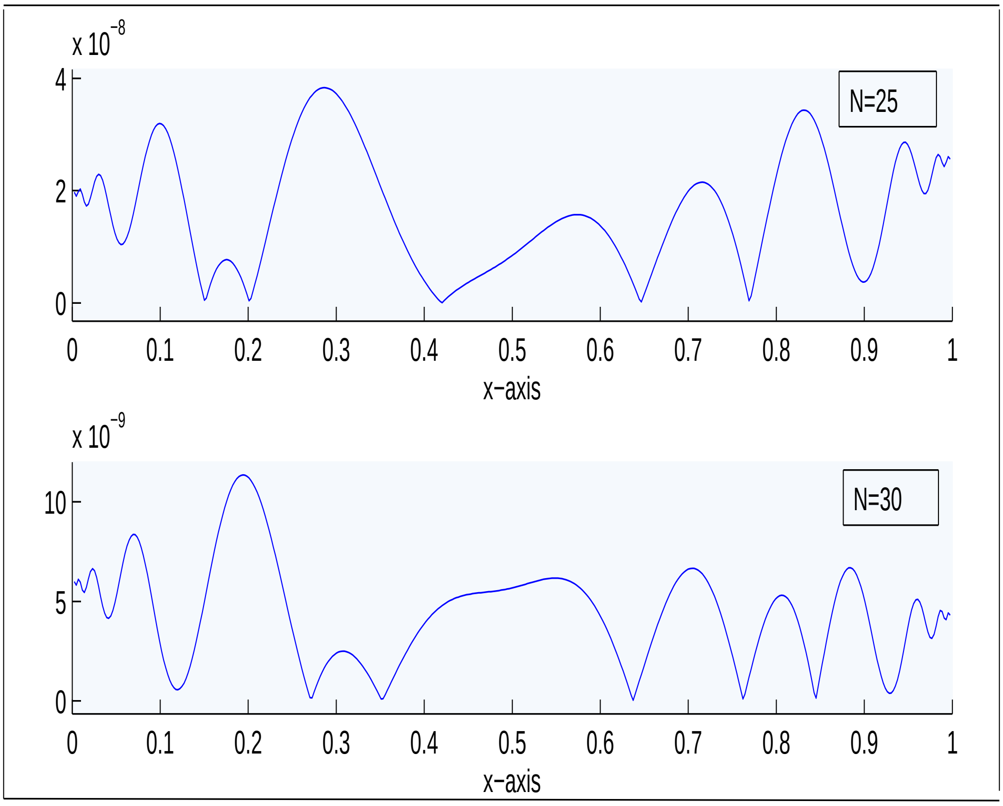

FIGURE 1. Plot of $E_N(x)$ with $N = 25$ and $N = 30$ for Example 2.

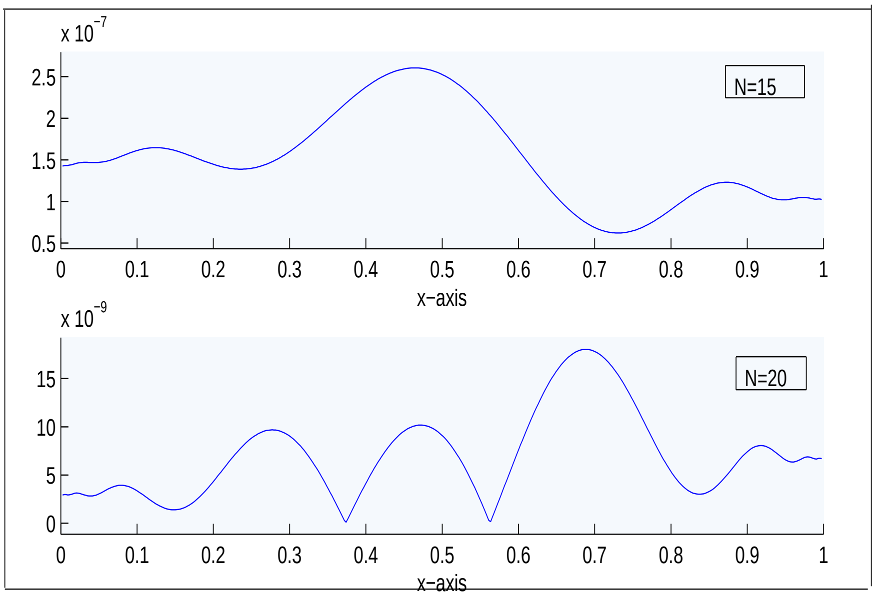

FIGURE 2. Plot of $E_N(x)$ with $N = 15$ and $N = 20$ for Example 3.

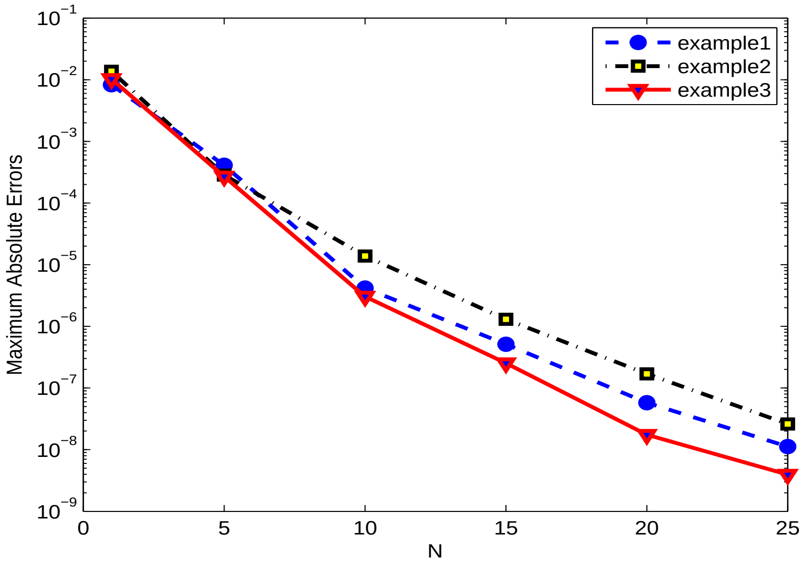

FIGURE 3. Comparison of maximum absolute errors e_N for different values of N for Examples 1,2,3.

According to these experiments, we find that the presented method provides very accurate results even for small N. Also if N increases, then the errors become smaller quickly.

5. Conclusion

The sinc-collocation method is used to solve the Duffing equation involving both integral and non-integral forcing terms. Properties of the sinc function are utilized to reduce the computation of this problem to some algebraic equations. Several examples are given and the numerical results demonstrate the reliability and efficiency of the new method proposed in the current paper for solving this type of problem.

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