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THEORY AND SIMULATION OF POLYCENTRIC CURVES AND UNARY GEARS

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Lucrarea prezintă curbele generate de traiectoria unui punct rigid, aflat pe o dreaptă, care se rostogolește, fără alunecare, pe cazuri particulare de curbe-bază. În cazul rulării pe poligoane, şi numai în cazul curbelor de tip poligon, se poate utiliza şi interiorul acestuia. Rezultatul acestui mod de generare este o evolventă (involută) având forma unei curbe închise. În legătură cu această clasă evolvente (involute), în această lucrare, autorii propun utilizarea noțiunii denumite curbe gevolvente. Totodată, în lucrare se prezintă principalele tipuri de gevolvente şi proprietățile acestora.

In the present paper is presented the case of the curves generated by any fixed point on line, which rolls, without slipping, over particular basic curves. In the case of rolling over polygons, and strictly necessary in the case of curved polygon, the interior contour can by used. The result is an evolvent (involute) shaped like a closed curve. Referring to this class of evolvents, the author proposed, in this paper, the term gevolvents. In the paper are presented the main types of gevolvents and some of their properties

Keywords: Evolvent, Gevolvent, Evolute, Unary gears, Simulation, Matlab.

1. Introduction

Huygens had given the definition of evolute and evolvent a long time ago, in 1673, as follows: *The evolute of a plane curve is the locus of the curve's centers of curvature.*

The evolute is, in the same time, the "wrapping curve" of the family of lines perpendicular (normal lines) to the reference curve.

If a line ***L****, rolls (as a tangent), without slipping, along a fixed curve (base-curve)* ***B****, any fixed point* ***P*** *on* ***L*** *is an* **evolvent** *(involute) of* ***B***, Fig. 1.

It can be shown that if ***B*** is the evolute of ***E***, then ***E*** is an evolvent (involute) of ***B***. All involutes of are parallel.

Some observations can be made at this point:

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- Any plane curve has an evolute.
- All evolvents of a plane curve are equidistant (parallels).

There will be presented, next, a few considerations, resulting from an analysis over the problems related to evolutes and evolvents.

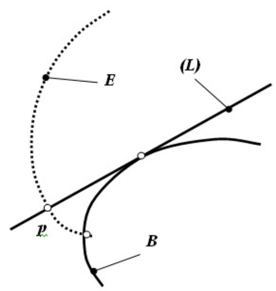


Fig. 1. The generating evolvent (involute)

2. Particular types of evolvents

The chapter titles will be numbered, if necessary, and will be written in small characters (12 pts), bold.

The presentation will be clear and concise and the symbols used therein will be specified in a symbol list (if necessary). In the paper it will be used the measurement units International System. In the paper, there will be no apparatus or installation descriptions.

2.1. The evolvent of circle

Let the particular case of a curve, degenerated into a point ***A***. In this case, according to the last definition, any point ***P*** fixed on line ***L*** which rolls, without slipping, along a *base-curve*, will generate an *evolvent*. It is obvious, because the point is dimensionless, that the result will be a *circle* ***E,*** Fig. 2.

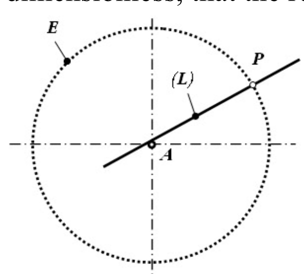


Fig. 2. The evolute of circle is a point

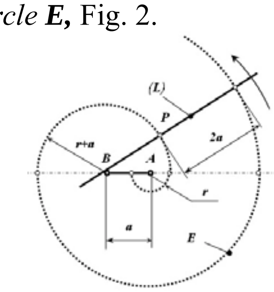


Fig. 3. The evolvent of a line segment

This leads us to the following observations:

- The evolute of circle degenerates into a point. In another words, ***the evolute of circle is a point***, the center of the circle;
- The evolvents of a point form a family of concentric circles.

2.2. The evolvent of a line segment

Let the case of a line segment ***AB*** and a line ***L***, which rolls along this particular curve.

The rolling becomes, in this particular case, a successively “balancing” over the ends of the segment, Fig. 3. This generates an evolvent shaped as a spiral, consisting in series o circle arcs.

The spiral has the property that the distance between any consecutive two spirals is constant and equal to the double of the line segment length ***a***. The line rolling, in this case, is done using successively, the two “sides” of the line segment.

2.3. The evolvent of triangle

Let the triangle ***ABC*** and the line ***L*** that rolls along this particular base curve. The problem has two particular cases:

- When rolling is done on the outside of ***ABC*** contour;
- When the line rolls on the inside of ***ABC*** contour.

When the line rolls along the outside of the ***ABC*** contour, Fig. 4, it generates a spiral, obtained from the successive “balancing" over the edges ***A, B*** and ***C***, then again ***A***, ***B***, ***C*** … etc. In this case, the spiral, also, has the property that the distance between any two successive spirals is constant and equal to ***a+b+c***, Fig. 2, where *a*=[*AB*], *b*=[*BC*], *c*=[*CA*]. The rolling of the line ***L*** can also be done in the inside of the ***ABC*** triangle, Fig. 2 and consists in a successive “balancing” over the edges ***A***, ***B***, ***C***, … , etc. The result is a closed evolvent. We will describe in detail this unusual way of evolvents generation, Fig. 2. First, point ***P*** describes a circle arc, with the center in point ***A***, then, when ***L*** reaches to have the same direction as the segment [***AC***], the balancing center becomes the point ***C***. When ***L*** has the same direction as [***CB***], the center becomes ***B***, until ***L*** coincides with [***BA***], when the balancing center becomes the point ***A*** again and so on. The generated evolvent is a *closed curve*. Indeed, if the generation is started with the initial radius ***r***, around the point ***A***, it is continued with:

$$
\begin{aligned}
&[CP2]=[CP3]=r+b, \text{ and then, successively:}\\
&[BP3]=[BP4]=r+b-a;\\
&[AP4]=[AP5]=r+b-a+c;\\
&[CP5]=[CP6]=r+b-a+c-b=r-a+c;\\
&[BP6]=[BP1]=r-a+c+a=r+c;\\
&[AP1]=[AP2]=r+c-c=r.
\end{aligned} \tag{1}
$$

To completely generate the evolvent of the triangle, the line has to execute a double roll along the contour. This particularity explains why is obtained a closed curve. This because each side of the triangle is used twice in the rolling procedure (first the length of a side increases the radius of the generating point, then decreases it). If the initial radius of generation ***r***, isn't big enough and at least one length of the following segments:

$$[\boldsymbol{AP_1}]=[\boldsymbol{AP_2}],\ [\boldsymbol{BP_3}]=[\boldsymbol{BP_4}],\ [\boldsymbol{CP_5}]=[\boldsymbol{CP_6}] \tag{2}$$

is negative (resulting from the above relation) then the evolvent contains a couple of turning points, forming "fish tail" shapes, Fig. 5.

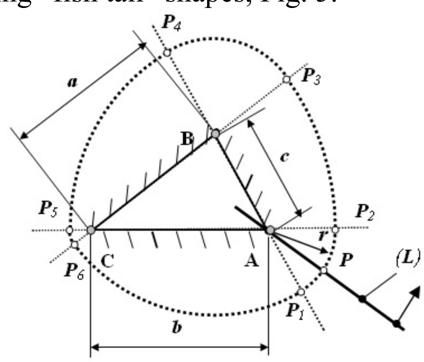


Fig. 4. The evolvent of a triangle

Continuing the analysis, in other cases, there can be obtained 4 turning points evolvents and in a special case, when the initial radius ***r*** is negative (from inside the triangle), it can result a evolute with 6 turning points, Fig. 6;

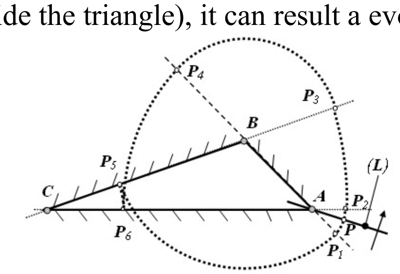


Fig. 5. The evolvent with "fish tail" shapes

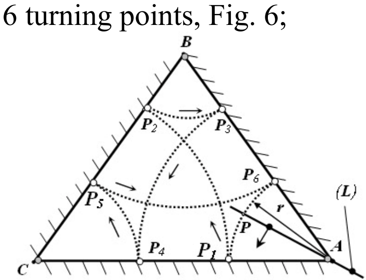


Fig. 6. The evolvent with all "fish tail" shapes

In any point, the normal line to the evolvent intersects again the evolvent also perpendicular. This property remains even when the evolvent has turning points. The distance measured between any two points of the contour, placed, according to the above observations, on the same normal line, is constant – the property of *constant dimension* ***d***, Fig. 7 and 8);

- The family of evolvents for a certain triangle consists only in equidistant (parallel) curves;

- The dimension ***d*** depends on the radius ***r*** the generation starts with and the length of the triangle sides. The mathematical relation is: ***d=2r+b+c-a***, where ***r*** is the initial radius, ***b***, ***c***, are sides adjacent to the edge the generation begins from, and ***a*** is the opposite side to the above mentioned edge;

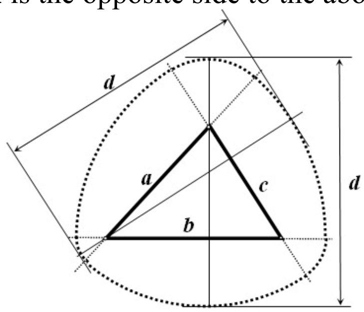


Fig. 7. The property of constant dimension d, example 1

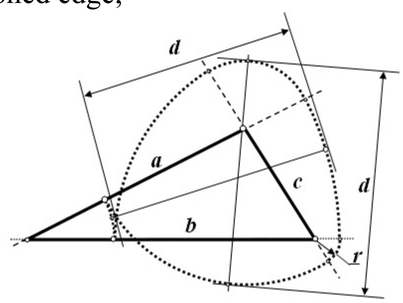


Fig. 8. The property of constant dimension d, example 2

- In the particular case of the evolvent generation corresponding to an equilateral triangle, with ***a=b=c***, it result that ***d=2r+a*** and the obtained evolvent contains only two circle 60^o arcs, having, one, the radius ***r*** and the other the radius ***r+a***. Due to its properties, this curve might find practical applications.

The case ***r=0*** is also known, in the mechanism theory, as the “Reuleaux square” problem, the curvilinear triangle having the property of permanently being in contact with a square having the length of its side equal to ***a***, Fig. 9.

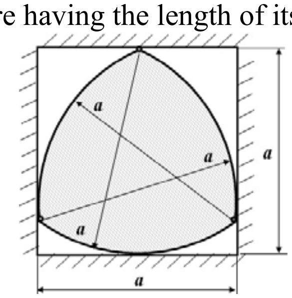


Fig. 9. The “Reuleaux square” problem

This property has found its utility in a special procedure of square holes machining. Based on the observations above presented in this paper, the mentioned procedure has been essentially improved [1,2].

- The property of constant dimension gives these evolvents excellent possibilities for dimensional shape measuring with universal measuring instruments (caliper, micrometer, etc). The variation of the dimension ***d*** indicates the errors from the theoretical shape and the difference from the desired dimension ***d*** indicates an uncompleted generation.

Based on the above observations, some analogies with the properties of the evolvent of circle can be made:

- The property of *constant dimension* is analogous to the property of *constant chord* or *length over n teeth,* which is the synthetic validation indicator of the accuracy of evolventic cylindrical denture machining;

- The cases of generation with turning points are analogous to the cases of *interference*, met in the evolventic denture generation. The effect originates in the generation below the base curve level, or in the second case, below the level of the triangle used as base curve for evolvent generation.

2.4. The evolvent of quadrilateral

Let the case of an ordinary quadrilateral ***ABCD***, Fig. 10. If the rolling takes place in the outside of the quadrilateral, it results a spiral evolvent, with the property that the distance between any two consecutive spirals is equal to the perimeter of the quadrilateral. This generation possibility does not show any other remarkable property. Proceeding the same as in generating the evolvent of having, as base curve, the interior contour of a triangle (see paragraph *2.3*) the result is an evolvent shaped as a closed curve only if the following condition is satisfied: ***a-b+c-d = 0.***

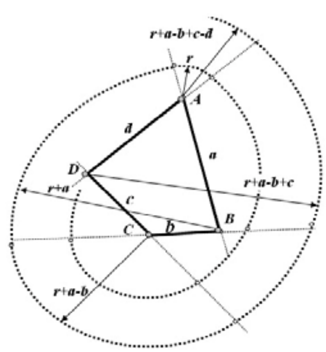


Fig. 10. The outside evolvent of quadrilateral

Some observations can be made about this type of evolvent:

- The generation of these evolvents can be done either starting from points ***A*** or ***C***, or starting from points ***B*** or ***D***. That way it is obtained two families of spiral evolvents, having different directions. For each case, the two families of evolvents contain only equidistant curves.

- The two families of evolvents have the property that the distance between any two consecutive spirals is constant and equal to ***a-b+c-d;***

- If ***a-b+c-d = 0*** the two families of spiral evolvents become two families of closed evolvents. As a result, starting from a base quadrilateral (square, rectangle, parallelogram, rhomb or long rhomb) there can be obtained, by rolling about the inside of these polygons, closed evolvents. In these cases, rolling once, over all edges and sides of the quadrilateral, it generates a complete evolvent. The evolvent obtained in this way does not have the property of *the constant dimension*.

2.5. The evolvent of pentagon

Generating evolvents using a pentagon as base curve is a special case only if rolling is done along the inside of the polygon and only when the polygon is convex, Fig. 11!

In this case, also, the rolling procedure is a bit less easy to understand and will be detailed next. Let ***P*** be the starting point of generation, placed at distance ***r*** from edge ***A***. After balancing over point ***A***, when reaching [***AB***]'s direction, balancing starts over point ***B*** with the radius equal to ***r+a***. When reaching the direction of ***[BC]*** segment, balancing moves around point ***C***, with the radius equal to ***r+a-b***, etc. Continuing the generation in the way presented above, it can be noticed that the convex pentagon is used for rolling twice, same as in the case of using, as base curve, the inside of triangle;

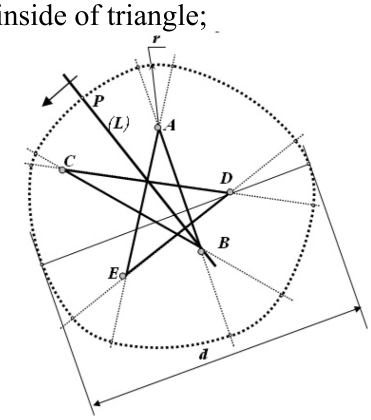


Fig. 11. The evolvent of pentagon

- Calculating the radii used to generate the evolvent of pentagon, it can be shown that, in this case also, same as in generating the interior evolvent of triangle, the radius which ends the generation is ***r+e***, so the resulting evolvent is a closed curve;

- The resulted evolvent has the property of constant dimension ***d=2r+a-b+c-d+e***;

- The family of curves, corresponding to the same convex pentagon used as base curve, is composed only of equidistant curves;

- Depending on the values ***r***, ***a***, ***b***, ***c***, ***d***, ***e***, there can be obtained also discontinuous evolvents, having 2, 4, …, 10 turning points and 1, 2, …, 5 corresponding "fish tails";

- In the particular case of using, for generation, a regular base pentagon, and ***r=0***, there results a evolvent having five sharp edges, which has a property similar to "Reuleaux' square", the property of remaining guided into a concave regular hexagon, having the distance between two parallel sides equal to ***a***, the side of the base pentagon, Fig. 12.

- It can be also noticed, in the same conditions as above, that the generated evolvent for this case stays guided in a square and in a rhomb at the same time, if the distance between two of their parallel sides is equal to ***a***, Fig. 12, dotted representation.

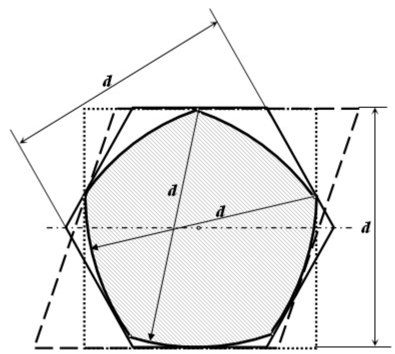


Fig. 12. The property of pentagon

2.6. The evolvents of polygon with more than five sides

The problem of generating evolvents by rolling along the outside or the inside of polygon having more than five sides has the following aspects:

- The generation using the exterior of the base polygon does not show mach interest because it has, as result, opened curves (spirals), having constant distance between any two consecutive spirals;

- The generation using the interior of the base polygon allows obtaining of more interesting evolvents only if the base polygons are convex, and:

a. if the sides number *n* is even, then it should have the following form $n = 2k$, $k = 2, 3, \ldots$. In this case, the evolvents have the properties shown in the section dedicated to the evolvents of quadrilaterals. This situation does not allow generation of evolvents with special properties;

b. if the number n is odd then the generation presents the necessity of using twice all the sides and edges of the base polygon, being obtained closed evolvents with the property of a *constant dimension***.**

2.7. The evolvents of curvilinear polygons inscribed other polygons

In this case, the term "curvilinear polygon" refers to the geometric figure which has, as sides, various curve segments. In this paragraph we consider only those curvilinear polygons having the same edges as another polygon and their curvilinear sides don't have inflexion points, Fig. 13. For these particular geometric figures, used as base curves, we have the following observations:

- in these case there can be obtained only evolvents generated by rolling ***L*** lines only inside the curvilinear polygons;

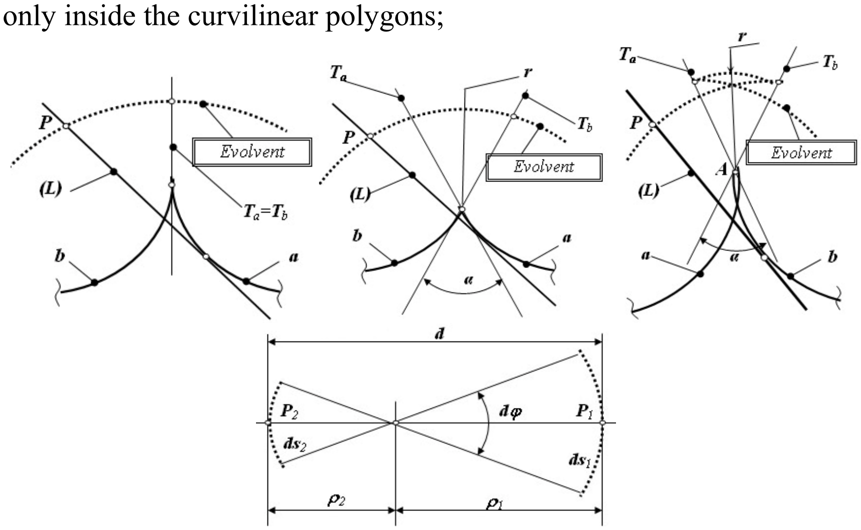


Fig. 13. The evolvents of curvilinear polygons inscribed other polygons

- if, in one of the polygon's corners, the tangents to the adjacent sides form an angle α, Fig. 13 a, the evolvent will contain a circle arc, placed between two tangents T_a and T_b;

- if in one corner the tangents to the adjacent curvilinear sides are one and the same, Fig. 13 b, then the evolvent contains only a transition, without circle arc;

- if the two tangents form a negative angle α, the evolvent has, around the corresponding edge, a segment having two turning points, Fig. 13 c.

The most interesting case of the there presented, is the one corresponding to Fig. 13 b, when the curvilinear polygon's edges are also the edges of a equiangular polygon and its sides are symmetric relative to the median perpendicular of the polygon's sides.

3. Gevolvent's generation

In all usual or particular cases of evolvent, the common element is that the curves are obtained by rolling without friction along a base curve.

The evolvent has the shape of a spiral which starts from the base curve and has the tendency to move away from the base curve, towards infinite. The result of rolling inside the contour of a polygon, may it be ordinary, equilateral or curvilinear, as showed above, in 2.2,…,2.7 has a totally different form. In these cases the evolvent is a *closed curve*.

This is the reason why we considered necessary that, for this type of special evolvents, a new name to be given. We suggest, in this paper, for these curves, the name ***gevolvents***.

By definition, *the gevolvents* are the evolvent is the locus any fixed point on a line which rolls, without slipping, along the inside of a polygon, resulting a *closed gyrating curve.*Related to the *gevolvent curves* there are a few observations:

- the gevolvents of curvilinear polygons having an odd number of sides, have the property of *constant dimension*.

This property is due to the fact that the generator point, placed on the rolling line ***L***, uses the same base curve twice, resulting opposite points;

- based on the above observation, the following theorem can be demonstrated: *the circumference of the gevolvents having the property of constant dimension is equal to πd, when* **d** *is the constant dimension.*

Let the case of a local differential generation area of a gevolvent curve, having the property of *constant dimension*. The lengths of the curvature radii ρ_1 and ρ_2, corresponding to point $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ as follows

$$ds_1 = \rho_1 \cdot d\varphi; \qquad ds_2 = \rho_2 \cdot d\varphi \tag{3}$$

The length ***L*** of the gevolvent circumference can be calculated integrating the relations above, when φ variates from *0* to π:

$$L = \int_0^\pi (ds_1 + ds_2) = \int_0^\pi (\rho_1 + \rho_2) \cdot d\varphi = = \int_0^\pi d \cdot d\varphi = d \cdot \int_0^\pi d\varphi = \pi \cdot d \tag{4}$$

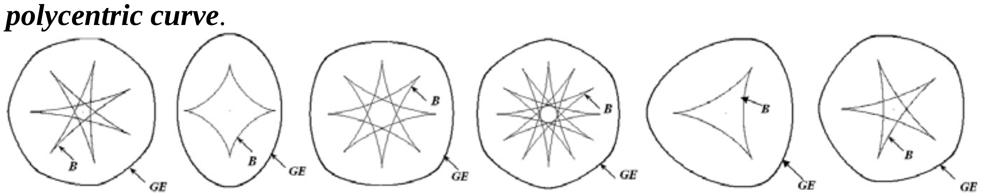
The theorem, concerning the circumference calculation formula, can be also generalized, for the case of the gevolvents not having the property of constant dimension, using in this case, for ***d***, the name of ***mean diameter*** $\boldsymbol{d_m}$:

$$L = \pi \cdot d_m \quad (5)$$

where: d_m is the average of the minimal and maximal dimensions, measured between two parallel lines, tangent to the closed gevolvent contours.

A special class of gevolvent is obtained when the base curvilinear polygon are hypocycloid result in the following conditions: the radius of the base circle is equal to ***en*²**, the roller circle radius is equal to ***en (n-1)/2***, where ***n*** is a natural number ≥***1*** and ***e*** (*the eccentricity*) is a positive real number. For ***n = 2***, ***3***, ***4*** results gevolvents having the shape as shown in Figure 14.

The study indicated in [3] suggests, for this gevolvents, the name of ***polycentric curve***.



a) b) c) d) e) f)

Fig. 14. The polycentric curves (Legend: GE = gevolvent; B = base curve)

Fig. 15. Polycentric gear simulation i= +-3/4 a

Fig. 16. External polycentric unary gear simulation , i=+1

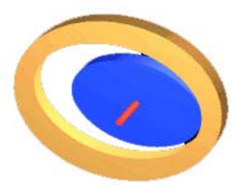


Fig. 17. Internal unary gear simulation, i=+1

Fig. 18. External circular gear simulation , i=+1 a

Fig. 19. External circular gear simulation , i=+1 b

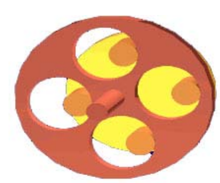


Fig. 20. Internal circular gear simulation , i=+1 a

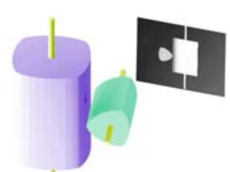


Fig. 21. Polycentric gear simulation i= +-3/4 (with plan view of the gear)

4. Unary and polycentric gears simulation

In figures 15 to 21 the simulation of the unary and polycentric gears are presented using the original theory presented in the paper. The designs of the presented simulations were developed using Matlab and 3D Studio Max software. More simulations were developed for research activities and also are used in didactical activities in order to understand better the proposed gear functioning. Simulation results show effectiveness of the proposed trajectory generation methods. These trajectories can be used for mobile robots, industrial robots and educational platforms planning and can be created algorithms for industrial mechatronics systems. In figure 22 are presented some applications of the described curves.

Fig. 22. Transmission parts realized using unary and polycentric gears

5. Conclusions

The above-presented theoretical aspects are only the beginning of the research concerning this type of special evolvents, called gevolvents. The importance of these curves is equally theoretical as practical, due to the possibility they offer to implement new category of surfaces having the section shaped as a gevolvent curve. The possibility of implementing these surfaces to computer numerical control machines - CNC, will surly lead to an extended use in a large series of technical applications in mechatronics and robotics field.

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