EXISTENCE, UNIQUENESS AND SUCCESSIVE APPROXIMATIONS FOR (λ, ψ)-HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

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The focus of this paper is on investigating a particular type of nonlinear (λ, ψ)-Hilfer fractional differential equations, and analyzing their existence results. Our approach involves utilizing Banach's fixed point theorem, and we also explore the global convergence of successive approximations to provide additional insights into the topic. To further illustrate our findings, we provide some examples that supplement our main results.

Keywords: Implicit differential equations, fractional differential equations, (λ, ψ)-Hilfer fractional derivative, existence, fixed point, global convergence, successive approximations.

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1. Introduction

Fractional calculus extends differentiation and integration to non-integer orders, gaining attention in theoretical studies and practical applications across research domains. Its versatility has made it a crucial tool in the field. Recently, there has been a significant increase in research on fractional calculus, exploring various outcomes under different conditions and forms of fractional differential equations and inclusions. For more details on the applications of fractional calculus, the reader is directed to the books of Herrmann [14], Hilfer [15], Kilbas *et al.* [16] and Samko *et al.* [29]. Agrawal [4] introduced some generalizations of fractional integrals and derivatives and presented some of their properties. In [5, 6], Benchohra *et al.* demonstrated the existence, uniqueness, and stability results for various classes of problems with different conditions with some form of extension of the well-known Hilfer fractional derivative which unifies the Riemann-Liouville and Caputo fractional derivatives.

In a recent publication [11], Diaz introduced novel definitions for the special functions λ-gamma and λ-beta. Those interested can find more information in other sources such as [9, 18, 19]. Sousa *et al.* presented the ψ-Hilfer fractional derivative in another work [33], highlighting important properties related to this type of fractional operator. Further insights and results based on this operator can be explored in papers like [3, 32] and their references. Inspired by the cited papers, we have introduced a new extension of the renowned Hilfer fractional derivative [28]. This extension, called the λ-generalized ψ-Hilfer fractional

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derivative, enabled us to establish a generalized version of Gr"onwall's lemma and explore various types of Ulam stability. Additionally, we have thoroughly investigated qualitative and quantitative results for different classes of fractional differential problems [17,21–27], all made possible by this new generalized fractional operator. More details can be found in [5,6].

Several research studies have investigated the convergence of successive approximations for nonlinear functional equations and the global convergence of successive approximations for functional differential equations [1,2,30,31]. Browder [7] established a generalization of the classical Picard-Banach contraction principle, utilizing the convergence of successive approximations in 1968. In a similar vein, Chen [8] employed the method of successive approximations to analyze the existence of solutions for functional integral equations in 1981. Czlapiiǹski [10] investigated the global convergence of successive approximations of the Darboux problem for partial functional differential equations with infinite delay, while Faina [12] studied the generic property of global convergence of successive approximations for functional differential equations with infinite delay.

Motivated by the aforementioned publications, in this paper, we study the following problem:

$$\begin{cases} \left({}_{k}^{H}\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}\right)(\delta) = \varsigma\left(\delta, \mathfrak{w}(\delta), \left({}_{k}^{H}\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}\right)(\delta)\right), \quad \delta \in \nabla := (\theta_1, \theta_2], \\ \left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}\right)(\theta_1^+) = \mathfrak{w}_{\theta_1}, \end{cases} \tag{1.1}$$

where ${}_{k}^{H}\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}$ and $\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}$ are, respectively, the λ-generalized ψ-Hilfer fractional derivative of order $\zeta_1 \in (0, \lambda)$ and type $\zeta_2 \in [0, 1]$, and λ-generalized ψ-fractional integral of order $\lambda(1 - \zeta_3)$, where $\zeta_3 = \frac{1}{\lambda}(\zeta_2(\lambda - \zeta_1) + \zeta_1)$, $\lambda > 0$, $\varsigma : [\theta_1, \theta_2] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions, and $\mathfrak{w}_{\theta_1} \in \mathbb{R}$.

This paper is structured as follows: In Section 2, we introduce the notations and offer an overview of the (λ, ψ)-Hilfer fractional derivatives that we will utilize throughout the manuscript. In Section 3, we present an existence result of the problem (**1**.1) based on Banach's fixed point theorem. In Section 4, we examine a result on the global convergence of successive approximations. Finally, in the last section, we provide various examples to reinforce the obtained results.

2. Preliminaries

First, we will present the weighted spaces, notations, definitions, and preliminary concepts that will be used in this paper. Let $0 < \theta_1 < \theta_2 < \infty$, $\nabla = [\theta_1, \theta_2]$, and let $\zeta_1 \in (0, \lambda)$, $\varrho \in [0, 1]$, $\lambda > 0$ and $\zeta_3 = \frac{1}{\lambda}(\varrho(\lambda - \zeta_1) + \zeta_1)$.

By $C(\nabla, \mathbb{R})$ we denote the Banach space of all continuous functions from ∇ into $\mathbb{R}$ with the norm

$$\|\mathfrak{w}\|_\infty = \sup\{|\mathfrak{w}(\delta)| : \delta \in \nabla\}.$$

Let $AC^\beta(\nabla, \mathbb{R})$ and $C^\beta(\nabla, \mathbb{R})$ be the spaces of β-times absolutely continuous and β-times continuously differentiable functions on ∇, respectively.

Consider the weighted Banach space

$$C_{\zeta_3;\psi}(\nabla) = \left\{\mathfrak{w} : (\theta_1, \theta_2] \to \mathbb{R} : \delta \to \Phi_{\zeta_3}^{\psi}(\delta, \theta_1)\mathfrak{w}(\delta) \in C(\nabla, \mathbb{R})\right\},$$

where $\Phi_{\zeta_3}^{\psi}(\delta, \theta_1) = (\psi(\delta) - \psi(\theta_1))^{1-\zeta_3}$, with the norm

$$\|\mathfrak{w}\|_{C_{\zeta_3;\psi}} = \sup_{\delta \in \nabla}\left|\Phi_{\zeta_3}^{\psi}(\delta, \theta_1)\mathfrak{w}(\delta)\right|,$$

and

$$C^{\beta}_{\zeta_3;\psi}(\nabla)=\left\{\mathfrak{w}\in C^{\beta-1}(\nabla):\mathfrak{w}^{(\beta)}\in C_{\zeta_3;\psi}(\nabla)\right\},\beta\in\mathbb{N},$$
$$C^{0}_{\zeta_3;\psi}(\nabla)=C_{\zeta_3;\psi}(\nabla),$$

with the norm

$$\|\mathfrak{w}\|_{C^{\beta}_{\zeta_3;\psi}}=\sum_{i=0}^{\beta-1}\|\mathfrak{w}^{(i)}\|_{\infty}+\|\mathfrak{w}^{(\beta)}\|_{C_{\zeta_3;\psi}}.$$

Consider the space $X^p_\psi(\theta_1,\theta_2)$ of those real-valued Lebesgue measurable functions μ on $[\theta_1,\theta_2]$ with $\|\mu\|_{X^p_\psi}<\infty$, and the norm

$$\|\mu\|_{X^p_\psi}=\left(\int_{\theta_1}^{\theta_2}\psi'(\delta)|\mu(\delta)|^p d\delta\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[\theta_1,\theta_2]$ where ψ' is continuous on $[\theta_1,\theta_2]$ and $1\le p\le\infty$.

Definition 2.1 ([11]). *The λ-gamma function is given by*

$$\Gamma_\lambda(\delta)=\int_0^\infty s^{\delta-1}e^{-\frac{s^\lambda}{\lambda}}ds,\delta>0.$$

When $\lambda\to1$ then $\Gamma(\delta)=\Gamma_\lambda(\delta)$, and some other useful relations are $\Gamma_\lambda(\delta)=\lambda^{\frac{\delta}{\lambda}-1}\Gamma\left(\frac{\delta}{\lambda}\right)$, $\Gamma_\lambda(\delta+\lambda)=\delta\Gamma_\lambda(\delta)$ and $\Gamma_\lambda(\lambda)=1$. Moreover, the λ-beta function is given as

$$B_\lambda(\delta,\xi)=\frac{1}{\lambda}\int_0^1 s^{\frac{\delta}{\lambda}-1}(1-s)^{\frac{\xi}{\lambda}-1}ds,$$

so that $B_\lambda(\delta,\xi)=\frac{1}{\lambda}B\left(\frac{\delta}{\lambda},\frac{\xi}{\lambda}\right)$ and $B_\lambda(\delta,\xi)=\frac{\Gamma_\lambda(\delta)\Gamma_\lambda(\xi)}{\Gamma_\lambda(\delta+\xi)}$.

Now, we give the definition to the integral fractional operator used throughout this paper and some of its properties.

Definition 2.2 (λ-generalized ψ-fractional integral [20]). *Let $\mu\in X^p_\psi(\theta_1,\theta_2)$, $\psi(\delta)>0$ be an increasing function on $(\theta_1,\theta_2]$ and $\psi'(\delta)>0$ be continuous on (θ_1,θ_2). The generalized λ-fractional integral operators of a function μ of order $\zeta_1>0$ is defined by:*

$$\mathfrak{I}^{\zeta_1,\lambda;\psi}_{\theta_1+}\mu(\delta)=\int_{\theta_1}^{\delta}\bar{\Phi}^{\lambda,\psi}_{\zeta_1}(\delta,s)\psi'(s)\mu(s)ds,$$

with $\lambda>0$ and $\bar{\Phi}^{\lambda,\psi}_{\zeta_1}(\delta,s)=\dfrac{(\psi(\delta)-\psi(s))^{\frac{\zeta_1}{\lambda}-1}}{\lambda\Gamma_\lambda(\zeta_1)}$.

Theorem 2.1 ([23, 24]). *Let $\mu\in X^p_\psi(\theta_1,\theta_2)$, $\zeta_1>0$ and $\lambda>0$. Then $\mathfrak{I}^{\zeta_1,\lambda;\psi}_{\theta_1+}\mu\in C([\theta_1,\theta_2],\mathbb{R})$.*

Lemma 2.1 ([23,24]). *Let $\zeta_1>0$, $\varrho>0$ and $\lambda>0$. Then, the semigroup properties that follow are met:*

$$\mathfrak{I}^{\zeta_1,\lambda;\psi}_{\theta_1+}\mathfrak{I}^{\varrho,\lambda;\psi}_{\theta_1+}\mu(\delta)=\mathfrak{I}^{\zeta_1+\varrho,\lambda;\psi}_{\theta_1+}\mu(\delta)=\mathfrak{I}^{\varrho,\lambda;\psi}_{\theta_1+}\mathfrak{I}^{\zeta_1,\lambda;\psi}_{\theta_1+}\mu(\delta).$$

Lemma 2.2 ([23,24]). *Let $\zeta_1,\varrho>0$ and $\lambda>0$. Then, we get*

$$\mathfrak{I}^{\zeta_1,\lambda;\psi}_{\theta_1+}\bar{\Phi}^{\lambda,\psi}_{\varrho}(\delta,\theta_1)=\bar{\Phi}^{\lambda,\psi}_{\zeta_1+\varrho}(\delta,\theta_1).$$

Theorem 2.2 ([23, 24]). *Let* $0 < \theta_1 < \theta_2 < \infty, \zeta_1, \varrho > 0, 0 \leq \zeta_3 = \frac{1}{\lambda}(\varrho(\lambda - \zeta_1) + \zeta_1) < 1$, $\lambda > 0$ *and* $\mathfrak{w} \in C_{\zeta_3;\psi}(\nabla)$. *If* $\dfrac{\zeta_1}{\lambda} > 1 - \zeta_3$, *then*

$$\left(\mathfrak{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\mathfrak{w}\right)(\theta_1) = \lim_{\delta\to\theta_1^+}\left(\mathfrak{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\mathfrak{w}\right)(\delta) = 0.$$

Definition 2.3 (λ-generalized ψ-Hilfer derivative [23, 24]). *Let* $\alpha - 1 < \dfrac{\zeta_1}{\lambda} \leq \alpha$ *with* $\alpha \in \mathbb{N}$, $-\infty \leq \theta_1 < \theta_2 \leq \infty$ *and* $\mu, \psi \in C^{\alpha}([\theta_1, \theta_2], \mathbb{R})$ *where* ψ *is increasing and* $\psi'(\delta) \neq 0$, *for all* $\delta \in [\theta_1, \theta_2]$. *The* λ*-generalized* ψ*-Hilfer fractional derivatives* ${}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}(\cdot)$ *of a function* μ *of order* ζ_1 *and type* $0 \leq \varrho \leq 1$, *with* $\lambda > 0$ *is given by:*

$$\begin{aligned}{}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mu(\delta) &= \left(\mathfrak{J}_{\theta_1+}^{\varrho(\lambda\alpha-\zeta_1),\lambda;\psi}\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^{\alpha}\left(\lambda^{\alpha}\mathfrak{J}_{\theta_1+}^{(1-\varrho)(\lambda\alpha-\zeta_1),\lambda;\psi}\mu\right)\right)(\delta)\\ &= \left(\mathfrak{J}_{\theta_1+}^{\varrho(\lambda\alpha-\zeta_1),\lambda;\psi}\delta_{\psi}^{\alpha}\left(\lambda^{\alpha}\mathfrak{J}_{\theta_1+}^{(1-\varrho)(\lambda\alpha-\zeta_1),\lambda;\psi}\mu\right)\right)(\delta),\end{aligned}$$

where $\delta_{\psi}^{\alpha} = \left(\dfrac{1}{\psi'(\delta)}\dfrac{d}{d\delta}\right)^{\alpha}$.

Lemma 2.3 ([23, 24]). *Let* $\delta > \theta_1$, $0 < \dfrac{\zeta_1}{\lambda} < 1, 0 \leq \varrho \leq 1, \lambda > 0$. *Then for* $0 < \zeta_3 < 1; \zeta_3 = \frac{1}{\lambda}(\varrho(\lambda - \zeta_1) + \zeta_1)$, *we have*

$$\left[{}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\left(\Phi_{\zeta_3}^{\psi}(s, \theta_1)\right)^{-1}\right](\delta) = 0.$$

Theorem 2.3 ([23, 24]). *If* $\mu \in C_{\zeta_3;\psi}^{\alpha}[\theta_1, \theta_2], \alpha - 1 < \dfrac{\zeta_1}{\lambda} < \alpha$, $0 \leq \varrho \leq 1$, *where* $\alpha \in \mathbb{N}$ *and* $\lambda > 0$, *then*

$$\left(\mathfrak{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\ {}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mu\right)(\delta) = \mu(\delta) - \sum_{i=1}^{\alpha}\frac{(\psi(\delta) - \psi(\theta_1))^{\zeta_3-i}}{\lambda^{i-\alpha}\Gamma_k(\lambda(\zeta_3 - i + 1))}\left\{\delta_{\psi}^{\alpha-i}\left(\mathfrak{J}_{\theta_1+}^{\lambda(\alpha-\zeta_3),\lambda;\psi}\mu(\theta_1)\right)\right\},$$

where

$$\zeta_3 = \frac{1}{\lambda}\left(\varrho(\lambda\alpha - \zeta_1) + \zeta_1\right).$$

If $\alpha = 1$, *we have*

$$\left(\mathfrak{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\ {}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mu\right)(\delta) = \mu(\delta) - \frac{(\psi(\delta) - \psi(\theta_1))^{\zeta_3-1}}{\Gamma_k(\varrho(\lambda - \zeta_1) + \zeta_1)}\mathfrak{J}_{\theta_1+}^{(1-\varrho)(\lambda-\zeta_1),\lambda;\psi}\mu(\theta_1).$$

Lemma 2.4 ([23, 24]). *Let* $\zeta_1 > 0, 0 \leq \varrho \leq 1$, *and* $\mathfrak{w} \in C_{\zeta_3;\psi}^{1}(\nabla)$, *where* $\lambda > 0$. *Then for* $\delta \in (\theta_1, \theta_2]$, *we have*

$$\left({}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\ \mathfrak{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\mathfrak{w}\right)(\delta) = \mathfrak{w}(\delta).$$

3. Existence of Solutions

Initially, we present the following theorem in order to convert our system (**1**.1) into a fractional integral equation.

Theorem 3.1. *Let* $\zeta_3 = \dfrac{\varrho(\lambda - \zeta_1) + \zeta_1}{\lambda}$, *where* $\lambda > 0$, $0 < \zeta_1 < \lambda, 0 \leq \varrho \leq 1$, *and let* $\varphi(\cdot) \in C(\nabla, \mathbb{R})$. *The function* $\mathfrak{w}$ *satisfies the initial value problem for* λ*-generalized* ψ*-Hilfer fractional differential equations:*

$$\left({}^{H}_{k}\mathfrak{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mathfrak{w}\right)(\delta) = \varphi(\delta), \quad \delta \in (\theta_1, \theta_2], \tag{3.1}$$

$$\left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}\right)(\theta_1^+)=\mathfrak{w}_{\theta_1},\tag{3.2}$$

if and only if it verifies the following integral equation:

$$\mathfrak{w}(\delta)=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)}+\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta),\quad \delta\in(\theta_1,\theta_2].\tag{3.3}$$

Proof. Assume that $\mathfrak{w}$ satisfies the equations (**3**.1)-(**3**.2). We apply $\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}(\cdot)$ on both sides of equation (**3**.1) to obtain

$$\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\ {}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mathfrak{w}\right)(\delta)=\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta),$$

and using Theorem **2**.3, we get

$$\mathfrak{w}(\delta)=\frac{\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}(\theta_1)}{\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)\Gamma_k(\lambda\zeta_3)}+\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta).\tag{3.4}$$

From the initial condition (**3**.2), we get

$$\begin{aligned}\mathfrak{w}(\delta)&=\frac{\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}_{\theta_1}}{\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)\Gamma_k(\lambda\zeta_3)}+\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta)\\&=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)}+\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta).\end{aligned}$$

For the converse, let us now prove that if $\mathfrak{w}$ satisfies equation (**3**.3), then it satisfies (**3**.1)-(**3**.2). We apply the operator ${}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}(\cdot)$ on equation (**3**.3) to get

$$\left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mathfrak{w}\right)(\delta)={}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\left(\frac{\mathfrak{w}_{\theta_1}}{\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)\Gamma_k(\lambda\zeta_3)}\right)+\left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\varrho;\psi}\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta).$$

Using Lemma **2**.3 and Lemma **2**.4, we get (**3**.1). Now we apply the operator $\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}(\cdot)$ to equation (**3**.3), to obtain

$$\left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}\right)(\delta)=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)}\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\left(\frac{1}{\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)}\right)+\left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta).$$

Now, using Lemma **2**.1 and **2**.2, we get

$$\left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3),\lambda;\psi}\mathfrak{w}\right)(\delta)=\mathfrak{w}_{\theta_1}+\left(\mathcal{J}_{\theta_1+}^{\lambda(1-\zeta_3)+\zeta_1,\lambda;\psi}\varphi\right)(\delta).\tag{3.5}$$

Using Theorem **2**.2 with $\delta\to\theta_1$, we obtain (**3**.2). This completes the proof. □

As a consequence of Theorem **3**.1, we have the following result:

Lemma 3.1. *Let* $\zeta_3=\dfrac{\zeta_2(\lambda-\vartheta)+\vartheta}{\lambda}$ *where* $0<\vartheta<\lambda$ *and* $0\le\zeta_2\le1$. *Then* $\mathfrak{w}\in C_{\zeta_3;\psi}(\nabla)$ *satisfies the system* (**1**.1) *if and only if* $\mathfrak{w}$ *is the fixed point of the operator* $\mathcal{H}:C_{\zeta_3;\psi}(\nabla)\to C_{\zeta_3;\psi}(\nabla)$ *defined by:*

$$\mathcal{H}(\mathfrak{w})(\delta)=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)}+\left(\mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi}\varphi\right)(\delta),\tag{3.6}$$

where $\varphi\in C_{\zeta_3;\psi}(\nabla)$ *such that* $\varphi(\delta)=\varsigma(\delta,\mathfrak{w}(\delta),\varphi(\delta))$.

We may employ Theorem **2**.1 to easily demonstrate that for $\mathfrak{w}\in C_{\zeta_3;\psi}(\nabla)$, we have $\mathcal{H}(\mathfrak{w})\in C_{\zeta_3;\psi}(\nabla)$, where $\mathcal{H}$ is the operator defined in (**3**.6).

The hypotheses:

(Cd_1) The function $\varsigma:\nabla\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$; is continuous.

(Cd_2) There exist constants $\imath > 0$ and $0 < \jmath < 1$ such that

$$|\varsigma(\delta, \mathfrak{w}, \mathfrak{w}_1) - \varsigma(\delta, \Im, \Im_1)| \leq \imath|\mathfrak{w} - \Im| + \jmath|\mathfrak{w}_1 - \Im_1|$$

for any $\mathfrak{w}, \mathfrak{w}_1, \Im, \Im_1 \in \mathbb{R}$ and $\delta \in \nabla$.

Remark 3.1. *We note that for any $\mathfrak{w}, \Im \in \mathbb{R}$, and each $\delta \in \nabla$, hypothesis* (Cd_1) *implies that*

$$|\varsigma(\delta, \mathfrak{w}, \Im)| \leq \imath|\mathfrak{w}| + \jmath|\Im| + \varsigma^*,$$

where $\varsigma^* = \sup_{\delta\in\nabla} \varsigma(\delta, 0, 0)$.

We can now declare and demonstrate our existence result for problem (**1**.1). The first result is based on Banach's fixed point theorem [13].

Theorem 3.2. *Suppose that* (Cd_1) *and* (Cd_2) *hold. If*

$$\ell := \frac{\imath\Gamma_\lambda(\lambda\zeta_3)\left(\psi(\theta_2) - \psi(\theta_1)\right)^{\frac{\zeta_1}{\lambda}}}{(1-\jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)} < 1, \tag{3.7}$$

then problem (**1**.1) *has a unique solution on* ∇.

Proof. Consider the operator $\mathcal{H}$ defined in (**3**.6). Let $\mathfrak{w}, \Im \in C_{\zeta_3;\psi}(\nabla)$. Then, for $\delta \in \nabla$ we have

$$|\mathcal{H}(\mathfrak{w})(\delta) - \mathcal{H}(\Im)(\delta)| \leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta} \left(\psi(\delta) - \psi(s)\right)^{\frac{\zeta_1}{\lambda}-1} \psi'(s)|\varphi_1(s) - \varphi_2(s)|ds$$

where $\varphi_1, \varphi_2 \in C_{\zeta_3;\psi}(\nabla)$ such that

$$\varphi_1(\delta) = \varsigma\left(\delta, \mathfrak{w}(\delta), \varphi_1(\delta)\right),$$
$$\varphi_2(\delta) = \varsigma\left(\delta, \Im(\delta), \varphi_2(\delta)\right).$$

By hypothesis (Cd_2) we have

$$|\varphi_1(s) - \varphi_2(s)| \leq \frac{\imath}{1-\jmath}|\mathfrak{w}(\delta) - \Im(\delta)|.$$

Therefore, for each $\delta \in \nabla$

$$\begin{aligned} |\mathcal{H}(\mathfrak{w})(\delta) - \mathcal{H}(\Im)(\delta)| &\leq \frac{\imath}{(1-\jmath)\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta} \left(\psi(\delta) - \psi(s)\right)^{\frac{\zeta_1}{\lambda}-1} \psi'(s)|\mathfrak{w}(s) - \Im(s)|ds \\ &\leq \frac{\imath\|\mathfrak{w} - \Im\|_{C_{\zeta_3;\psi}}}{(1-\jmath)} \mathcal{J}_{\theta_1+}^{\zeta_1,\lambda;\psi} \left(\psi(\delta) - \psi(\theta_1)\right)^{\zeta_3-1}. \end{aligned}$$

By Lemma **2**.2, we have

$$|\mathcal{H}(\mathfrak{w})(\delta) - \mathcal{H}(\Im)(\delta)| \leq \left[\frac{\imath\Gamma_\lambda(\lambda\zeta_3)}{(1-\jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)} \left(\psi(\delta) - \psi(\theta_1)\right)^{\frac{\zeta_1+\lambda\zeta_3}{\lambda}-1}\right] \|\mathfrak{w} - \Im\|_{C_{\zeta_3;\psi}}.$$

Hence,

$$\begin{aligned} \left|\left(\psi(\delta) - \psi(\theta_1)\right)^{1-\zeta_3}\left(\mathcal{H}(\mathfrak{w})(\delta) - \mathcal{H}(\Im)(\delta)\right)\right| &\leq \left[\frac{\imath\Gamma_\lambda(\lambda\zeta_3)\left(\psi(\delta) - \psi(\theta_1)\right)^{\frac{\zeta_1}{\lambda}}}{(1-\jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)}\right] \|\mathfrak{w} - \Im\|_{C_{\zeta_3;\psi}} \\ &\leq \left[\frac{\imath\Gamma_\lambda(\lambda\zeta_3)\left(\psi(\theta_2) - \psi(\theta_1)\right)^{\frac{\zeta_1}{\lambda}}}{(1-\jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)}\right] \|\mathfrak{w} - \Im\|_{C_{\zeta_3;\psi}}, \end{aligned}$$

which implies that

$$\|\mathcal{H}\mathfrak{w} - \mathcal{H}\Im\|_{C_{\zeta_3;\psi}} \leq \left[\frac{\imath\Gamma_\lambda(\lambda\zeta_3)\left(\psi(\theta_2) - \psi(\theta_1)\right)^{\frac{\zeta_1}{\lambda}}}{(1-\jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)}\right] \|\mathfrak{w} - \Im\|_{C_{\zeta_3;\psi}}.$$

Hence, we get

$$\|\mathcal{H}\mathfrak{w}-\mathcal{H}\Im\|_{C_{\zeta_3;\psi}}\le \ell\|\mathfrak{w}-\Im\|_{C_{\zeta_3;\psi}}.$$

Consequently, by Banach's fixed point theorem, the operator $\mathcal{H}$ has a unique fixed point, which is the unique solution of our problem (**1**.1) on ∇. □

4. Successive Approximations and Uniqueness Results

This section is devoted to giving the main result of the global convergence of successive approximations of our problem (**1**.1). We will study the solution in $C_{\zeta_3;\psi}(\nabla)$ of our problem.

Set $J_\mu := [\theta_1, \mu\theta_2]$ for any $\mu \in [0,1]$. In what follows, we need the following hypotheses:

(H_1)**:** There exist a constant $\varkappa > 0$ and a continuous function $h : \nabla \times [0,\varkappa]\times[0,\varkappa] \longrightarrow \mathbb{R}_+$, such that $h(\delta,\cdot,\cdot)$ is nondecreasing for all $\delta\in\nabla$ and the inequality

$$|\varsigma(\delta,\mathfrak{w}_1,\bar{\mathfrak{w}_1})-\varsigma(\delta,\mathfrak{w}_2,\bar{\mathfrak{w}_2})|\le h\left(\delta,|\mathfrak{w}_1-\mathfrak{w}_2|,|\bar{\mathfrak{w}_1}-\bar{\mathfrak{w}_2}|\right) \qquad (\mathbf{4}.1)$$

holds for $\delta\in\nabla$ and $\mathfrak{w}_1,\mathfrak{w}_2,\bar{\mathfrak{w}_1},\bar{\mathfrak{w}_2}\in\mathbb{R}$, with $|\mathfrak{w}_1-\mathfrak{w}_2|\le\varkappa$ and $|\bar{\mathfrak{w}_1}-\bar{\mathfrak{w}_2}|\le\varkappa$.

(H_2)**:** $R\equiv 0$ is the only function in $C_{\zeta_3;\psi}(J_\xi,[0,\varkappa])$ which satisfies the integral inequality

$$R(\delta)\le \frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi^{\psi}_{\zeta_3}(\delta,\theta_1)} + \frac{1}{\lambda\Gamma_\lambda(\zeta_1)}\int_{\theta_1}^{\xi\theta_2}\frac{\psi'(s)\varsigma\left(s,R(s),\left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}R\right)(s)\right)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}}ds,$$

with $\mu\le\xi\le1$.

For $\delta\in\nabla$, we define the successive approximations of the problem (**1**.1) as follows:

$$\mathfrak{w}_0(\delta)=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi^{\psi}_{\zeta_3}(\delta,\theta_1)},$$

$$\mathfrak{w}_{\beta+1}(\delta)=\frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi^{\psi}_{\zeta_3}(\delta,\theta_1)} + \frac{1}{\lambda\Gamma_\lambda(\zeta_1)}\int_{\theta_1}^{\delta}\frac{\psi'(s)\varsigma\left(s,\mathfrak{w}_\beta(s),\left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(s)\right)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}}ds.$$

Theorem 4.1. *Assume that the hypotheses (H_1)-(H_2) hold. Then, the successive approximations $\mathfrak{w}_\beta$; $\beta\in\mathbb{N}$ are well defined and converge to the unique solution of the problem* (**1**.1) *uniformly in $C_{\zeta_3;\psi}(\nabla)$.*

Proof. Since the function ς is continuous, then the successive approximations are well defined. Differentiating the two sides of the successive approximations $\mathfrak{w}_\beta$; $\beta\in\mathbb{N}$ by using the λ-generalized ψ-Hilfer fractional derivative of order ζ_1, by Lemma **2**.3, Lemma **2**.4 and Theorem **2**.3, we have

$$\left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_0\right)(\delta)=0,\quad \theta\in\Theta,$$

$$\left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_{\beta+1}\right)(\delta)=\varsigma\left(\delta,\mathfrak{w}_\beta(\delta),\left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta)\right),\quad \theta\in\Theta.$$

And since $\mathfrak{w}_\beta\in C_{\zeta_3;\psi}(\nabla)$, then there exist two constants $\delta_1,\delta_2>0$ such that

$$\|\mathfrak{w}_\beta\|_{C_{\zeta_3;\psi}}\le\delta_1 \text{ and } \|{}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\|_{C_{\zeta_3;\psi}}\le\delta_2.$$

Let $\delta_1, \delta_2 \in \nabla$, $\delta_1 < \delta_2$. Then,

$$\begin{aligned}
&\left| (\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}\, \mathfrak{w}_\beta(\delta_2) - (\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}\, \mathfrak{w}_\beta(\delta_1)\right| \\
&\quad \leq \left| \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta_2} \frac{\psi'(s)\varsigma(s,\mathfrak{w}_{\beta-1}(s),({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1})(s))}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} ds \right. \\
&\qquad \left. - \frac{(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta_1} \frac{\psi'(s)\varsigma(s,\mathfrak{w}_{\beta-1}(s),({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1})(s))}{(\psi(\delta_1)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} ds \right| \\
&\quad \leq \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\delta_1}^{\delta_2} \frac{\psi'(s)|\varsigma(s,\mathfrak{w}_{\beta-1}(s),({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1})(s))|}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} ds \\
&\qquad + \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta_1} \left| \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} - \frac{(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_1)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \right| \psi'(s) \\
&\qquad \times |\varsigma(s,\mathfrak{w}_{\beta-1}(s),({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1})(s))| ds \\
&\quad \leq \sup_{(\delta,\mathfrak{w},\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} \left| (\psi(\delta)-\psi(\theta_1))^{1-\zeta_3}\, \varsigma(\delta,\mathfrak{w},\Im)\right| \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{\lambda\Gamma_\lambda(\zeta_1)} \\
&\qquad \times \int_{\delta_1}^{\delta_2} \frac{\psi'(s)\,(\psi(\delta)-\psi(\theta_1))^{\zeta_3-1}}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} ds \\
&\qquad + \frac{\sup_{(\delta,\mathfrak{w},\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} \left| (\psi(\delta)-\psi(\theta_1))^{1-\zeta_3}\, \varsigma(\delta,\mathfrak{w},\Im)\right|}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta_1} \left| \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \right. \\
&\qquad \left. - \frac{(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_1)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \right| \psi'(s)\,(\psi(\delta)-\psi(\theta_1))^{\zeta_3-1}\, ds.
\end{aligned}$$

By Lemma **2**.2, we have

$$\begin{aligned}
&\left| (\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}\, \mathfrak{w}_\beta)(\delta_2 - (\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}\, \mathfrak{w}_\beta(\delta_1)\right| \\
&\quad \leq \frac{\sup_{(\delta,\mathfrak{w},\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} \left| (\psi(\delta)-\psi(\theta_1))^{1-\zeta_3}\, \varsigma(\delta,\mathfrak{w},\Im)\right|}{\lambda\Gamma_\lambda(\zeta_1+\lambda\zeta_3)} \\
&\qquad \times \Gamma_\lambda(\lambda\zeta_3)\,(\psi(\theta_2)-\psi(\theta_1))^{1-\zeta_3}\,(\psi(\delta_2)-\psi(\delta_1))^{\frac{\zeta_1}{\lambda}} \\
&\qquad + \frac{\sup_{(\delta,\mathfrak{w},\Im)\in\nabla\times[0,\delta_1]\times[0,\delta_2]} \left| (\psi(\delta)-\psi(\theta_1))^{1-\zeta_3}\, \varsigma(\delta,\mathfrak{w},\Im)\right|}{\lambda\Gamma_\lambda(\zeta_1)} \\
&\qquad \times \int_{\theta_1}^{\delta_1} \left| \frac{(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_2)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} - \frac{(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}}{(\psi(\delta_1)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \right| \psi'(s)\,(\psi(\delta)-\psi(\theta_1))^{\zeta_3-1}\, ds.
\end{aligned}$$

As $\delta_1 \longrightarrow \delta_2$ the right hand side of the above inequality tends to zero. On the other hand, we have

$$\begin{aligned}&\left|(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta_2)-(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta_1)\right|\\&\quad\le\Big|(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}\varsigma(\delta_2,\mathfrak{w}_{\beta-1}(\delta_2),({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_{\beta-1})(\delta_2))\\&\qquad-(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}\varsigma(\delta_1,\mathfrak{w}_{\beta-1}(\delta_1),({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_{\beta-1})(\delta_1))\Big|\\&\quad\longrightarrow 0,\ \text{as}\ \delta_1\longrightarrow\delta_2.\end{aligned}$$

Thus,

$$\left|(\psi(\delta_2)-\psi(\theta_1))^{1-\zeta_3}\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta_2)-(\psi(\delta_1)-\psi(\theta_1))^{1-\zeta_3}\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta_1)\right|\longrightarrow 0,$$
$$\text{as}\ \delta_1\longrightarrow\delta_2.$$

As a result, the sequences $\{\mathfrak{w}_\beta(\delta);\ \beta\in\mathbb{N}\}$ and $\left\{\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta);\ \beta\in\mathbb{N}\right\}$ are equicontinuous on ∇.

Let

$$\vartheta:=\{\mu\in[0,1]:\{\mathfrak{w}_\beta(\delta);\ \beta\in\mathbb{N}\}\ \text{converges uniformly on}\ J_\mu\}.$$

If $\vartheta=1$, then we have the global convergence of successive approximations. Suppose that $\tau<1$, then the sequence $\{\mathfrak{w}_\beta(\delta);\ \beta\in\mathbb{N}\}$ converges uniformly on J_ϑ. As this sequence is equicontinuous, it converges uniformly to a continuous function $\tilde{\mathfrak{w}}(\delta)$. In the case that we prove that there exists $\xi\in(\vartheta,1]$ that $\{\mathfrak{w}_\beta(\delta);\ \beta\in\mathbb{N}\}$ converges uniformly on J_ξ, this will yield a contradiction.

Put $\mathfrak{w}(\delta)=\tilde{\mathfrak{w}}(\delta)$ for $\delta\in J_\vartheta$. From (H_1), there exist a constant $\varkappa>0$ and a continuous function $h:\nabla\times[0,\varkappa]\times[0,\varkappa]\longrightarrow\mathbb{R}_+$ ensuring inequality (**4**.1). Also, there exist $\xi\in[\vartheta,1]$ and $\beta_0\in\mathbb{N}$, such that for all $\delta\in J_\xi$ and $\beta,\alpha>\beta_0$, we have

$$|\mathfrak{w}_\beta(\delta)-\mathfrak{w}_m(\delta)|\le\varkappa,$$

and

$$\left|\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta)-\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\alpha\right)(\delta)\right|\le\varkappa.$$

For all $\delta\in J_\xi$, put

$$R^{(\beta,\alpha)}(\delta)=|\mathfrak{w}_\beta(\delta)-\mathfrak{w}_m(\delta)|,$$

$$R_{\overline{\lambda}}(\delta)=\sup_{\beta,\alpha\ge\overline{\lambda}}R^{(\beta,\alpha)}(\delta),$$

$$\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}R^{(\beta,\alpha)}\right)(\delta)=\left|\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_\beta\right)(\delta)-\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}\mathfrak{w}_m\right)(\delta)\right|,$$

and

$$\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}R_{\overline{\lambda}}\right)(\delta)=\sup_{\beta,\alpha\ge\overline{\lambda}}\left({}^{H}_{k}\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+}R^{(\beta,\alpha)}\right)(\delta),$$

Since the sequence $R_{\overline{\lambda}}(\delta)$ is non-increasing, it is convergent to a function $R(\delta)$ for each $\delta\in J_\xi$. From the equicontinuity of $\{R_{\overline{\lambda}}(\delta)\}$, it follows that $\lim_{\overline{\lambda}\to\infty}R_{\overline{\lambda}}(\delta)=R(\delta)$ uniformly on

J_ξ. Furthermore, for $\delta \in J_\xi$ and $\beta, \alpha \geq \overline{\lambda}$, we have

$$\begin{aligned}
R^{(\beta,\alpha)}(\delta) &= |\mathfrak{w}_\beta(\delta) - \mathfrak{w}_m(\delta)| \\
&\leq \sup_{s\in[0,\delta]} |\mathfrak{w}_\beta(s) - \mathfrak{w}_m(s)| \\
&\leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \\
&\quad \times \left| \varsigma\left(s, \mathfrak{w}_{\beta-1}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1}\right)(s)\right) - \varsigma\left(s, \mathfrak{w}_{\alpha-1}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\alpha-1}\right)(s)\right)\right| ds \\
&\leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \\
&\quad \times \left| \varsigma\left(s, \mathfrak{w}_{\beta-1}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1}\right)(s)\right) - \varsigma\left(s, \mathfrak{w}_{\alpha-1}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\alpha-1}\right)(s)\right)\right| ds.
\end{aligned}$$

Then, by inequality (**4**.1), we have

$$\begin{aligned}
&R^{(\beta,\alpha)}(\delta) \\
&\leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} \\
&\times h\left(s, |\mathfrak{w}_{\beta-1}(s) - \mathfrak{w}_{\alpha-1}(s)|, \left|\left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\beta-1}\right)(s) - \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\alpha-1}\right)(s)\right|\right) ds \\
&\leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} h\left(s, R^{(\beta-1,\alpha-1)}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}R^{(\beta-1,\alpha-1)}\right)(s)\right) ds.
\end{aligned}$$

Thus,

$$R_{\overline{\lambda}}(\delta) \leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} h\left(s, R_{\overline{\lambda}-1}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}R_{\overline{\lambda}-1}\right)(s)\right) ds.$$

By the Lebesgue dominated convergence theorem we have

$$R(\delta) \leq \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} h\left(s, R(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}R\right)(s)\right) ds.$$

Then, by (H_2) we get $R \equiv 0$ on J_ξ, which yields that $\lim_{\overline{\lambda}\to\infty} R_{\overline{\lambda}}(\delta) = 0$ uniformly on J_ξ. Thus, $\{\mathfrak{w}_\lambda(\theta)\}_{\lambda=1}^{\infty}$ is a Cauchy sequence on Θ_ξ. Consequently, $\{\mathfrak{w}_{\overline{\lambda}}(\delta)\}_{\overline{\lambda}=1}^{\infty}$ is uniformly convergent on J_ξ, which yields the contradiction.

Also, $\{\mathfrak{w}_{\overline{\lambda}}(\delta)\}_{\overline{\lambda}=1}^{\infty}$ converges uniformly on ∇ to a continuous function $\mathfrak{w}_*(\delta)$. By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{\overline{\lambda}\to\infty} \frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)} + \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} h\left(s, \mathfrak{w}_{\overline{\lambda}}(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_{\overline{\lambda}}\right)(s)\right) ds \\
&= \frac{\mathfrak{w}_{\theta_1}}{\Gamma_k(\lambda\zeta_3)\Phi_{\zeta_3}^{\psi}(\delta,\theta_1)} + \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\delta} \frac{\psi'(s)}{(\psi(\delta)-\psi(s))^{1-\frac{\zeta_1}{\lambda}}} h\left(s, \mathfrak{w}_*(s), \left({}_k^H\mathcal{D}_{\theta_1+}^{\zeta_1,\zeta_2;\psi}\mathfrak{w}_*\right)(s)\right) ds,
\end{aligned}$$

for all $\delta \in \nabla$. This means that $\mathfrak{w}_*$ is a solution of the problem (**1**.1).

Let us now prove the uniqueness result of the problem (**1**.1). Let $\mathfrak{w}_1$ and $\mathfrak{w}_2$ be two solutions of (**1**.1). As above, put

$$\widehat{\vartheta} := \{\mu \in [0,1] : \mathfrak{w}_1(\delta) = \mathfrak{w}_2(\delta) \text{ for } \delta \in J_\mu\},$$

and suppose that $\widehat{\vartheta} < 1$. There exist a constant $\varkappa > 0$ and a comparison function $h : J_{\widehat{\vartheta}} \times [0, \varkappa] \times [0, \varkappa] \longrightarrow \mathbb{R}_+$ verifying inequality (**4**.1). We take $\xi \in (\mu, 1)$ such that

$$|\mathfrak{w}_1(\delta) - \mathfrak{w}_2(\delta)| \le \varkappa,$$

and

$$\left| \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_1 \right)(\delta) - \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_2 \right)(\delta) \right| \le \varkappa.$$

for $\delta \in J_\xi$. Then, for all $\delta \in J_\xi$, we have

$$\begin{aligned} &|\mathfrak{w}_1(\delta) - \mathfrak{w}_2(\delta)| \\ &\le \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta) - \psi(s))^{1-\frac{\zeta_1}{\lambda}}} \\ &\times \left| \varsigma(\tau, \mathfrak{w}_0(s), \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_0 \right)(\tau)) - \varsigma(\tau, \mathfrak{w}_1(\tau), \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_1 \right)(s)) \right| ds \\ &\le \frac{1}{\lambda\Gamma_\lambda(\zeta_1)} \int_{\theta_1}^{\xi\theta_2} \frac{\psi'(s)}{(\psi(\delta) - \psi(s))^{1-\frac{\zeta_1}{\lambda}}} \\ &\times h\left(s, |\mathfrak{w}_0(s) - \mathfrak{w}_1(s)|, \left| \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_0 \right)(s) - \left({}^H_k\mathcal{D}^{\zeta_1,\zeta_2;\psi}_{\theta_1+} \mathfrak{w}_1 \right)(s) \right| \right) ds. \end{aligned}$$

Again, by (H_2) we get $\mathfrak{w}_1 - \mathfrak{w}_2 \equiv 0$ on J_ξ. This gives us $\mathfrak{w}_1 = \mathfrak{w}_2$ on J_ξ, which gives a contradiction. Consequently, $\widehat{\vartheta} = 1$ and the solution of the problem (**1**.1) is unique on ∇. □

5. Some Examples

We give now some examples that illustrate our obtained results throughout the paper.

Example 5.1. *Taking $\zeta_2 \to 0$, $\zeta_1 = \frac{1}{2}$, $\xi = \frac{1}{2}$, $\theta_1 = 0$, $\theta_2 = 1$, $\psi(\delta) = e^{-\delta}$ and $\mathfrak{w}_0 = 0$, we obtain the following problem:*

$$\begin{cases} \left({}^H_1\mathcal{D}^{\frac{1}{2},0;\psi}_{0+} \mathfrak{w} \right)(\delta) = \dfrac{e^{-\delta}}{190\,(1 + |\mathfrak{w}(\delta)|)} + \dfrac{e^{-\delta}}{130\left(1 + \left| \left({}^H_1\mathcal{D}^{\frac{1}{2},0;\psi}_{0+} \mathfrak{w} \right)(\delta) \right| \right)}; \ \delta \in \nabla := [0,1], \\ \left(\mathcal{J}^{\frac{1}{2},1;\psi}_{0+} \mathfrak{w} \right)(0^+) = 0. \end{cases} \tag{5.1}$$

Set

$$\varsigma(\delta, \mathfrak{w}, \Im) = \frac{e^{-\delta}}{190\,(1 + |\mathfrak{w}|)} + \frac{e^{-\delta}}{130\,(1 + |\Im|)}; \ \delta \in [0,1], \ \mathfrak{w}, \Im \in \mathbb{R}.$$

We have

$$C_{\zeta_3;\psi}(\nabla) = C_{\frac{1}{2};\psi}(\nabla) = \left\{ \mathfrak{w} : (0,1] \to \mathbb{R} : \delta \to e^{-\delta}\mathfrak{w}(\delta) \in C(\nabla, \mathbb{R}) \right\}.$$

Since the continuous function $\varsigma \in C_{\frac{1}{2};\psi}(\nabla)$. For any $\mathfrak{w}, \widetilde{\mathfrak{w}}, \mathfrak{w}, \widetilde{\mathfrak{w}} \in \mathbb{R}$, and $\delta \in [0,1]$, we have

$$|\varsigma(\delta, \mathfrak{w}, \Im) - \varsigma(\delta, \widetilde{\mathfrak{w}}, \widetilde{\Im})| \le \frac{1}{190}|\mathfrak{w} - \widetilde{\mathfrak{w}}| + \frac{1}{130}|\Im - \widetilde{\Im}|.$$

Hence hypothesis (Cd$_2$) *is satisfied with*

$$\imath = \frac{1}{190} \quad \text{and} \quad \jmath = \frac{1}{130}.$$

*Next, the condition (**3**.7) is verifies with $\lambda = 1$, $\zeta_3 = \frac{1}{2}$ and $\zeta_1 = \frac{1}{2}$. Indeed,*

$$\frac{\imath\Gamma_\lambda(\lambda\zeta_3)\,(\psi(\theta_2) - \psi(\theta_1))^{\frac{\zeta_1}{\lambda}}}{(1 - \jmath)\Gamma_\lambda(\zeta_1 + \lambda\zeta_3)} = \frac{\frac{1}{190}\Gamma_1(\frac{1}{2})\,(e^2 - 1)^{\frac{1}{2}}}{(1 - \frac{1}{130})\Gamma(1)} < 1.$$

*Some calculations indicate that all of the requirements of Theorem **3**.2 are verified. Thus, (**5**.1) has a unique solution.*

Example 5.2. *Taking $\zeta_2 \to 0$, $\zeta_1 = \frac{1}{2}$, $\xi = \frac{1}{2}$, $\theta_1 = 0$, $\theta_2 = 1$, $\psi(\delta) = \frac{\sqrt{\pi-\delta}}{\pi}$ and $\mathfrak{w}_0 = 0$, we consider the following problem involving the λ-generalized ψ-Hilfer fractional derivative:*

$$\begin{cases} \left({}_1^H\mathfrak{D}_{0+}^{\frac{1}{2},0;\psi}\mathfrak{w}\right)(\delta) = \dfrac{\frac{\sqrt{\pi-\delta}}{\pi}(7e^\delta + \sqrt{2}\delta^7 + 33)}{79e^{\delta+1}\left(1 + |\mathfrak{w}(\delta)| + \left|\left({}_1^H\mathfrak{D}_{0+}^{\frac{1}{2},0;\psi}\mathfrak{w}\right)(\delta)\right|\right)}, \quad \delta \in := [0,\pi], \\ \\ \left(\mathfrak{J}_{0+}^{\frac{1}{2},1;\psi}\mathfrak{w}\right)(0^+) = 0. \end{cases} \tag{5.2}$$

Set

$$\varsigma(\delta, \mathfrak{w}(\delta), ({}_1^H\mathfrak{D}_{0+}^{\frac{1}{2},0;\psi}\mathfrak{w})(\delta)) = \frac{\frac{\sqrt{\pi-\delta}}{\pi}(7e^\delta + \sqrt{2}\delta^7 + 33)}{79e^{\delta+1}\left(1 + |\mathfrak{w}(\delta)| + \left|\left({}_1^H\mathfrak{D}_{0+}^{\frac{1}{2},0;\psi}\mathfrak{w}\right)(\delta)\right|\right)},$$

where $\zeta_1 = \frac{1}{2}$.
We have

$$C_{\zeta_3;\psi}(\nabla) = C_{\frac{1}{2};\psi}(\nabla) = \left\{\mathfrak{w} : (0,\pi] \to \mathbb{R} : \delta \to \frac{\sqrt{\pi-\delta}}{\pi}\mathfrak{w}(\delta) \in C(\nabla, \mathbb{R})\right\}.$$

For each $\mathfrak{w}_1, \bar{\mathfrak{w}_1}, \mathfrak{w}_2, \bar{\mathfrak{w}_2} \in \mathbb{R}$ and $\delta \in [0,\pi]$, we have

$$|\varsigma(\delta, \mathfrak{w}_1, \mathfrak{w}_2) - \varsigma(\delta, \bar{\mathfrak{w}_1}, \bar{\mathfrak{w}_2})| \leq \frac{7e^\delta + \sqrt{2}\delta^7 + 33}{79e^{\delta+1}}\left[|\mathfrak{w}_1 - \bar{\mathfrak{w}_1}| + |\mathfrak{w}_2 - \bar{\mathfrak{w}_2}|\right].$$

Therefore, (H_1) is verified for all $\delta \in [0,\pi]$, $\varkappa > 0$ and the comparison function $h : \nabla \times [0,\varkappa] \times [0,\varkappa] \longrightarrow \mathbb{R}_+$ is defined by:

$$h(\delta, \mathfrak{w}_1, \mathfrak{w}_2) = \frac{7e^\delta + \sqrt{2}\delta^7 + 33}{79e^{\delta+1}}(\mathfrak{w}_1 + \mathfrak{w}_2).$$

Moreover, we have

$$\lim_{\delta_1 \longrightarrow \delta_2} \left(\varsigma(\delta_2, \mathfrak{w}_1, \mathfrak{w}_2) - \varsigma(\delta_1, \mathfrak{w}_1, \mathfrak{w}_2)\right) = 0.$$

*Thus, ς is equicontinuous. Consequently, Theorem **4**.1 means that the successive approximations $\mathfrak{w}_\beta$; $\beta \in \mathbb{N}$, defined by*

$$\mathfrak{w}_0(\delta) = 0, \quad \theta \in [0,\pi],$$

$$\mathfrak{w}_{\beta+1}(\delta) = -\frac{1}{\Gamma(\frac{1}{2})}\int_0^\delta \frac{7e^\delta + \sqrt{2}\delta^7 + 33}{2\pi^2\left(\frac{\sqrt{\pi-\delta}}{\pi} - \frac{\sqrt{\pi-s}}{\pi}\right)^{\frac{1}{2}} 79e^{\delta+1}\left(1 + |\mathfrak{w}(\delta)| + \left|\left({}_1^H\mathfrak{D}_{0+}^{\frac{1}{2},0;\psi}\mathfrak{w}\right)(\delta)\right|\right)}ds,$$

*converges uniformly on $[0,\pi]$ to the unique solution of the problem (**5**.2).*

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