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SOLUTION OF THE FUNCTIONAL EQUATION $f \circ f = g$ FOR NON INJECTIVE g

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Se rezolvă ecuaţia funcţională $f \circ f = g$ în cazul când funcţa cunoscută g are două intervale de monotonie de sens opus. Remarcăm că în cazul clasic funcţia cunoscută g este strict monotonă. Prima secţiune a articolului are un caracter preliminar. În a doua secţiune este formulată problema care este, apoi, soluţionată într-o manieră constructivă.

We solve the functional equation $f \circ f = g$ in case the given function g has two intervals of opposite monotonicity. Notice that in the classical framework the given function g is strictly monotone. The first section has a preliminary character. In the second section the problem is formulated and solved in a constructive manner.

Keywords: homeomorphism, strictly monotone function, generalized inverse, fixed point.

MSC2000: 39B12, 26A48.

1. Preliminary part

Throughout the paper $\mathbb{R}$ will be the set of real numbers.

For $\emptyset \neq A \subset X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$, we shall write $f \downarrow$ on A (respectively $f \uparrow$ on A) to denote the fact that f is strictly decreasing (respectively strictly increasing) on A.

For non empty sets X, Y and injective function $f : X \to Y$, the generalized inverse of f is the function $f^{-1} : f(X) \to X$ given via $f^{-1}(y) = x$, where $x \in X$ is uniquely determined by the condition $f(x) = y$.

For non empty X, $f : X \to X$ and natural number n, we define the function $f^n : X \to X$ as follows. In case $n = 0$, $f^0 = 1_X$, where $1_X(x) = x$, for any $x \in X$. In case $n > 0$, $f^n = f \circ f \circ \ldots \circ f$ (n times). Assume, supplementarily, that f is injective. For any natural $n > 0$, we define $f^{-n} =$ the generalized inverse of f^n. Hence $f^{-n} : X_n \to X$, where $X_0 = X$, $X_1 = f(X)$, $X_2 = f(X_1) = f^2(X)$, $\ldots$, $X_n = f(X_{n-1}) = f^n(X)$.

We shall use the following result: if $\emptyset \neq A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ is monotone and $f(A)$ is an interval, then f is continuous.

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For $\emptyset \neq A \subset \mathbb{R}$ and $f : A \to A$, we denote

$$\mathrm{Fix}(f) = \{x \in A \mid f(x) = x\}.$$

The elements of $\mathrm{Fix}(f)$ are called fixed points of f. Notice that, in case f is continuous, it follows that $\mathrm{Fix}(f)$ is closed in A.

2. Formulation and Solution of the Problem, Fundamental Lemma.

We shall begin with the formulation of the Problem.

Formulation of the Problem

Assume $g : \mathbb{R} \to \mathbb{R}$ is a continuous function having the property that there exists a real number x_0 such that $g \downarrow$ on $(-\infty, x_0]$ and $g \uparrow$ on $[x_0, \infty)$. We assume also that $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to-\infty} g(x) = \infty$.

We want to find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = g(x)$ for any $x \in \mathbb{R}$. We shall call such a function (in case it exists) *a solution* of the functional equation

$$f \circ f = g. \tag{2.1}$$

Fundamental Lemma. *Assume x_0 is a real number, b is such that $x_0 < b \leq \infty$ and $G : [x_0, b) \to [x_0, b)$ is a continuous, strictly increasing function having the property that $\lim\limits_{x\to b} G(x) = b$.*

Then there exists a continuous, strictly increasing function $h : [x_0, b) \to \to [x_0, b)$ such that $h \circ h = G$.

Proof. 1. Case $\mathrm{Fix}(G) = \emptyset$.

It follows that $G(x) > x$ for any $x \in [x_0, b)$. Indeed, $G(x_0) \neq x_0$ implies $G(x_0) > x_0$ and the existence of $x \in (x_0, b)$ such that $G(x) < x$ would imply $G(u) = u$ for some $u \in (x_0, x)$, impossible.

The construction of h which is sketched further is canonical (for the special case when G is a homeomorphism, see [3], [5] and [4]). Write $x_2 = G(x_0) > x_0$. The first parameter of the construction is an (arbitrary) number x_1 such that $x_0 < x_1 < x_2$. The second parameter of the construction is a strictly increasing homeomorphism $\varphi : [x_0, x_1] \to [x_1, x_2]$.

Now we define the sequence $(x_n)_n$ as follows: $x_{2n} = G^n(x_0)$ and $x_{2n+1} = G^n(x_1)$, for all natural $n \geq 0$. Using $x_0 < x_1 < x_2$, we prove via mathematical induction that $(x_n)_n$ is strictly increasing.

Of course $\lim\limits_n x_n = L \leq b$. If $L < b$, we get $x_{2n+2} = G^{n+1}(x_0) = G(G^n(x_o)) = G(x_{2n}) \underset{n}{\to} G(L) = L$, false.

Consequently $L = b$ and this implies

$$\bigcup_{n=0}^{\infty} [x_n, x_{n+1}] = [x_0, b),$$

which enables us to construct $h : [x_0, b) \to [x_0, b)$ via

$$h(x) = \begin{cases} G^n \circ \varphi \circ G^{-n}(x), & \text{if } \ x \in [x_{2n}, x_{2n+1}] \\ G^{n+1} \circ \varphi^{-1} \circ G^{-n}(x), & \text{if } \ x \in (x_{2n+1}, x_{2n+2}], \end{cases}$$

for all natural $n \geq 0$. The definition is correct, because $[x_{2n}, x_{2n+1}]$ and $(x_{2n+1}, x_{2n+2}]$ are subsets of $[x_0, b)_n$ (see the Preliminary Part).

One can check that the function h has the desired properties.

2. Case $\mathrm{Fix}(G) \neq \emptyset$.

The set $\mathrm{Fix}(G)$ is closed in $[x_0, b)$, hence $[x_0, b) \setminus \mathrm{Fix}(G)$ is open in $[x_0, b)$.

There are two possibilities: either $x_0 \in \mathrm{Fix}(G)$ or $x_0 \notin \mathrm{Fix}(G)$.

We study first the situation when $x_0 \in \mathrm{Fix}(G)$. The case $\mathrm{Fix}(G) = [x_0, b)$ (i.e. $G(x) = x$ for all $x \in [x_0, b)$) is trivial: we can take $h(x) = x$ for all $x \in [x_0, b)$. Assume that $\mathrm{Fix}(G) \neq [x_0, b)$. We have $[x_0, b) \setminus \mathrm{Fix}(G) = (x_0, b) \setminus \mathrm{Fix}(G)$ and the last set is open and non empty. One can write

$$(x_0, b) \setminus \mathrm{Fix}(G) = \bigcup_{n \in I} (a_n, b_n),$$

where I is at most countable and (a_n, b_n) are mutually disjoint open non empty intervals.

For any $n \in I$, one has $G(a_n) = a_n$, $G(b_n) = b_n$ and $G_n : (a_n, b_n) \to (a_n, b_n)$, $G_n(x) = G(x)$, is a strictly increasing homeomorphism.

Using canonical procedures (see [3], [5] and [4]), one can find $H_n : (a_n, b_n) \to (a_n, b_n)$ such that $H_n \circ H_n = G_n$ and H_n is a strictly increasing homeomorphism. We define, $h : [x_0, b) \to [x_0, b)$,

$$h(x) = \begin{cases} x, & \text{if } \ x \in [x_0, b) \setminus \left(\bigcup_{n \in I} (a_n, b_n) \right) = \mathrm{Fix}(G) \\ H_n(x), & \text{if } \ x \in (a_n, b_n). \end{cases}$$

Clearly, the function h is strictly increasing, continuous and $h \circ h = G$.

It remains to study the situation $x_0 \notin \mathrm{Fix}(G)$. Notice first the existence of $x_0 < x_1 < b$ such that $[x_0, x_1] \cap \mathrm{Fix}(G) = \emptyset$. Indeed, if $[x_0, x] \cap \mathrm{Fix}(G) \neq \emptyset$, for any $x_0 < x < b$, we can find a strictly decreasing sequence $(a_n)_n$, $x_0 < a_n < b$, such that $G(a_n) = a_n$ for any n and $a_n \underset{n}{\longrightarrow} x_0$. Passing to n-limit, we get $G(x_0) = x_0$, contradiction.

Now $[x_0, b) \setminus \mathrm{Fix}(G) = ((x_0, b) \setminus \mathrm{Fix}(G)) \cup \{x_0\} = \left(\bigcup_{n \in I} (a_n, b_n) \right) \cup \{x_0\}$, where I is at most countable and (a_n, b_n) are mutually disjoint and non empty open intervals. According to the previous remark, one of the intervals (a_n, b_n) has the form (x_0, β), where $x_0 < \beta \leq b$. Hence we can write

$$[x_0, b) \setminus \mathrm{Fix}(G) = [x_0, \beta) \cup \left(\bigcup_{n \in I_1} (a_n, b_n) \right)$$

where I_1 is at most countable and the intervals $[x_0, \beta)$, (a_n, b_n) are mutually disjoint.

One has $\beta < b$ (in case $\beta = b$ one has $\mathrm{Fix}(G) = \emptyset$) and the set I_1 can be empty.

It is seen that $G(x) > x$ on $[x_0, \beta)$. Using the case 1 we find a strictly increasing and continuous $H : [x_0, \beta) \to [x_0, \beta)$ such that $H \circ H = G_1$, where $G_1 : [x_0, \beta) \to [x_0, \beta)$ is given via $G_1(x) = G(x)$ (use the fact that $G(\beta) = \beta$).

In case $I_1 \neq \emptyset$, for any $n \in I_1$ we find H_n such that $H_n \circ H_n = G_n$ (see the situation $x_0 \in \mathrm{Fix}(G)$). Now, it is possible to define the desired $h : [x_0, b) \to [x_0, b)$ given as follows (in case $I_1 = \emptyset$, the second determination of h disappears):

$$h(x) = \begin{cases} H(x), & \textit{if} \;\; x \in [x_0, \beta) \\ H_n(x), & \textit{if} \;\; x \in (a_n, b_n)\,, \; n \in I_1 \\ x, & \textit{if} \;\; x \in \mathrm{Fix}(G). \end{cases}$$

□

Once we haven established the previous result, we can pass to the solution of the problem.

Solution of the Problem

A. Preliminary Remarks

1. If the function f is a solution, one must have $f \downarrow$ on $(-\infty, x_0]$ and $f \uparrow$ on $[x_0, \infty)$.

a) Because of the strict monotonicity of g on $(-\infty, x_0]$ and $[x_0, \infty)$ the restrictions $f\left|_{(-\infty,\, x_0]}\right.$, $f\left|_{[x_0,\, \infty)}\right.$ must be injective functions.

b) Because the function g is not strictly increasing one must have either $f \downarrow$ on $(-\infty, x_0]$ and $f \uparrow$ on $[x_0, \infty)$ or $f \uparrow$ on $(-\infty, x_0]$ and $f \downarrow$ on $[x_0, \infty)$. The second situation cannot occur, because its occurence would imply the fact that the function f has a finite maximum, hence the function g would have a finite maximum.

2. If the function f is a solution, we have the inclusions $f\left((-\infty, x_0]\right) \subset [x_0, \infty)$ and $f\left([x_0, \infty)\right) \subset [x_0, \infty)$.

Indeed, assume first that $f\left((-\infty, x_0]\right) \not\subset [x_0, \infty)$. Hence $f\left((-\infty, x_0]\right) = [f(x_0), b) \not\subset [x_0, \infty)$, where $b = \lim\limits_{x \to -\infty} f(x)$. We have $b = \infty$. Indeed, in case $b < \infty$, it follows that $\lim\limits_{x \to -\infty} f\left(f(x)\right) = \lim\limits_{x \to -\infty} g(x) = f(b)$, false. Consequently $f(x_0) < x_0$ and we get the non degenerate interval $[f(x_0), x_0] \subset (-\infty, x_0] \cap f\left((-\infty, x_0]\right)$. Let $a < b$ in $(-\infty, x_0]$ such that $f(a) > f(b)$ are in $(-\infty, x_0] \cap f\left((-\infty, x_0]\right)$. It follows that $f\left(f(a)\right) = g(a) < f\left(f(b)\right) = g(b)$ which is false, because $g \downarrow$ on $(-\infty, x_0)$.

Now, assume that $f\left([x_0, \infty)\right) \not\subset [x_0, \infty)$. Hence $f\left([x_0, \infty)\right) = [f(x_0), b) \not\subset [x_0, \infty)$, where $b = \lim\limits_{x \to \infty} f(x)$. We have $b = \infty$ as previously. It follows that $f(x_0) < x_0$ and we get the nondegenerate interval $[f(x_0), x_0] \subset (-\infty, x_0) \cap$

$f([x_0, \infty))$. Let $a < b$ in $[x_0, \infty)$ such that $f(a) < f(b)$ are in $(-\infty, x_0] \cap f([x_0, \infty))$. We get $f(f(a)) = g(a) > f(f(a)) = g(b)$, which is false, because $g \uparrow$ on $[x_0, \infty)$.

3. We have $f((-\infty, x_0]) = f([x_0, \infty)) = [f(x_0), \infty) \subset [x_0, \infty)$.

Indeed, due to strict monotony we have $f((-\infty, x_0]) = [f(x_0), b)$, where $b = \lim_{x \to -\infty} f(x)$. We have seen that $b = \infty$. The same for $\lim_{x \to \infty} f(x) = \infty$ and $f([x_0, \infty)) = [f(x_0), \infty)$

4. Using remarks 2 and 3, we get for a solution f: $f(x_0) \geq x_0$.

Hence

$$f(f(x_0)) = g(x_0) \geq f(x_0) \geq x_0.$$

Because $g(x_0) \geq f(x_0) \geq x_0$, we have

$$g(x_0) = x_0 \Leftrightarrow f(x_0) = x_0.$$

Consequently

$$g(x_0) > x_0 \Leftrightarrow f(x_0) > x_0.$$

5. The preceding remark tells us that not all functions g as in the formulation of the problem can furnish a solution f. Namely, such a (suitable) function g must have the property $g(x_0) \geq x_0$. i.e. $g(\mathbb{R}) \subset [x_0, \infty)$.

As a consequence, we obtain the following fact: for such a function g the set $\mathrm{Fix}(g) \cap (-\infty, x_0]$ has at most one point. Namely we have

$$\mathrm{Fix}(g) \cap (-\infty, x_0] \subset \{x_0\}.$$

Indeed, let us assume there exists a fixed point $a < x_0$ of the function g. Then $a = g(a) > g(x_0)$. Hence $g(x_0) < a < x_0$ and this is impossible, because $g(x_0) \geq x_0$.

6. For any solution f one has $\mathrm{Fix}(f) = \mathrm{Fix}(g)$.

Indeed, the inclusion $\mathrm{Fix}(f) \subset \mathrm{Fix}(g)$ being trivial, let $a \in \mathrm{Fix}(g)$. From the remark 5 we obtain $a \geq x_0$. Hence, using the remark 2, we have a and $f(a)$ in $[x_0, \infty)$. Assume, by absurdum, $f(a) > a$ (respectively $f(a) < a$). Then $f(f(a)) = g(a) = a > f(a)$ (respectively $f(f(a)) = g(a) = a < f(a)$), contradiction.

B. Construction of the solution.

We start with the given function g as in part A and such that $g(x_0) \geq x_0$ (as we have seen, this conditions is necessary for the existence of the solution; actually, it is sufficient too). We can construct the strictly increasing and continuous $G : [x_0, \infty) \to [x_0, \infty)$, $G(x) = g(x)$. Using G and the Fundamental Lemma, we can construct the function $h : [x_0, \infty) \to [x_0, \infty)$.

We have $h(x_0) \geq x_0$, hence $h(h(x_0)) = G(x_0) \geq h(x_0)$ and this implies $h([x_0, \infty)) = [h(x_0), \infty) \supset [G(x_0), \infty) = G([x_0, \infty))$. Considering the generalized inverse $h^{-1} : [h(x_0), \infty) \to [x_0, \infty)$ one can define, for any $x \in [x_0, \infty)$, $h^{-1}(G(x))$.

Theorem (Existence and Uniqueness of the Solution).

Let g be a function as in part A with $g(x_0) \geq x_0$.

Existence. *There exists a solution f of the equation* (2.1). *The form of f is the following*:

a) *Construct G and h as previously.*

b) *Define $f : \mathbb{R} \to \mathbb{R}$ via*

$$f(x) = \begin{cases} h(x), & \text{if } x \in [x_0, \infty) \\ h^{-1}(g(x)), & \text{if } x \in (-\infty, x_0). \end{cases}$$

Uniqueness. *All solutions are of the above form.*

Proof. **Existence.** We have $g((-\infty, x_0]) = [g(x_0), \infty) = g([x_0, \infty)) = G([x_0, \infty))$ and we have seen that one can write $h^{-1}(y)$ for any $y \in G([x_0, \infty))$, hence one can write $h^{-1}(g(x))$ for any $x \in (-\infty, x_0]$.

Now we prove that the function f is a solution.

a) The function f thus constructed is continuous.

This assertion is clear on (x_0, ∞) (because h is continuous) and on $(-\infty, x_0)$ (because h^{-1} is continuous, being increasing and having an interval as range). Clearly f is right continuous at x_0. On the other hand, one has, using the continuity of h^{-1} and g,

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = \lim_{\substack{x \to x_0 \\ x < x_0}} h^{-1}(g(x)) = h^{-1}(g(x_0)).$$

One must have

$$h^{-1}(g(x_0)) = h(x_0) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = f(x_0)$$

and this is obvious.

Notice that we proved also the equality $f(f(x_0)) = g(x_0)$.

b) It remains to prove that $f(f(x)) = g(x)$ for any $x \in \mathbb{R}$, $x \neq x_0$.

If $x > x_0$, one has $f(x) = h(x)$ and $f(f(x)) = h(h(x)) = G(x) = g(x)$, because $h(x) \in [x_0, \infty)$.

If $x < x_0$, one has $f(x) = h^{-1}(g(x)) \in [x_0, \infty)$.

Consequently $f(f(x)) = h(h^{-1}(g(x))) = g(x)$.

The proof of the existence part is finished.

Uniqueness. For a given g as previously, let f be a solution. Remarks 1, 2 and 3 show that $f \downarrow$ in $(-\infty, x_0]$, $f \uparrow$ in $[x_0, \infty)$ and

$$f((-\infty, x_0]) = f([x_0, \infty)) = [f(x_0), \infty) \subset [x_0, \infty).$$

We define $h : [x_0, \infty) \to [x_0, \infty)$, $h(x) = f(x)$ and $G : [x_0, \infty) \to [x_0, \infty)$ via $G(x) = g(x)$. Then, because $h(h(x)) = G(x)$ for any $x \in [x_0, \infty)$ and h is continuous and strictly increasing, it follows that h is generated by G like in the Fundamental Lemma. It is seen that $f(x) = h(x)$ for any $x \in [x_0, \infty)$. All it remains to be proved is that $f(x) = h^{-1}(g(x))$ for any $x \in (-\infty, x_0]$.

Put $y = f(x) \in [x_0, \infty)$. It follows that $g(x) = f(f(x)) = f(y) = h(y) = h(f(x))$. Hence $f(x) = h^{-1}(g(x))$. □

Example. Let a be a real number. We consider the function

$$F : \mathbb{R} \to \mathbb{R}, F(x) = |x| + a.$$

Then $F \downarrow$ on $(-\infty, 0]$ and $F \uparrow$ on $[0, \infty)$. One has $F(0) = a$. We want to take F in the role of g, hence x_0 will be equal to 0. The condition $F([0,\infty)) \subset [0,\infty)$, i.e. $[a,\infty) \subset [0,\infty)$ is equivalent to $a \geq 0$ (this means $F(0) \geq 0$).

Consequently, we consider a number $a \geq 0$ and the function $g : \mathbb{R} \to \mathbb{R}$ given via

$$g(x) = |x| + a.$$

We want to solve the equation (2.1). Notice that in case $a < 0$ the problem has no solution.

The fixed points of g are given by the equation

$$|x| + a = x. \tag{2.2}$$

For $x \geq 0$ equation (2.2) becomes $a = 0$ and for $x < 0$ equation (2.2) becomes $x = \frac{a}{2}$, impossible.

Consequently, in case $a > 0$ there are no fixed points and in case $a = 0$ all the points $x \geq 0$ are fixed points.

Case $a > 0$. The function g generates the strictly increasing and continuous function $G : [0,\infty) \to [0,\infty)$ given via $G(x) = |x| + a = x + a$.

As we have seen, the solution depends upon and arbitrary strictly increasing and continuous function $h : [0,\infty) \to [0,\infty)$ such that $h \circ h = G$. Such a function is e.g. given via $h(x) = x + \frac{a}{2}$.

In this case $h^{-1} : \left[\frac{a}{2}, \infty\right) \to [0,\infty)$, $h^{-1}(y) = y - \frac{a}{2}$. The general solution of (2.1) is given via

$$f(x) = \begin{cases} h(x), & \text{if } x \in [0,\infty) \\ h^{-1}(-x+a), & \text{if } x \in (-\infty, 0). \end{cases}$$

In the particular case $h(x) = x + \frac{a}{2}$ we have the solution

$$f(x) = \begin{cases} x + \dfrac{a}{2}, & \text{if } x \in [0,\infty) \\ -x + \dfrac{a}{2}, & \text{if } x \in (-\infty, 0), \end{cases}$$

i.e. $f(x) = |x| + \frac{a}{2}$.

Case $a = 0$. The function g generates the strictly increasing and continuous function $G : [0,\infty) \to [0,\infty)$, $G(x) = x$.

We look for strictly increasing and continuous functions $h : [0,\infty) \to [0,\infty)$ such that $h \circ h = G$, i.e. $h(h(x)) = x$ for any $x \in [0,\infty)$. We shall see that the unique possibility is $h(x) = x$ for any $x \in [0,\infty)$, hence $h : [0,\infty) \to [0,\infty)$ is a homeomorphism and $h = h^{-1}$.

Indeed, if $h(t) > t$ (respectively $h(t) < t$) we obtain $h(h(t)) = t > h(t)$ (respectively $h(h(t)) = t < h(t)$), contradiction.

Consequently, the unique solution is $f:\mathbb{R}\to\mathbb{R}$ given via

$$f(x)=\begin{cases} x, & \text{if } x\geq 0\\ |x|, & \text{if } x<0.\end{cases}$$

Finally it is seen that the unique solution is $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|x|$.

Final Remark. Considering $F:\mathbb{R}\to\mathbb{R}$, $F(x)=|x|+a$, for $a<0$, we take the function $\varphi:\mathbb{R}\to\mathbb{R}$, $\varphi(x)=|x|+\frac{a}{2}$. We can see that the function φ does not satisfy the equation $\varphi\circ\varphi=F$. Indeed, for $x\in\left[0,-\frac{a}{2}\right)$, one has $\varphi(x)=x+\frac{a}{2}<0$ and $\varphi\left(\varphi(x)\right)=-x-\frac{a}{2}+\frac{a}{2}=-x\neq|x|+a$.

Conclusion. We could solve in a constructive manner the equation $f\circ f=g$ in case the given function g is not injective. Up to now, the aforementioned equation was solved in case the given function g was strictly monotone (see the classical monograph [5], the classical paper [4] and [3]).

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