ON ELECTROMAGNETIC WAVELETS

Teodora Daniela CHICIOREANU[1]

Semnalele sunt purtătoare de informație, codificată sau nu. Informația, ca și energia, are proprietatea generică de a-și putea modifica forma în mod repetat, fără a-și pierde esența. Teoria seriilor Fourier și a transformării Fourier constituie un subiect central în matematică și în științele inginerești. Prin transformata Fourier, un semnal în timp (din $L^1 \cap L^2$) își modifică doar modul de reprezentare (în frecvență), putând fi oricând recuperat. Folosind tehnicile frecvențiale, informația este transportată cu viteza luminii și decodificată la receptor.

Signals carry codified and uncodified information. Information, as well as energy, has the generic property of being able to change its form repeatedly, without losing its essence. Fourier's theory of series and Fourier transformation plays a central role in mathematics and engineering sciences. Through the Fourier transform, a time signal $L^1 \cap L^2$ changes only its means of representation in the frequency domain, and thus it can be retrieved at any time. Using sequential techniques, the information is light-speed transported and decodified at the reception point.

Keywords: signal, electromagnetic waves, wavelets, radar, sonar

1. The complex solutions of Maxwell's equations

Any electromagnetic wave in the free space (without any perturbations or genetic restrictions) can be described by a pair of vector spaces dependent upon the 4-vector $\vec{v} = (\underline{x}, t)$; with $\underline{x} = (x_1, x_2, x_3)$ denoting the current position and t-the time.

This pair of vectors is $(\vec{E}, \vec{B}): R^4 \to R^3 \times R^3$, where $\vec{E}(\underline{x}, t) =$ the electric field and $\vec{B}(\underline{x}, t) =$ the magnetic field. In this respect, there are known the Maxwell's equations satisfied by these two fields:

$$\text{rot}\,\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \tag{1}$$

---

[1] Lector,Training Teaching personnel and Social and Human Sciences Department -University "Politehnica" of Bucharest, ROMANIA

$$\mathrm{rot}\vec{B} - \frac{\partial}{\partial t}\vec{E} = 0 \tag{2}$$

$$\mathrm{div}\vec{E} = 0 \ , \ \mathrm{div}\vec{B} = 0 \tag{3}$$

By introducing complex variables, it follows the application $\vec{Z}:\Re^4 \to C^3$, $\vec{Z} = \vec{B} + i\vec{E}$; the equations (1) and (2) can be written concisely as

$$\frac{\partial}{\partial t}\vec{Z} = i \cdot \mathrm{rot}\vec{Z} \tag{4}$$

and (3) becomes

$$\mathrm{div}\vec{Z} = 0 \tag{5}$$

Proposition 1.

The complex electromagnetic field $\vec{Z}$ represents the solution of the wave equation

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\vec{Z} = 0 \tag{6}$$

Demonstration

According to (4),

$$\frac{\partial}{\partial t}(\frac{\partial}{\partial t}\vec{Z}) \ = \ i\frac{\partial}{\partial t}(\mathrm{rot}\vec{Z}) \ = \ i\mathrm{rot}(\frac{\partial}{\partial t}\vec{Z}) \ = \ i\mathrm{rot}(i\mathrm{rot}\vec{Z}) \ = \ =i^2\mathrm{rot}(\mathrm{rot}\vec{Z}) \ =$$

$$-\mathrm{rot}(\mathrm{rot}\vec{Z}) = -grad(\mathrm{div}\vec{Z}) + \Delta\vec{Z} \overset{cf(5)}{=} \Delta\vec{Z} \text{ and hence (6)}$$

We should remember that for any fast descendent function $f(\underline{x})$, $f:\Re^n \to \Re$ there can be defined its n-dimensional Fourier transform $\hat{f}:R^n \to C$, $\hat{f}(\underline{\omega}) = \int_{\Re^n} f(\underline{x}) \cdot e^{-i<\underline{\omega},\underline{x}>}d\underline{x}$

The function $\hat{f}$ is continuous, limited and tends from 0 to the infinite on any direction. In addition, there is the inversation formula $f(\underline{x}) = \frac{1}{(2\pi)^n}\int_{\Re^n}\hat{f}(\underline{\omega}) \cdot e^{i<\underline{\omega},\underline{x}>}d\underline{\omega}$, for $\underline{x} \in R^n$.

More specifically, the $\vec{Z}$ field has a 4-dimensional Fourier transform and we intend to find the solution of the (1)+(2)+(3) system or of its equivalent, the (4)+(5) one of the equation (6) as an inverse 4-dimensional Fourier transform :

$$\vec{Z}(\underline{x},t) = \frac{1}{(2\pi)^4} \int_{\mathfrak{R}^4} \stackrel{\wedge}{\vec{Z}}(\underline{p}) e^{i<\underline{p},\underline{r}>} d\underline{p} \tag{7}$$

where $\underline{p} = (p_1, p_2, p_3, p_0) = (\hat{p}, p_0)$ represents the 4-vector spatial wave with the frequency p_0 ; and we considered

$$\langle \vec{p}, \vec{r} \rangle = p_0 t - p_1 x_1 - p_2 x_2 - p_3 x_3 = p_0 t - \vec{p} \cdot \vec{x}.$$

We define $\omega = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}$ as the absolute frequency value, conse quently $\langle \underline{p}, \underline{p} \rangle = p_0^2 - \omega^2$. With $K = \left\{ \underline{p} \in R^4 \middle| \underline{p} \neq 0, p_0^2 = \omega^2 \right\}$,

also called "the light cone", equation (6) becomes $\left(p_0^2 - \omega^2\right) \cdot \stackrel{\wedge}{\vec{Z}}(\underline{p}) = 0$ and this equation e (within tempered distribution) results in

$$\stackrel{\wedge}{\vec{Z}}(\underline{p}) = 2\pi \cdot \delta\left(p_0^2 - \omega^2\right) \cdot f(\underline{p}),$$

where $f : K \to \mathbb{C}^3$ denotes a function of L^2.

But generally $\delta(t^2 - b^2) = \frac{1}{2b}(\delta(t - b) + \delta(t + b))$ for $b > 0$, so

$\delta(p_0^2 - \omega^2) = \frac{1}{2\omega}(\delta(p_0 - \omega) + \delta(p_0 + \omega))$. Then according to (7) and via the filter formula ($\int \varphi(t)\delta(t - b)dt = \varphi(b)$), it leads to

$$\vec{Z}(\underline{x},t) = \frac{1}{(2\pi)^3} \int_{\mathfrak{R}^3} e^{-i(x_1 p_1 + x_2 p_2 + x_3 p_3)} \cdot \frac{1}{2\omega}(e^{i\omega t} \cdot f_+(p_1, p_2, p_3) + e^{-i\omega t} \cdot f_-(p_1, p_2, p_3))dp_1 dp_2 dp_3$$

Where we noted
$$f_+(p_1, p_2, p_3) = f(p_1, p_2, p_3, \omega)$$
and
$$f_-(p_1, p_2, p_3) = f(p_1, p_2, p_3, -\omega).$$

This gives raise to:

Proposition 2.

The solution of the equation $\left(-\dfrac{\partial^2}{\partial t^2} + \Delta \right)\vec{Z} = 0$ is

$$\vec{Z}(\underline{x},t) = \frac{1}{16\pi^3\omega}\int_K f(\underline{p})\cdot e^{i<\underline{p},\underline{r}>}dp_1 dp_2 dp_3 ,\qquad (8)$$

where $f: K \to C^3$ denotes a function of L^2, defined within the light cone.

2. The connection to the electromagnetic wavelets

By convention $\vec{Z}:\Re^4 \to C^3$ denotes the previously defined complex electromagnetic field. To measure it with a punctual meands in the Minkonski space-time M_4 we assume that the impulse response of the instrument is a $\psi:\Re \to C$, and the trajectory of the instrument finds its parameters in M_4 as follows:

$$x_1(u) = a_1 + uv_1,$$
$$x_2(u) = a_2 + uv_2,$$
$$x_3(u) = a_3 + uv_3$$

and

$$t(u) = t_0 + uv_0$$

where $u \in \Re$ denotes the parameter, $a = (a_1, a_2, a_3)$ the initial position, t_0 the initial moment, and $\vec{v} = (v_1, v_2, v_3, v_0)$ the speed of the instrument. The value resulting from measuring the $\vec{Z}$ field with the specified instrument will be

$$\vec{Z}_\psi(\underline{a},t_0) = \int_\Re \vec{Z}(a_1 + uv_1, a_2 + uv_2, a_3 + uv_3, t_0 + uv_0)\cdot \overline{\psi}(u)du$$

Retaining only the temporal component, we produce
$\vec{Z}_\psi(t_0) = \int_\Re \vec{Z}(t_0 + uv_0)\cdot \overline{\psi}(u)du$ and by changing the variable $t_0 + uv_0 = t$ we come to

$$\vec{Z}_\psi(t_0) = \int_\Re \vec{Z}(t)\cdot \overline{\psi}(\frac{t - t_0}{v_0})\cdot \frac{1}{|v_0|}dt \qquad (9)$$

Let us remember that a 1D wavelet is a function $\psi : R \to C$ of $L^1 \cap L^2$ with energy 1, and so its Fourier transform $\hat{\psi}$ satisfies the relations $\hat{\psi}(0) = 0$ and

$$\int_{\Re} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty .$$

Establishing such a wavelet for any signal $f \in L^2$ and any real parameters a, b $(a \neq 0)$ there can be associated the coefficients of f in relation to ψ as resulting from $c_f(a,b) = \frac{1}{\sqrt{|a|}} \int_R f(t) \cdot \hat{\psi}(\frac{t-b}{a}) dt$. Once this family of complex numbers $c_f(a,b)$, also called the integral transform of f by the wavelet ψ, we can retrieve f.

The (9) relation can be then rewritten as :

$$\vec{Z}_{\psi}(t_0) = \int_{\Re} \frac{1}{\sqrt{|v_0|}} c_{\vec{Z}}(t_0, v_0) , \qquad (10)$$

where $c_{\vec{Z}}$ denotes the vector of the coefficients of $\vec{Z}$ in relation to the wavelet ψ.

By using the reverse formula and with the knowledge of the coefficients $c_{\vec{Z}}(t_0, v_0)$, we can retrieve $\vec{Z}(t_0)$.

Note. The above mentioned findings apply to the acoustic wavelets as well, which unlike the electromagnetic wavelets are scalar (not vector fields) and moreover propagate in a real physical medium (and not in a vacuum).

3. An application of the wavelets to the radar (or sonar) signals

The purpose of radar/sonar is to obtain information about moving objects, related to velocity and location.

Radar and sonar technology respectively allow for obtaining this kind of information about flying objects as well as submarines by analysing the electromagnetic or acoustic wavelets reflected by the specific objects.

Object positioning results from measuring the delay time between the sent signal and its echo, taking into account the Doppler effect due to the movement of objects; the comparison between the sent signal and its echo leads to an approximate estimation of the D distance between the object and the location of the radar station and of the radial speed v along the view line.
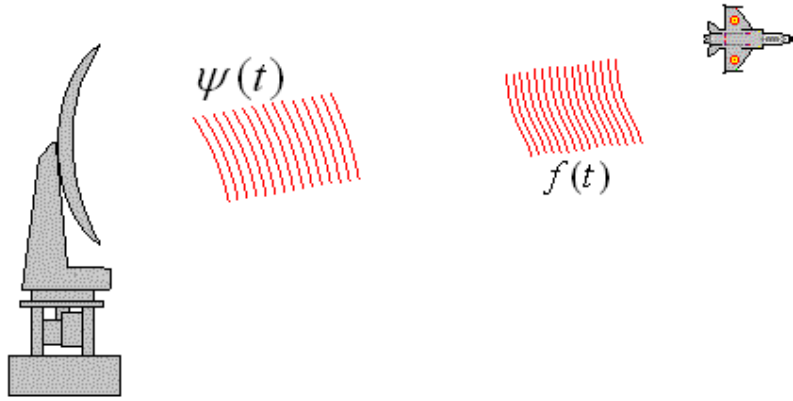
Fig 1. illustration radar

The radar signal represents a time function $\psi(t)$, where $\psi : \Re \to \Re$ to signify the electric tension in the transmission antenna.

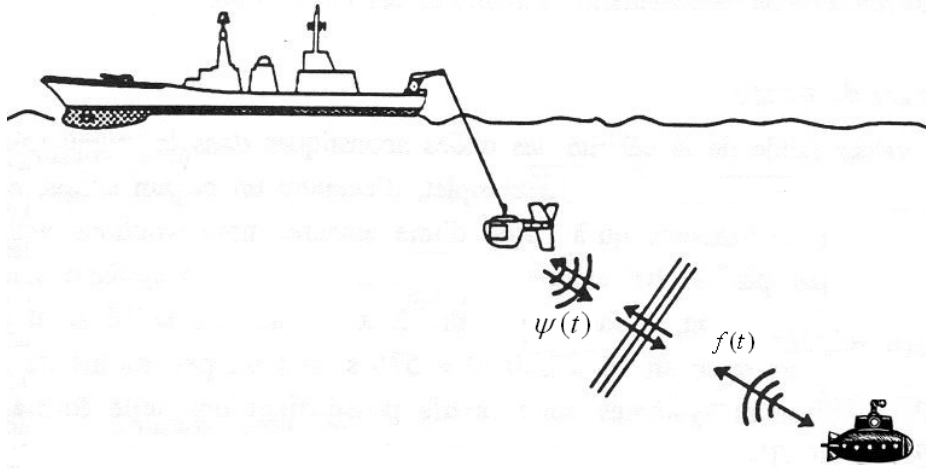

Fig 2. illustration sonar

The antenna converts $\psi(t)$ into an electromagnetic radar wave $\vec{Z}(\underline{x},t) = \vec{B}(\underline{x},t) + i\vec{E}(\underline{x},t)$, thus satisfying relation (6), that equals Maxwell's equations. The explicit connection between $\vec{Z}(\underline{x},t)$ and $\psi(t)$ does not constitute the purpose of this study.

After the object reflexion, the echo-electromagnetic wave is translated by the same antenna into a real signal $f(t)$. The next statement will establish the connection between $f(t)$ and $\psi(t)$.

Electromagnetic waves travel at the speed of light. If the followed flying object is in a state of spell ($v = 0$) at a D distance from the radar station, then

$$f(t) = \alpha \cdot \psi(t - \frac{2D}{c})$$ (11)

where α denotes a constant – positive - real which depends on the object's reflectivity and the diminishing of the signal ($c \cong 300.000.000\ {}^{m}\!/_{s}$).

If the object is relatively small and travels at a radial speed v, then the distance between the object and the station at the moment t is $D(t) = D_0 + vt$, where $D_0 = D(0)$. If $v > 0$ then the object moves away and if $v < 0$ then it draws near.

If $\tau(t)$ denotes the delay of the echo that arrives at the moment t, then

$$c \cdot \tau(t) = 2D(t - \frac{\tau(t)}{2})$$ (12)

since

$$c \cdot \tau(t) = 2D_0 + 2v(t - \frac{\tau(t)}{2}),$$

which means

$$(c + v) \cdot \tau(t) = 2D_0 + 2vt,$$

and so

$$\tau(t) = \frac{2D_0 + 2vt}{c + v}.$$

Relations (11) and (12) show that

$$f(t) = \alpha\psi(t - \tau(t)) = \alpha\psi(t - \frac{2D_0 + 2vt}{c + v})$$

Now we can determine the real parameters a and b so as

$$t - \frac{2D_0 + 2vt}{c + v} = \frac{t - b}{a},$$

for any t. The identification of the coefficients of t and the free terms show that

$$a = \frac{c + v}{c - v},$$

$$b = \frac{2aD_0}{c + v},$$

therefore

$$v = c \cdot \frac{a - 1}{a + 1} \quad \text{and} \quad D_0 = \frac{bc}{a + 1}$$ (13)

As $v < c$ it follows $a > 0$. If $v > 0$ then $a > 1$ so if the object is moving away, we obtain the relation:

$$f(t) = \alpha\psi(\frac{t-b}{a}) = \alpha\sqrt{a} \cdot \psi_{a,b}(t) \tag{14}$$

For energy it writes

$$E(f) = \alpha\sqrt{a} \cdot E(\psi_{a,b}) = \alpha\sqrt{a} \cdot E(\psi)$$

Proposition 3.

The signal $f(t)$ transformed after reflexion from the radon signal $\psi(t)$ satisfies the relation $f(t) = \alpha\psi(\frac{t-b}{a}) = \alpha\sqrt{a} \cdot \psi_{a,b}(t)$ where $a = \frac{c+v}{c-v}$, $b = \frac{2aD_0}{c+v}$, where v denotes the radial speed of the object, D_0 denotes the initial distance from the station and c denotes light speed.

The parameters a and b can be approximately determined by comparing the sent $\psi(t)$ signal to the received echo-signal. As soon as a and b are known the parameters v, D0 can be determined in accordance with the relations (13).

Therefore all desired characteristics can be measured (object's speed and location) and the trajectory of the object (traced via radar/sonar) can be determined.

R E F E R E N C E S

[1] *C.K.Chui*, Wavelets - a mathematical tool for signal processing, SIAM 1998

[2] *G. Kaiser*, A friendly guide to wavelets, Birkhauser 1999

[3] *D.F.Walnut,* An introduction to wavlet analysis, 2002