MINIMIZATION PROBLEM OF A VARIATIONAL INEQUALITY ON A FAMILY OF SET-VALUED MAPPINGS

Mohammad Eslamian[1], Kourosh Nourouzi[2]

In this paper we propose a new iterative scheme for a finite family of quasi-nonexpansive set-valued mappings by the general viscosity iterative method. We establish the strong convergence for the iterative scheme to prove the existence of a unique solution for the variational inequality which is the optimality condition for the minimization problem. Our results generalize and improve some results of Xu (2003) and Marino, Xu (2006).

Keywords: Minimization problem, Variational inequality, Fixed point, Quasi-nonexpansive set-valued mapping.
MSC2010: 47H10, 47H09

1. Introduction

Let A be a strongly positive bounded linear operator on a real Hilbert space H, that is, there exists $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \qquad (x \in H).$$

For a nonexpansive mapping T from a nonempty subset C of H into itself ($\|Tx - Ty\| \leq \|x - y\|$, for each $x, y \in C$), a typical problem is to minimize the quadratic function

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.1}$$

over the set of all fixed points $F(T)$ of T. In 2003, Xu [1] showed that the sequence $\{x_n\}$ defined by the iterative method

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n x_0, \qquad (n \geq 0), \tag{1.2}$$

with the initial guess $x_0 \in H$ converges strongly to the unique solution of the minimization problem (1.1) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions.

The viscosity approximation method for nonexpansive mappings was given in [2] and followed in [3] . More precisely, for a contraction f on H, starting with an

[1] Corresponding author, Young Researchers Club, Babol Branch, Islamic Azad University, Babol, Iran, Email: mhmdeslamian@gmail.com

[2] Department of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran, Email:nourouzi@kntu.ac.ir

arbitrary initial point $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \qquad (n \geq 0) \tag{1.3}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.3) strongly converges to the unique solution $x^\star$ of the variational inequality

$$\langle (I - f)x^\star, x - x^\star \rangle \geq 0, \qquad (x \in F(T)).$$

A combination of the iterative method (1.2) with the viscosity approximation (1.3) is given in [4] via considering the general iterative method

$$x_{n+1} = a_n \gamma f(x_n) + (I - a_n A)Tx_n, \qquad (n \geq 0). \tag{1.4}$$

It is shown that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^\star, x - x^\star \rangle \geq 0, \qquad (x \in F(T))$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for every $x \in H$).

Also, recently Yao and Postolache [5] presented a new iterative methods for variational inequalities and fixed point problems. In recent years, the methods of approximating of fixed points of set-valued nonexpansive mappings have been studied by many authors (see, for example, [6-10] and the references therein).

In this paper we introduce a new iterative process by the general viscosity iterative method for a finite family of quasi-nonexpansive set-valued mappings. We prove the strong convergence for the iterative process to prove the existence of a unique solution for the variational inequality which is the optimality condition for the minimization problem. Our results in this paper are new even for single valued mappings and generalize and improve some results of Xu [1], and Marino, Xu [4].

We start with some preliminaries which will be needed in this paper. Throughout the paper H will denote a real Hilbert space and C denote a nonempty closed, convex subset of H, unless otherwise stated. We will write $x_n \longrightarrow x$ ($x_n \rightharpoonup x, resp.$) if $\{x_n\}$ converges strongly (weakly, resp.) to x. For every element $x \in H$ there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$, for each $y \in C$. The metric projection P_C of H onto C is a nonexpansive mapping. It is known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 1. *([4]) If $x \in H$ and $z \in C$, then $z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0$, for each $y \in C$.*

Lemma 2. ([11]) *For each* $x_1, \cdots, x_m \in H$ *and* $\alpha_1, \cdots, \alpha_m \in [0,1]$ *with* $\sum_{i=1}^m \alpha_i = 1$ *the equality*

$$\|\alpha_1 x_1 + ... + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\le i<j\le m} \alpha_i\alpha_j \|x_i - x_j\|^2,$$

holds.

Lemma 3. ([1]) *Let* $\{\gamma_n\}$ *be a sequence in (0,1) and* $\{\delta_n\}$ *be a sequence in* $\mathbb{R}$ *satisfying*

(i) $\sum_{n=1}^\infty \gamma_n = \infty$,
(ii) $\limsup_{n\longrightarrow\infty} \delta_n \le 0$ *or* $\sum_{n=1}^\infty |\gamma_n\delta_n| < \infty$.

If $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n,$$

for each $n \ge 0$, *then* $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 4. ([4]) *Suppose that* A *is a strongly positive linear bounded operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \le \|A\|^{-1}$. *Then* $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.

Lemma 5. ([12]) *Let* $\{u_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{u_{n_i}\}$ *of* $\{u_n\}$ *such that* $u_{n_i} < u_{n_i+1}$ *for all* $i \ge 0$. *For every* $n \ge n_0$, *define an integer sequence* $\{\tau(n)\}$ *by*

$$\tau(n) = \max\{k \le n : u_k < u_{k+1}\}.$$

Then $\tau(n) \longrightarrow \infty$ *as* $n \longrightarrow \infty$ *and* $\max\{u_{\tau(n)}, u_n\} \le u_{\tau(n)+1}$, *for every* $n \ge n_0$.

A subset $C \subset H$ is called proximinal if for each $x \in H$ there exists an element $y \in C$ such that

$$\| x - y \| = dist(x, C) = inf\{\| x - z \| : z \in C\}.$$

We denote by $CB(C), K(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of C, respectively. The Hausdorff metric $\mathfrak{h}$ on $CB(H)$ is defined by

$$\mathfrak{h}(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(H)$.

For a set-valued mapping $T : H \longrightarrow 2^H$ an element $x \in H$ is said to be a fixed point of T if $x \in Tx$. The set of all fixed points of T will be denoted by $F(T)$.

Definition 1.1. A set-valued mapping $T : H \longrightarrow CB(H)$ is called

(i) nonexpansive if

$$\mathfrak{h}(Tx, Ty) \le \|x - y\|, \qquad (x, y \in H).$$

(ii) quasi-nonexpansive if $F(T) \ne \emptyset$ and $\mathfrak{h}(Tx, Tp) \le \|x - p\|$ for every $x \in H$ and $p \in F(T)$.

Definition 1.2. Let $T : C \longrightarrow CB(C)$ be a set-valued mapping. The mapping $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $\lim_{n\longrightarrow\infty} dist(x_n, Tx_n) = 0$, imply $x \in Tx$.

2. General iterative process

Let $T : C \longrightarrow C$ be a quasi-nonexpansive mapping. Itoh and Takahashi [13] showed that the set of all fixed points $F(T)$ is closed and convex. Now we have the following generalization in the setting of set-valued mappings.

Lemma 6. *Let $T : C \longrightarrow P(C)$ be a set-valued mapping such that P_T is quasi-nonexpansive, where $P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}$. Then $F(T)$ is closed and convex.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \longrightarrow z$ as $n \longrightarrow \infty$. Since $P_T p_n = \{p_n\}$ and P_T is quasi-nonexpansive, we have

$$\begin{aligned} dist(z, P_T z) &\le d(z, p_n) + dist(p_n, P_T z) \\ &\le d(z, p_n) + \mathfrak{h}(P_T p_n, P_T z) \\ &\le 2\, d(z, p_n) \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned}$$

This implies that z is a fixed point of T. To see that why $F(T)$ is convex, let $x, y \in F(T)$, $\alpha \in [0, 1]$ and $z = \alpha x + (1 - \alpha)y$. Since $P_T x = \{x\}$ and $P_T y = \{y\}$, if $w \in P_T z$, then we have

$$\begin{aligned} \|w - z\|^2 &= \|\alpha(w - x) + (1 - \alpha)(w - y)\|^2 \\ &= \alpha\|w - x\|^2 + (1 - \alpha)\|w - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha \cdot dist(w, P_T x)^2 + (1 - \alpha) \cdot dist(w, P_T y)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\le \alpha \cdot \mathfrak{h}(P_T z, P_T x)^2 + (1 - \alpha) \cdot \mathfrak{h}(P_T z, P_T y)^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\le \alpha\|z - x\|^2 + (1 - \alpha)\|z - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\le \alpha(1 - \alpha)^2\|y - x\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 = 0. \end{aligned}$$

Therefore $z = w \in P_T z \subset T(z)$ and so $z \in F(T)$. □

Lemma 7. *Let $T : C \longrightarrow K(C)$ be a set-valued mapping such that P_T is nonexpansive. If $x_n \rightharpoonup w$ and $\lim_{n\longrightarrow\infty} dist(x_n, P_T x_n) = 0$, then $w \in Tw$.*

Proof. For each $n \ge 1$, we can choose $y_n \in P_T w$ such that $\|x_n - y_n\| = dist(x_n, P_T w)$. Since $P_T w$ is compact, the sequence $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ with $\lim_{k\longrightarrow\infty} y_{n_k} = v \in P_T w$. Now

$$\begin{aligned}\|x_{n_k}-v\| &\leq \|x_{n_k}-y_{n_k}\|+\|y_{n_k}-v\|\\ &= dist(x_{n_k},P_Tw)+\|y_{n_k}-v\|\\ &\leq dist(x_{n_k},P_Tx_{n_k})+\mathfrak{h}(P_Tx_{n_k},P_Tw)+\|y_{n_k}-v\|\\ &\leq dist(x_{n_k},P_Tx_{n_k})+\|x_{n_k}-w\|+\|y_{n_k}-v\|,\end{aligned}$$

for each k. Therefore

$$\limsup_{k\longrightarrow\infty}\|x_{n_k}-v\|\leq\limsup_{k\longrightarrow\infty}\|x_{n_k}-w\|.$$

Since H satisfies the Opial property, we get $w=v\in P_Tw\subset Tw$, i.e., $w\in F(T)$. □

Now we present an example of a set-valued mapping such that P_T is quasi-nonexpansive, but T is not quasi-nonexpansive.

Example 1. Let $I=[0,1]$, $H=L^2(I)$, and $C=\{f\in H : f(x)\geq 0, \forall x\in I\}$. Let $T: C\longrightarrow CB(C)$ be defined by

$$T(f)=\{g\in C : f(x)\leq g(x)\leq 2\,f(x)\}.$$

Then we have

$$P_T(f)=\{g\in T(f), \|g-f\|_2=dist(T(f),f)\}=\{f\}$$

and hence

$$\mathfrak{h}(P_T(f_1),P_T(f_2))\leq\|f_1-f_2\|_2, \qquad (f_1,f_2\in C).$$

Therefore P_T is quasi-nonexpansive. Now putting $f_1\equiv 0$ and $f_2\equiv 1$ we have $T(f_1)=0$ and $T(f_2)=\{g\in C : 1\leq g(x)\leq 2\}$. Hence

$$\mathfrak{h}(T0,T1)=\|2\|_2=2>1=\|0-1\|_2,$$

which shows that T is not quasi-nonexpansive.

Now we give the main result.

Theorem 2.1. *Let $T_i : C\longrightarrow K(C)$, $i=1,2,...,m$ be a finite family of set-valued mappings such that for each $1\leq i\leq m$, P_{T_i} is quasi-nonexpansive and $I-P_{T_i}$ is demiclosed at zero. Let $\mathcal{F}=\bigcap_{i=1}^m F(T_i)\neq\emptyset$. Suppose that f is a contraction from H into itself with constant $b\in(0,1)$ and A is a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$ and $0<\gamma<\frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by an arbitrary $x_0\in C$ and*

$$\begin{cases} y_n=b_{n,0}x_n+b_{n,1}z_{n,1}+b_{n,2}z_{n,2}+...+b_{n,m}z_{n,m},\\ x_{n+1}=a_n\gamma fx_n+(I-a_nA)y_n,\end{cases} \tag{2.1}$$

for every $n\geq 0$, where $\sum_{i=0}^m b_{n,i}=1$, $z_{n,i}\in P_{T_i}(x_n)$ and the sequences $\{a_n\}$ and $\{b_{n,i}\}$ satisfy the following conditions:

(i) $a_n\subset(0,1)$, $\lim_{n\longrightarrow\infty}a_n=0$, $\sum_{n=1}^\infty a_n=\infty$,

(ii) $\{b_{n,i}\}\subset[c,1)\subset(0,1), i=0,1,...,m.$

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \mathcal{F}$ *which solves the variational inequality*

$$\langle (A - \gamma f)q, x - q\rangle \geq 0, \qquad (x \in \mathcal{F}). \tag{2.2}$$

Proof. By Lemma 2.1 each $F(T_i)$ is closed and convex, so the projection $P_{\mathcal{F}}$ is well-defined. Putting $Q = P_{\mathcal{F}}$, we show that $Q(I - A + \gamma f)$ is a contraction from C into itself. In fact, for any $x, y \in C$ we have

$$\begin{aligned}\|Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y)\| &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\|\\ &\leq \|(I - A)x - (I - A)y\| + \gamma\|fx - fy\|\\ &\leq (1 - \overline{\gamma})\|x - y\| + \gamma b\|x - y\|\\ &\leq (1 - (\overline{\gamma} - \gamma b)\|x - y\|.\end{aligned}$$

So there exists a unique element $q \in C$ such that $q = P_{\mathcal{F}}(I - A + \gamma f)q$, which by Lemma 1.1 is equivalent to

$$\langle (I - A + \gamma f)q - q, q - p\rangle \geq 0 \qquad (p \in \mathcal{F}).$$

Since $\lim_{n\longrightarrow\infty} a_n = 0$, we can assume that $a_n \in (0, \|A\|^{-1})$, for all $n \geq 0$. By Lemma 1.4 we have $\|I - a_nA\| \leq 1 - a_n\overline{\gamma}$. Now we show that $\{x_n\}$ is bounded. Choose $p \in \mathcal{F}$. Since for each $1 \leq i \leq m$, P_{T_i} is quasi-nonexpansive we have

$$\begin{aligned}&\|y_n - p\| = \|b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + ... + b_{n,m}z_{n,m} - p\|\\ &= b_{n,0}\|x_n - p\| + b_{n,1}\|z_{n,1} - p\| + b_{n,2}\|z_{n,2} - p\| + ... + b_{n,m}\|z_{n,m} - p\|\\ &\leq b_{n,0}\|x_n - p\| + b_{n,1}\, dist(z_{n,1}, P_{T_1}p) + b_{n,2}\, dist(z_{n,2}, P_{T_2}p) + ... \qquad (2.3)\\ &+ \quad b_{n,m}\, dist(z_{n,m}, P_{T_m}p)\\ &\leq b_{n,0}\|x_n - p\| + b_{n,1}\mathfrak{h}(P_{T_1}x_n, P_{T_1}p) + b_{n,2}\mathfrak{h}(P_{T_2}x_n, P_{T_2}p) + ... \qquad (2.4)\\ &+ \quad b_{n,m}\mathfrak{h}(P_{T_m}x_n, P_{T_m}p)\\ &\leq b_{n,0}\|x_n - p\| + b_{n,1}\|x_n - p\| + b_{n,2}\|x_n - p\| + ... + b_{n,m}\|x_n - p\|\\ &\leq \|x_n - p\|, \qquad (2.5)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - p\| &= \|a_n(\gamma fx_n - Ap) + (I - a_nA)(y_n - p)\|\\ &\leq a_n\|\gamma fx_n - Ap\| + \|I - a_nA\|\|y_n - p\|\\ &\leq a_n\|\gamma fx_n - Ap\| + (1 - a_n\overline{\gamma})\|x_n - p\|\\ &\leq a_n\gamma\|fx_n - fp\| + a_n\|\gamma fp - Ap\| + (1 - a_n\overline{\gamma})\|x_n - p\|\\ &\leq a_n\gamma b\|x_n - p\| + a_n\|\gamma fp - Ap\| + (1 - a_n\overline{\gamma})\|x_n - p\|\\ &\leq (1 - a_n(\overline{\gamma} - \gamma b))\|x_n - p\| + a_n\|\gamma fp - Ap\|\\ &= (1 - a_n(\overline{\gamma} - \gamma b))\|x_n - p\| + a_n(\overline{\gamma} - \gamma b)\tfrac{\|\gamma fp - Ap\|}{\overline{\gamma} - \gamma b}.\end{aligned}$$

It follows, by induction, that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f p - Ap\|}{\overline{\gamma} - \gamma b}\}.$$

Next, we show that for $i = 1, 2, ..., m$, $\lim_{n \longrightarrow \infty} dist(x_n, P_{T_i} x_n) = 0$. Indeed, using Lemma 1.2, for $p \in \mathcal{F}$ we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + ... + b_{n,m}z_{n,m} - p\|^2 \\
&\leq b_{n,0}\|x_n - p\|^2 + b_{n,1}\|z_{n,1} - p\|^2 + b_{n,2}\|z_{n,2} - p\|^2 \\
&+ .... + b_{n,m}\|z_{n,m} - p\|^2 - \textstyle\sum_{i=1}^m b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 \\
&\leq b_{n,0}\|x_n - p\|^2 + b_{n,1}\, dist(z_{n,1}, P_{T_1}p)^2 + b_{n,2}\, dist(z_{n,2}, P_{T_2}p)^2 \\
&+ .... + b_{n,m}\, dist(z_{n,m}, P_{T_m}p)^2 - \textstyle\sum_{i=1}^m b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 \\
&\leq b_{n,0}\|x_n - p\|^2 + b_{n,1}\mathfrak{h}(P_{T_1}x_n, P_{T_1}p)^2 + b_{n,2}\mathfrak{h}(P_{T_2}x_n, P_{T_2}p)^2 \\
&+ .... + b_{n,m}\mathfrak{h}(P_{T_m}x_n, P_{T_m}p)^2 - \textstyle\sum_{i=1}^m b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 \\
&\leq b_{n,0}\|x_n - p\|^2 + b_{n,1}\|x_n - p\|^2 + b_{n,2}\|x_n - p\|^2 \\
&+ .... + b_{n,m}\|x_n - p\|^2 - \textstyle\sum_{i=1}^m b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 \\
&\leq \|x_n - p\|^2 - \textstyle\sum_{i=1}^m b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2.
\end{aligned}$$

Hence for $i = 1, 2, ..., m$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|a_n(\gamma f x_n - Ap) + (I - a_n A)(y_n - p)\|^2 \\
&\leq a_n^2\|\gamma f x_n - Ap\|^2 + (1 - a_n\overline{\gamma})^2\|y_n - p\|^2 + 2a_n(1 - a_n\overline{\gamma})\|\gamma f x_n - Ap\|\|y_n - p\| \\
&\leq a_n^2\|\gamma f x_n - Ap\|^2 + (1 - a_n\overline{\gamma})^2\|x_n - p\|^2 + 2a_n(1 - a_n\overline{\gamma})\|\gamma f x_n - Ap\|\|x_n - p\| \\
&- (1 - a_n\overline{\gamma})b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
(1 - a_n\overline{\gamma})b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+ 2a_n(1 - a_n\overline{\gamma})\|\gamma f x_n - Ap\|\|x_n - p\| + a_n^2\|\gamma f x_n - Ap\|^2.
\end{aligned} \tag{2.6}$$

In order to prove that $x_n \longrightarrow q$ as $n \longrightarrow \infty$, we consider two possible cases.

Case 1. Assume that $\{\|x_n - q\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - q\|\}_{n \geq n_0}$ is either nondecreasing or nonincreasing. Since $\{\|x_n - q\|\}$ is bounded, it is convergent. Since $\lim_{n \longrightarrow \infty} a_n = 0$ and $\{f x_n\}$, $\{x_n\}$ are bounded, from (2.6) we have

$$\lim_{n \longrightarrow \infty} (1 - a_n\overline{\gamma})b_{n,0}b_{n,i}\|x_n - z_{n,i}\|^2 = 0,$$

and by the assumption we get

$$\lim_{n \longrightarrow \infty} \|x_n - z_{n,i}\| = 0.$$

Hence for $i = 1, 2, ..., m$, we have

$$\lim dist(x_n, P_{T_i} x_n) \leq \|x_n - z_{n,i}\| \longrightarrow 0, \quad n \longrightarrow \infty.$$

Next, we show that

$$\lim sup_{n \longrightarrow \infty} \langle (A - \gamma f)q, q - x_n \rangle \leq 0.$$

To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i \longrightarrow \infty} (\langle A - \gamma f)q, q - x_{n_i} \rangle = \lim sup_{n \longrightarrow \infty} \langle (A - \gamma f)q, q - x_n \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Since $I - P_{T_i}$ is demiclosed at zero, we have $w \in \mathcal{F}$. Since $q = P_{\mathcal{F}}(I - A + \gamma f)q$ and $w \in \mathcal{F}$, by Lemma 1.1 it follows that

$$\lim sup_{n \longrightarrow \infty} \langle (A - \gamma f)q, q - x_n \rangle = \lim_{i \longrightarrow \infty} (\langle A - \gamma f)q, q - x_{n_i} \rangle = (\langle A - \gamma f)q, q - w \rangle \leq 0.$$

From

$$x_{n+1} - q = a_n(\gamma f x_n - Aq) + (I - a_n A)(y_n - q),$$

and (2.3), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|(I - a_n A)(y_n - q)\|^2 + 2a_n \langle \gamma f x_n - Aq, x_{n+1} - q \rangle \\
&\leq (1 - a_n \overline{\gamma})^2 \|x_n - q\|^2 + 2a_n \gamma \langle f x_n - fq, x_{n+1} - q \rangle + 2a_n \langle \gamma fq - Aq, x_{n+1} - q \rangle \\
&\leq (1 - a_n \overline{\gamma})^2 \|x_n - q\|^2 + 2a_n b \gamma \|x_n - q\| \|x_{n+1} - q\| + 2a_n \langle \gamma fq - Aq, x_{n+1} - q \rangle \\
&\leq (1 - a_n \overline{\gamma})^2 \|x_n - q\|^2 + a_n b \gamma (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + 2a_n \langle \gamma fq - Aq, x_{n+1} - q \rangle \\
&\leq ((1 - a_n \overline{\gamma})^2 + a_n b \gamma) \|x_n - q\|^2 + a_n \gamma b \|x_{n+1} - q\|^2 + 2a_n \langle \gamma fq - Aq, x_{n+1} - q \rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \tfrac{1 - 2a_n \overline{\gamma} + (a_n \overline{\gamma})^2 + a_n \gamma b}{1 - a_n \gamma b} \|x_n - q\|^2 + \tfrac{2a_n}{1 - a_n \gamma b} \langle \gamma fq - Aq, x_{n+1} - q \rangle \\
&= (1 - \tfrac{2(\overline{\gamma} - \gamma b)a_n}{1 - a_n \gamma b}) \|x_n - q\|^2 + \tfrac{(a_n \overline{\gamma})^2}{1 - a_n \gamma b} \|x_n - q\|^2 + \tfrac{2a_n}{1 - a_n \gamma b} \langle \gamma fq - Aq, x_{n+1} - q \rangle \\
&\leq (1 - \tfrac{2(\overline{\gamma} - \gamma b)a_n}{1 - a_n \gamma b}) \|x_n - q\|^2 + \tfrac{2(\overline{\gamma} - \gamma b)a_n}{1 - a_n \gamma b} (\tfrac{(a_n \overline{\gamma}^2)M}{2(\overline{\gamma} - \gamma b)} + \tfrac{1}{\overline{\gamma} - \gamma b}) \langle \gamma fq - Aq, x_{n+1} - q \rangle) \\
&= (1 - \gamma_n) \|x_n - q\|^2 + \gamma_n \delta_n,
\end{aligned}$$

where

$$M = sup\{\|x_n - q\|^2 : n \geq 0\}, \qquad \gamma_n = \frac{2(\overline{\gamma} - \gamma b)a_n}{1 - a_n \gamma b},$$

and

$$\delta_n = \frac{(a_n \overline{\gamma}^2)M}{2(\overline{\gamma} - \gamma b)} + \frac{1}{\overline{\gamma} - \gamma b} \langle \gamma fq - Aq, x_{n+1} - q \rangle.$$

It is easily seen that $\gamma_n \longrightarrow 0, \sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \longrightarrow \infty} \delta_n \leq 0$. Hence, by Lemma 1.3 the sequence $\{x_n\}$ converges strongly to q.

Case 2. Assume that $\{\|x_n - q\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N}; k \leq n : \|x_k - q\| < \|x_{k+1} - q\|\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and for all $n \geq n_0$,

$$\|x_{\tau(n)} - q\| < \|x_{\tau(n)+1} - q\|.$$

From (2.6) we obtain that

$$\lim_{n \longrightarrow \infty} \|x_{\tau(n)} - z_{\tau(n),i}\| = 0.$$

Following an argument similar to that in *Case* 1 we have

$$\limsup_{n \longrightarrow \infty} \langle f(q) - q, x_{\tau(n)+1} - q\rangle \leq 0.$$

And by similar argument we have

$$\|x_{\tau(n)+1} - q\|^2 \leq (1 - \eta_{\tau(n)})\|x_{\tau(n)} - q\|^2 + \eta_{\tau(n)}\delta_{\tau(n)},$$

where $\eta_{\tau(n)} \longrightarrow 0, \sum_{n=1}^{\infty} \eta_{\tau(n)} = \infty$ and $\limsup_{n\longrightarrow\infty} \delta_{\tau(n)} \leq 0$. Hence, by Lemma 1.3, we obtain $\lim_{n\longrightarrow\infty} \|x_{\tau(n)} - q\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_{\tau(n)+1} - q\| = 0$. Now Lemma 1.5 implies

$$0 \leq \|x_n - q\| \leq \max\{\|x_{\tau(n)} - q\|, \|x_n - q\|\} \leq \|x_{\tau(n)+1} - q\|.$$

Therefore $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}(I - A + \gamma f)q$. □

A mapping $T : C \longrightarrow CB(C)$ is $\star$-nonexpansive [14] if for every $x, y \in C$ and $u_x \in Tx$ with $d(x, u_x) = \inf\{d(x, z) : z \in Tx\}$, there exists $u_y \in Ty$ with $d(y, u_y) = \inf\{d(y, w) : w \in Ty\}$ such that

$$d(u_x, u_y) \leq d(x, y).$$

It is not hard to see that if T is $\star$-nonexpansive, then P_T is nonexpansive. It should be mentioned that the $\star$-nonexpansiveness is different from the nonexpansiveness for set-valued mappings, (see [15, 16, 17] for details).

Corollary 1. *Let $T_i : C \longrightarrow K(C)$, $i = 1, 2, ..., m$, be a finite family of $\star$-nonexpansive set-valued mappings. Let $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that f is a contraction from H into itself with constant $b \in (0, 1)$ and A is a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by an arbitrary element $x_0 \in C$ and*

$$\begin{cases} y_n = b_{n,0}x_n + b_{n,1}z_{n,1} + b_{n,2}z_{n,2} + ... + b_{n,m}z_{n,m}, \\ x_{n+1} = a_n\gamma f x_n + (I - a_nA)y_n, \end{cases}$$

for all $n \geq 0$, where $\sum_{i=0}^m b_{n,i} = 1$, $z_{n,i} \in P_{T_i}(x_n)$ and $\{a_n\}, \{b_{n,i}\}$ satisfy the following conditions:

(i) $a_n \subset (0, 1)$, $\lim_{n\longrightarrow\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $\{b_{n,i}\} \subset [c, 1) \subset (0, 1), i = 0, 1, ..., m$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$ which solves the variational inequality (2.2).

Proof. Since each T_i is $\star$-nonexpansive, so P_{T_i} is nonexpansive, for every i, $1 \leq i \leq m$. By Lemma 2.2, each mapping $I - P_{T_i}$ is demiclosed at zero. Now by using Theorem 2.4, we easily obtain the desired result. □

In Theorem 2.4, if we assume that each T_i, for $i = 1, 2, ..., m$ is single-valued and put $\gamma = 1$, $A = I$, we obtain the following corollary.

Corollary 2. *Let $T_i : C \longrightarrow C$, $i = 1, 2, ..., m$, be a finite family of quasi-nonexpansive mappings such that each $I - T_i$ is demiclosed at zero. Let $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Assume that f is a contraction from H into itself with constant $b \in (0, 1)$. Let $\{x_n\}$ be a sequence generated by an arbitrary element $x_0 \in C$ and*

$$\begin{cases} y_n = b_{n,0}x_n + b_{n,1}T_1x_n + b_{n,2}T_2x_n + ... + b_{n,m}T_mx_n, \\ x_{n+1} = a_n f x_n + (1 - a_n)y_n, \end{cases}$$

for all $n \geq 0$, where $\sum_{i=0}^{m} b_{n,i} = 1$, and $\{a_n\}, \{b_{n,i}\}$ satisfy the following conditions:

(i) $a_n \subset (0, 1)$, $\lim_{n \longrightarrow \infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,

(ii) $\{b_{n,i}\} \subset [c, 1) \subset (0, 1), i = 0, 1, ..., m$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$ which solves the variational inequality:

$$\langle q - fq, x - q \rangle \geq 0, \qquad (x \in \mathcal{F}).$$

Remark 2.2. All the results above hold, if we assume that T is quasi-nonexpansive and for all $p \in F(T), T(p) = \{p\}$.

2.1. Application

Recently, Kohsaka and Takahashi [18, 19] introduced an important class of mappings which they called the class of nonspreading mappings. More precisely, a mapping $T : C \longrightarrow C$ is called nonspreading if

$$2\,\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \qquad (x, y \in C).$$

Note that if T is nonespreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive and $I - T$ is demiclosed at zero (see [20] for details). Recently, Iemoto and Takahashi [20] obtained some fundamental properties for nonspreading mappings in Hilbert spaces. Now, as a conclusion of Theorem 2.4, we give the following corollary for a finite family of nonspreading mappings and a finite family of nonexpansive mappings.

Corollary 3. *Let $T_i : C \longrightarrow C$, $i = 1, 2, ..., m$, be a finite family of nonexpansive mappings and $S_i : C \longrightarrow C$, $i = 1, 2, ..., m$, be a finite family of nonspreading mappings such that $\mathcal{F} = \bigcap_{i=1}^{m} (F(T_i) \bigcap F(S_i)) \neq \emptyset$. Assume that f is a contraction from H into itself with constant $b \in (0, 1)$ and A is a strongly positive bounded linear* *operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by an arbitrary element $x_0 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^{m} \beta_{n,i} T_i x_n + \sum_{i=1}^{m} \gamma_{n,i} S_i x_n, \\ x_{n+1} = a_n \gamma f x_n + (I - a_n A) y_n, \end{cases}$$

for all $n \geq 0$, where $\alpha_n + \sum_{i=1}^{m} \beta_{n,i} + \sum_{i=1}^{m} \gamma_{n,i} = 1$, and $\{a_n\}, \{\alpha_n\}, \{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ satisfy the following conditions:

(i) $a_n \subset (0,1)$, $\lim_{n \longrightarrow \infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,

(ii) $\{\alpha_n\}, \{\beta_{n,i}\}, \{\gamma_{n,i}\} \subset [c,1) \subset (0,1)$ *for* $i = 1, 2, ..., m$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$ which solves the variational inequality (2.2).

Now, we supply an example to illustrate the main result of this paper.

Example 2. We consider the nonempty closed convex subset $C = [0, 3]$ of the Hilbert space $\mathbb{R}$. Define multivalued mappings $T_1, T_2 : C \longrightarrow K(C)$ as follows:

$$T_1(x) = [\frac{x}{5}, \frac{x}{2}], \qquad T_2(x) = [\frac{x}{3}, x].$$

Observe that

$$P_{T_1}(x) = \{\frac{x}{2}\}, \qquad P_{T_2}(x) = \{x\}.$$

Hence P_{T_1} and P_{T_2} are nonexpansive mappings. Define the contractive mapping $f(x) = \frac{2x}{3}$. Also we define operator $A(x) = \frac{x}{3}$. We see that A is a strongly positive bounded linear operator on H with coefficient $\frac{1}{3}$. We choose $\gamma = \frac{1}{3}$, $b_{n,i} = \frac{1}{3}, (i = 1, 2, 3)$ and $a_n = \frac{1}{n}$. Now, we have the following algorithm:

$$\begin{cases} x_1 \in C \\ y_n = \frac{5x_n}{6}, \\ x_{n+1} = \frac{2x_n}{9n} + \frac{(3n-1)}{3n} y_n : \qquad \forall n \geq 1. \end{cases}$$

Hence we have

$$x_{n+1} = \frac{2x_n}{9n} + \frac{(3n-1)}{3n} \frac{5x_n}{6} = \frac{(15n-1)}{18n} x_n.$$

We observe that for an arbitrary $x_1 \in C$, x_n is convergent to zero. We note that $\mathcal{F} = F(T_1) \bigcap F(T_2) = \{0\}$.

REFERENCES

1. H. K. Xu, *An iterative approach to quadratic optimization.* J. Optim. Theory Appl. **116**, (2003) 659-678.
2. A. Moudafi, *Viscosity approximation methods for fixed point problems.* J. Math. Anal. Appl. **241**, (2000) 46-55.
3. H. K. Xu, *Viscosity approximation methods for nonexpansive mappings.* J. Math. Anal. Appl. **298**, (2004) 279-291.
4. G. Marino, H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces.* J. Math. Anal. Appl. **318**, (2006) 43-52.

5. Y. Yao, M. Postolache, *Iterative Methods for Pseudomonotone Variational Inequalities and Fixed-Point Problems*. J. Optim. Theory Appl. **155**, (2012) 273-287.
6. N. Shahzad, H. Zegeye, *Strong convergence results for nonself multimaps in Banach spaces*. Proc. Amer. Math. Soc. **136**, (2008) 539-548.
7. Y. Song, H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces*. Nonlinear Anal. **70**, (2009) 1547-1556.
8. L. C. Ceng, Y. C. Yao, *Strong convergence of an iterative algorithm for nonself multimaps in Banach spaces*, Nonlinear Anal. **71**, (2009) 4476-4485.
9. A. Abkar, M. Eslamian, *Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT(0) spaces*. Nonlinear Anal. **75**, (2012) 1895-1903.
10. M. Eslamian, A. Abkar, *Strong convergence of a multi-step iterative process for relatively quasi-nonexpansive multivalued mappings and equilibrium problem in Banach spaces*. U.P.B. Sci. Bull., Series A. **74**, (2012) 117-130.
11. H. Zegeye, N. Shahzad, *Convergence of Manns type iteration method for generalized asymptotically nonexpansive mappings*. Comput. Math. Appl. **62**, (2011) 4007-4014.
12. P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Analysis, **16**, (2008) 899-912.
13. S. Itoh, W. Takahashi, *The common fixed point theory of single valued mappings and multivalued mappings*, Pacific J. Math., **79**, (1978) 493-508.
14. T. Husain, A. Latif, *Fixed points of multivalued nonexpansive maps*. Math. Japon. **33**, (1988) 385-391.
15. H. K. Xu, *On weakly nonexpansive and ⋆-nonexpansive multivalued mappings*. Math. Japon. **36**, (1991) 441-445.
16. V. M. Sehgal, P. S. Singh, *A generalization to multi functions of Fan best approximation theorem*. Proc. Amer. Math. Soc. **102**, (1988) 534-537.
17. N. Hussain, A. R. Khan, *Applications of the best approximation operator to ⋆-nonexpansive maps in Hilbert spaces*, Numeri. Func. Anal. Optim. **24**, (2003) 327-338
18. F. Kohsaka, W. Takahashi, *Fixed point theorems for a class of nonlinear mappings relate to maximal monotone operators in Banach spaces*. Arch. Math. (Basel). **91** , (2008) 166-177.
19. F. Kohsaka, W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*. SIAM J. Optim. **19**, (2008) 824-835.
20. S. Iemoto, W. Takahashi, *Approximating commom fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*. Nonlinear Anal. **71**, (2009) 2082-2089.