BAER'S THEOREM AND ITS CONVERSE IN THE VARIETY OF n-ABELIAN GROUPS

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In this article, we first prove an extension of Baer's theorem in the variety of n-Abelian groups. Then we establish the converse of the mentioned result when $G/Z_i^n(G)$ is finitely generated. Next we seek conditions which imply the finiteness of the n-center factor subgroup of an n-Abelian group G and finally we give some upper bounds, for the order of this factor subgroup.

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1. Introduction and Motivation

Because of the importance of Abelian groups in group theory, many generalizations have been considered. One of these generalizations is the concept of n-Abelian group which has been presented in 1944 by Levi [10], for the first time. These groups play an important role in our discussion and extensively have been analyzed in [5], [2], [3] and [1]. If n is an integer and $n \geq 1$, then a group G is said to be n-Abelian if $(xy)^n = x^n y^n$, for all elements x and y in G. It follows that $[x^n, y] = [x, y]^n = [x^n, y^n]$, for n-Abelian groups in which $[x, y] = x^{-1}y^{-1}xy$. It is clear that a group is 2-Abelian if and only if it is Abelian, while non Abelian n-Abelian groups do exist, for every $n > 2$. Here we use two other concepts, the n-potent and the n-center subgroups of a group G, that have been introduced by Fay and Waals [4]. For a positive integer n, the n-potent and the n-center subgroups of a group G are defined respectively, as follows:

$$G_n = \langle [x, y^n] | x, y \in G \rangle,$$
$$Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}.$$

It is easy to see that G_n is a fully invariant subgroup and $Z^n(G)$ is a characteristic subgroup of G.

A famous theorem of Schur asserts that for a group G the finiteness of $G/Z(G)$ implies the finiteness of G'. The proof of this theorem in fact has been stated by B.H. Neumann [11, Theorem 5.3]. He [12, End of page 237] mentioned that this result can be obtained from an implicit idea of I. Schur [16], and his proof used Schur's basic idea. Neumann [11] also provided a partial converse of the Schur's theorem as follows:
If G is finitely generated by k elements and G' is finite, then $G/Z(G)$ is finite and bounded by $|G/Z(G)| \leq |G'|^k$.
Recently P. Niroomand [13] generalized this result by proving that if G' is finite and $G/Z(G)$

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is finitely generated, then $G/Z(G)$ is finite and $|G/Z(G)| \leq |G'|^{d(G/Z(G))}$. B. Sury [17] gave a completely elementary short proof of a further generalization of the Niroomand's result. Also Yadav [19] provided some modifications of the converse of Schur's theorem. In 1952, Baer gave an important generalization of Schur's theorem which asserts that for a group G if $G/Z_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite.
Hall [6] constructed a nilpotent p-group which shows that the converse of Baer's theorem is not true. In 1986 Hekster [8] proved that the converse of Baer's theorem holds, for finitely generated groups. Recently Hatamian et al. [7] improved the mentioned result of Hekster and proved that, if G is a group, $\gamma_{n+1}(G)$ is finite and $G/Z_n(G)$ is finitely generated, then $G/Z_n(G)$ is finite and

$$\left|\frac{G}{Z_n(G)}\right| \leq |\gamma_{n+1}(G)|^{d(G/Z_n(G))^n}.$$

Definition 1.1. *A normal series* $1 = G_0 \leq G_1 \leq \cdots \leq G_t = G$ *of group* G *is called* n*-central series of length* t *if and only if*

$$\frac{G_{i+1}}{G_i} \leq Z^n\left(\frac{G}{G_i}\right).$$

Now we introduce the upper and lower n-central series of G which are two examples of n-central series.

Definition 1.2. *The upper* n*-central series of* G *is defined to be the series*

$$1 = Z_0^n(G) \leq Z_1^n(G) \leq \cdots \leq Z_t^n(G) \leq \cdots$$

where inductively

$$Z_{i+1}^n(G)/Z_i^n(G) = Z^n(G/Z_i^n(G)),$$

for $i \geq 0$*. So* $Z_1^n(G) = Z^n(G)$*.*

Definition 1.3. *Put* $\gamma_1^n(G) = G$*, and let* $\gamma_{i+1}^n(G)$ *be defined inductively, for* $i \geq 1$*, as the subgroup* $[\gamma_i^n(G), G^n]$*. It is clear that the series*

$$G = \gamma_1^n(G) \geq \gamma_2^n(G) \geq \cdots \geq \gamma_t^n(G) \geq \cdots$$

is an n*-central series of* G*. Note that* $\gamma_2^n(G)$ *is the* n*-potent subgroup of* G*.*

Here the subgroups

$$a^{G^n} = \{g^{-n}ag^n | g \in G\}$$

and

$$C(a^n) = \{g \in G | [a^n, g] = 1\}$$

are denoted by $CL^n(a)$ and $C^n(a)$, respectively.
In this article, we first prove that the finiteness of $G/Z_i^n(G)$ implies the finiteness of $\gamma_{i+1}^n(G)$. This is actually a version of Baer's theorem in the variety of n-Abelian groups. Then we prove some theorems to establish the converse of the mentioned result when $G/Z_i^n(G)$ is finitely generated. Next we seek conditions which imply that finiteness of n-center factor subgroup $G/Z^n(G)$ of an n-Abelian group G. Finally we give some upper bounds for the order of n-center factor subgroup of an n-Abelian group G.

2. Main Results

We begin this section by proving a generalization of Baer's theorem. This generalization is in fact a version of Baer's theorem in the variety of n-Abelian group. For this end we need the following lemma.

Lemma 2.1. *If* G *is an* n*-Abelian group, then* $\gamma_i(G^n) = \gamma_i^n(G)$*, for any positive integer* i*.*

Proof. It is clear that $\gamma_i(G^n) \leq \gamma_i^n(G)$. Since G is n-Abelian, so $[x_1, x_2^n] = [x_1^n, x_2^n]$, for any $x_1, x_2 \in G$. Therefore by induction we can plainly show that $[x_1, x_2^n, \dots, x_i^n] = [x_1^n, x_2^n, \dots, x_i^n]$, for any $x_1, x_2, \dots, x_i \in G$. □

Theorem 2.1. *Let G be an n-Abelian group and let $G/Z_i^n(G)$ be finite. Then $\gamma_{i+1}^n(G)$ is finite.*

Proof. Since $G/Z_i^n(G)$ is finite thus $(G^n Z_i^n(G))/Z_i^n(G)$ and so $G^n/(Z_i^n(G) \cap G^n)$ is finite. We can easily show that $Z_i^n(G) \cap G^n = Z_i(G^n)$. Hence $G^n/Z_i(G^n)$ is finite. Therefore Baer's theorem implies the finiteness of $\gamma_{i+1}(G^n)$ and so by Lemma 2.1, $\gamma_{i+1}^n(G)$ is finite. □

As we said before, there are many attempts to prove the converse of Baer's theorem. Therefore we are interested in finding conditions under which the converse of the above theorem holds. We collect here a few results to achieve the mentioned goal.

Lemma 2.2. *Let G be an n-Abelian group. Then $\gamma_i^n(G)/(C_G^n((\gamma_{i+1}^n(G)) \cap \gamma_i^n(G))$ can be embedded in $Aut(\gamma_{i+1}^n(G))$.*

Proof. For any $g \in G$, let $f_g : \gamma_{i+1}^n(G) \to \gamma_{i+1}^n(G)$, defined by $f_g(x) = x^{g^n}$, and let $f : \gamma_i^n(G) \to Aut(\gamma_{i+1}^n(G))$, defined by $f(g) = f_g$. It is easy to see that f is a homomorphism and $\ker f = C_G^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G)$. Hence the assertion follows. □

Theorem 2.2. *Let G be a finitely generated n-Abelian group and $i \geq 1$. If $\gamma_{i+1}^n(G)$ is finite, then $\gamma_i^n(G/Z^n(G))$ is finite.*

Proof. Suppose $G = \langle g_1, g_2, \dots, g_t \rangle$, for a natural number t. Therefore

$$Z^n(G) = \bigcap_{j=1}^{t} C^n(g_j).$$

Let $f_j : C_G^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G) \to \gamma_{i+1}^n(G)$, defined by $f_j(x) = [x, g_j^n]$. Clearly f_j is a well-defined homomorphism and we can show that

$$\ker f_j = C^n(g_j) \cap C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G).$$

Put $K_j = \ker f_j$. Since $\gamma_{i+1}^n(G)$ is finite, $[C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G) : K_j] < \infty$. On the other hand $\bigcap_{j=1}^t K_j = Z^n(G) \cap C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G) = Z^n(G) \cap \gamma_i^n(G)$. Therefore by Poincare's Lemma the following index is finite.

$$[C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G) : \bigcap_{j=1}^{t} K_j] = [C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G) : Z^n(G) \cap \gamma_i^n(G)].$$

The finiteness of $Aut(G)$ implies the finiteness of $\gamma_i^n(G)/(C^n(\gamma_{i+1}^n(G)) \cap \gamma_i^n(G))$ by Lemma 2.2. Hence $[\gamma_i^n(G) : Z^n(G) \cap \gamma_i^n(G)] < \infty$ and therefore by the following isomorphism, $\gamma_i^n(G/Z^n(G))$ is finite.

$$\frac{\gamma_i^n(G)}{Z^n(G) \cap \gamma_i^n(G)} \cong \frac{Z^n(G)\gamma_i^n(G)}{Z^n(G)} = \gamma_i^n\left(\frac{G}{Z^n(G)}\right).$$

□

Let $\mathcal{V}$ be the variety of groups defined by the word $v = \{[x_1, x_2^n, \dots, x_{c+1}^n]\}$. The following proposition gives the structures of verbal and marginal subgroups of n-Abelian groups, for the variety $\mathcal{V}$.

Proposition 2.1. *By the above assumption $V(G) = \gamma_{c+1}^n(G)$ and $V^*(G) = Z_c^n(G)$, for any n-Abelian group G.*

Proof. This is easy to show that $V(G) = \gamma^n_{c+1}(G)$. We show that $V^*(G) = Z^n_c(G)$. Let $x \in V^*(G)$ and e be the identity element of G, thus $[xe, g_1^n, \dots, g_c^n] = [e, g_1^n, \dots, g_c^n] = e$, for any $g_1, \dots, g_c \in G$. Therefore $x \in Z^n_c(G)$ and so $V^*(G) \le Z^n_c(G)$. We use induction on c to show that $Z^n_c(G) \le V^*(G)$. For $c = 1$ if $x \in Z^n(G)$, then for any $y, g \in G$, we have $[xy, g^n] = [x, g^n]^y [y, g^n] = [y, g^n]$, and

$$[y, (xg)^n] = [y^n, xg] = [y^n, g][y^n, x]^g = [y, g^n].$$

Therefore $x \in V^*(G)$.
Now suppose that for all $i < c$, $Z^n_i(G) \le V^*(G)$. We prove that $Z^n_c(G) \le V^*(G)$. Let $x \in Z^n_c(G)$. Then $xZ^n(G) \in Z^n_{c-1}(G/Z^n(G))$ and so by induction hypothesis, $xZ^n(G) \in V^*(G/Z^n(G))$. It follows that for all $g_j \in G$, $1 \le j \le c$ and all $1 \le i < c$, we have

$$[g_1, g_2^n, ..., (g_i x)^n, ..., g_c^n] \equiv [g_1, g_2^n, ..., g_i^n, ..., g_c^n] \ (\mathrm{mod}\ Z^n(G)).$$

Now let $X = [g_1, g_2^n, ..., (g_i x)^n, ..., g_c^n]$ and $Y = [g_1, g_2^n, ..., g_i^n, ..., g_c^n]$. Then $X = Yz$, for an element $z \in Z^n(G)$. Thus

$$[X, g^n_{c+1}] = [Yz, g^n_{c+1}] = [Y, g^n_{c+1}]^z [z, g^n_{c+1}] = ([Y, g_{c+1}]^n)^z = [Y, g^n_{c+1}].$$

So

$$[g_1, g_2^n, ..., (g_i x)^n, ..., g_c^n, g^n_{c+1}] = [g_1, g_2^n, ..., g_i^n, ..., g_c^n, g^n_{c+1}]. \quad (1)$$

The missing case is $i = c + 1$. Put $y = g_c^{-1}$ and $z = [g_1, g_2^n, ..., g^n_{c-1}]$. Then

$$[g_1, g_2^n, ..., g_c^n, x^n] = [z, y^{-n}, x^n],$$

and therefore by equality $[z, y^{-1}, x]^y [y, x^{-1}, z]^x [x, z^{-1}, y]^z = 1$ we have

$$[z, y^{-n}, x^n] = (([x^n, z^{-1}, y^n]^z)^{-1} ([y^n, x^{-n}, z]^{x^n})^{-1})^{y^{-n}}.$$

The equality $Z^n_c(G)/Z^n_{c-1}(G) = Z^n(G/Z^n_{c-1}(G))$ implies that $[Z^n_c(G), G^n] \le Z^n_{c-1}(G)$. It follows that

$$[g_1, g_2^n, ..., g_c^n, x^n] = [z, y^{-n}, x^n] = 1. \quad (2)$$

Now by (2) we are in a position to show that

$$[g_1, g_2^n, ..., g_c^n, g^n_{c+1} x^n] = [g_1, g_2^n, ..., g_c^n, g^n_{c+1}].$$

Of course

$$[g_1, g_2^n, ..., g_c^n, g^n_{c+1} x^n] = [g_1, g_2^n, ..., g_c^n, x^n][g_1, g_2^n, ..., g_c^n, g^n_{c+1}]^{x^n}.$$

So

$$([g_1, g_2^n, ..., g_c^n, g^n_{c+1} x^n])^{x^{-n}} = [g_1, g_2^n, ..., g_c^n, g^n_{c+1}]$$

and

$$([g_1, g_2^n, ..., g_c^n, g^n_{c+1} x^n])^{x^{-n}} = [g_1^{x^{-n}}, (g_2^n)^{x^{-n}}, ..., (g_c^n)^{x^{-n}}, (g^n_{c+1})^{x^{-n}} (x^n)^{x^{-n}}].$$

Therefore by (1), $[g_1, g_2^n, ..., g_c^n, x^n g^n_{c+1}] = [g_1, g_2^n, ..., g_c^n, g^n_{c+1}]$, which proves the proposition. □

Hekster [8] defined the concept of $\mathcal{V}$-isologism between groups for an arbitrary variety $\mathcal{V}$. We apply Proposition 2.1 to this definition, for the variety $\mathcal{V}$ defined by the word $v = \{[x_1, x_2^n, ..., x^n_{c+1}]\}$.

Definition 2.1. *Let G and H be n-Abelian groups. Then (n, i)-isoclinism between G and H is a pair of isomorphisms (α, β) with $\alpha : G/Z^n_i(G) \to H/Z^n_i(H)$ and $\beta : \gamma^n_{i+1}(G) \to \gamma^n_{i+1}(H)$, such that for all $g_1, ..., g_{i+1} \in G$, $\beta([g_1, g_2^n, ..., g^n_{i+1}]) = [h_1, h_2^n, ..., h^n_{i+1}]$, whenever $h_j \in \alpha(g_j Z^n_i(G))$, for $1 \le j \le i + 1$. Then we write $G \sim_{(n,i)} H$ and we say that G and H are (n, i)-isoclinic.*

Lemma 2.3. *Let G be an n-Abelian group and $H \leq G$. Then H is (n,i)-isoclinic to $HZ_i^n(G)$.*

Proof. See [8, Lemma 4.4]. □

Lemma 2.4. *If G is an n-Abelian group, then $[G : C^n(a)] = |CL^n(a)|$, for any $a \in G$.*

Proof. For any $a \in G$, $f : G/C(a^n) \to CL^n(a)$, defined by $f(gC(a^n)) = g^{-n}ag^n$, for all $g \in G$, is a correspondence between $G/C^n(a)$ and $CL^n(a)$. □

We have now accumulated all the information necessary to prove the following result which is a modification of the converse of Baer's theorem in the variety of n-Abelian groups.

Theorem 2.3. *Let G be an n-Abelian group such that $G/Z_i^n(G)$ is finitely generated. Then if $\gamma_{i+1}^n(G)$ is finite, then $G/Z_i^n(G)$ is finite and*

$$\left|\frac{G}{Z_i^n(G)}\right| \leq \left|\gamma_{i+1}^n(G)\right|^{(d(\frac{G}{Z_i^n(G)}))^i}$$

where $d(X)$ is the least number of generator of the group X.

Proof. For convenience, put $t = d(G/Z_i^n(G))$. We use induction on i. If $i = 1$ and $G/Z^n(G) = \langle x_1Z^n(G), x_2Z^n(G), ..., x_tZ^n(G)\rangle$, then $G = HZ^n(G)$ such that $H = \langle x_1, x_2, ..., x_t\rangle$. So by Lemma 2.3, G and H are $(n,1)$-isoclinic. Therefore $G/Z_1^n(G) \cong H/Z_1^n(H)$ and $\gamma_{1+1}^n(G) \cong \gamma_{1+1}^n(H)$. These isomorphisms allow us to apply H instead of G such that $d(H) = t$. Thus by Lemma 2.4 we have

$$[H : Z^n(H)] = [H : \cap_{i=1}^t C_H^n(x_i)] \leq \prod_{i=1}^t [H : C_H^n(x_i)] = \prod_{i=1}^t |x_i^{H^n}|.$$

Since $x_i^{h^n} = x_i[x_i, h^n]$, for any $h \in H$, thus $|x_i^{H^n}| \leq |x_iH_n| = |H_n|$ where $H_n = \gamma_2^n(H)$. Therefore $[H : Z^n(H)] \leq \prod_{i=1}^t |H_n| = |H_n|^t = |\gamma_2^n(H)|^t$. Now suppose that the assertion holds for $i - 1$. Let

$$G/Z_i^n(G) = \langle x_1Z_i^n(G), x_2Z_i^n(G), \ldots, x_tZ_i^n(G)\rangle$$

and $\gamma_{i+1}^n(G)$ be finite. Then G and H are (n,i)-isoclinic where $H = \langle x_1, \cdots x_t\rangle$. Therefore $\gamma_{i+1}^n(H)$ is finite. Thus by Theorem 2.2, $\gamma_i^n(H/Z^n(H))$ is finite. Hence the induction assumption holds, for $H/Z^n(H)$. So we have

$$\left|\frac{H}{Z_i^n(H)}\right| = \left|\frac{H/Z^n(H)}{Z_i^n(H)/Z^n(H)}\right| = \left|\frac{H/Z^n(H)}{Z_{i-1}^n(H/Z^n(H))}\right| \leq |\gamma_i^n\left(\frac{H}{Z^n(H)}\right)|^{t^{i-1}}. \quad (3)$$

Since $\bigcap_{i=1}^t C_H^n(x_i) = Z^n(H)$, thus $Z^n(H) \cap \gamma_i^n(H) = \bigcap_{i=1}^t C_{\gamma_i^n(H)}^n(x_i)$. So

$$\gamma_i^n\left(\frac{H}{Z^n(H)}\right) = \frac{\gamma_i^n(H)Z^n(H)}{Z^n(H)} \cong \frac{\gamma_i^n(H)}{Z^n(H) \cap \gamma_i^n(H)} = \frac{\gamma_i^n(H)}{\bigcap_{i=1}^t C_{\gamma_i^n(H)}^n}.$$

Now we have

$$\begin{aligned}
[\gamma_i^n(H) : \bigcap_{i=1}^t C_{\gamma_i^n(H)}^n(x_i)] &\leq \prod_{i=1}^t [\gamma_i^n(H) : C_{\gamma_i^n(H)}^n(x_i)] \\
&= \prod_{i=1}^t |x_i\gamma_{i+1}^n(H)| \\
&= |\gamma_{i+1}^n(H)|^t.
\end{aligned}$$

On the other hand, G and H are (n,i)-isoclinic. Therefore by (3),

$$\left|\frac{G}{Z_i^n(G)}\right| = \left|\frac{H}{Z_i^n(H)}\right| \leq (|\gamma_{i+1}^n(H)|^{t^{i-1}})^t = |\gamma_{i+1}^n(H)|^{t^i} = |\gamma_{i+1}^n(G)|^{t^i}.$$

□

Now an attempt will be made to study the finiteness of the n-center factor of n-Abelian groups under conditions different from Theorem 2.3.

Theorem 2.4. *Let G be an n-Abelian group having a normal subgroup A such that the index of $C_G^n(A)$ in G is finite and G/A is finitely generated by $g_1A,...,g_rA$ where $|g_i^{G^n}| < \infty$, for $1 \leq i \leq r$. Then $G/Z^n(G)$ is finite.*

Proof. Let G/A be generated by $g_1A,...,g_rA$, for some $g_i \in G$, where $1 \leq i \leq r$. Let $X = \{g_1,...,g_r\}$ and A be generated by a set Y. Then $G = \langle X \cup Y\rangle$ and $Z^n(G) = C_G^n(X) \cap C_G^n(Y)$. Note that $C_G^n(A) = C_G^n(Y)$. Therefore $C_G^n(Y)$ is of finite index in G. On the other hand, since $|g_i^{G^n}| < \infty$, for $1 \leq i \leq r$, $C_G^n(X)$ is also of finite index in G, by Lemma 2.4. Hence $G/Z^n(G) = G/(C_G^n(X) \cap C_G^n(Y))$ is finite. □

Theorem 2.5. *Let G be an n-Abelian and let $|\gamma_2^n(G)/(\gamma_2^n(G) \cap Z^n(G))| = t$. Then*

$$\left|\frac{G}{Z_2^n(G)}\right| \leq t^{2\log_2^t}.$$

Proof. For the proof we refer the reader to [15]. □

Yadav provided a modification of the converse of Schur's theorem in [19]. The next result is a generalization of Yadav's result.

Theorem 2.6. *For an n-Abelian group G, $G/Z^n(G)$ is finite if any one of the following holds.*
(i) $Z_2^n(G)/Z^n(Z_2^n(G))$ is finitely generated and $\gamma_2^n(G)$ is finite.
(ii) $G/Z^n(Z_2^n(G))$ is finitely generated and $G/(Z_2^n(G)\gamma_2^n(G))$ is finite.
(iii) $\gamma_2^n(G)$ is finite and $Z^n(Z_2^n(G)) = Z_2^n(G)$.
(iv) $\gamma_2^n(G)$ is finite and $Z_2^n(G) \leq \gamma_2^n(G)$.

Proof. Since $\gamma_2^n(G)$ is finite, it follows from Theorem 2.5, that $G/Z_2^n(G)$ is finite. Now using the fact that $Z_2^n(G)/Z^n(Z_2^n(G))$ is finitely generated, it follows that $G/Z^n(Z_2^n(G))$ is finitely generated. Take $A = Z^n(Z_2^n(G))$. Then note that A is a normal subgroup of G and since $Z_2^n(G) \leq C_G^n(A)$, so the index of $C_G^n(A)$ in G is finite. Hence $G/Z^n(G)$ is finite, by Theorem 2.4 and (i) follows.
Again take $A = Z^n(Z_2^n(G))$ and note that $Z_2^n(G)\gamma_2^n(G) \leq C_G^n(A)$. So proof of (ii) directly follows from Theorem 2.4. Proof of (iii) follows from Theorem 2.4 by taking $A = Z_2^n(G)$ and using Theorem 2.5. If $Z_2^n(G) \leq \gamma_2^n(G)$, then $Z^n(Z_2^n(G)) = Z_2^n(G)$. Thus (iv) follows from (iii). □

To close this article we give some bounds for the order of $G/Z^n(G)$, for an n-Abelian group G, which generalize some results of [19].

Theorem 2.7. *Let G be an n-Abelian group in which $[\gamma_2^n(G)Z^n(G) : Z^n(G)] = t$. Let $Z_2^n(G)/Z^n(G)$ be finitely generated by $x_1Z^n(G), x_2Z^n(G), \ldots, x_sZ^n(G)$ such that $exp([x_i, G^n])$ is finite, for any $1 \leq i \leq s$ and let $Z_2^n(G)/Z^n(G)$ has no element as $xZ^n(G)$ such that $x^n \in Z^n(G)$. Then*

$$\left|\frac{G}{Z^n(G)}\right| \leq t^{2\log_2 t} \prod_{i=1}^{s} \exp([x_i, G^n]).$$

Proof. From Theorem 2.5, $|G/Z_2^n(G)| \leq t^{2\log_2 t}$. By hypothesis $exp([x_i, G^n])$ is finite, for any $1 \leq i \leq s$. Suppose that $exp([x_i, G^n]) = t_i$. Since $x_i \in Z_2^n(G)$, so $[x_i^k, G^n] \subseteq Z^n(G)$, for any integer k. Then we can use induction on k to prove $[x_i^k, g^n] = [x_i, g^n]^k$. So $[x_i^{t_i}, g^n] = [x_i, g^n]^{t_i} = 1$, for any $1 \leq i \leq s$. Thus $x_i^{t_i} \in Z^n(G)$ and no lesser power of x_i than t_i can lie in $Z^n(G)$. Since $Z_2^n(G)/Z^n(G)$ has no element as $xZ^n(G)$ such that $x^n \in Z^n(G)$, therefore

$$\left|\left(\frac{Z_2^n(G)}{Z^n(G)}\right)^n\right| = \left|\frac{Z_2^n(G)}{Z^n(G)}\right|.$$

Hence $(Z_2^n(G)/Z^n(G))^n$ is Abelian and generated by

$$x_1Z^n(G), x_2Z^n(G), \ldots, x_sZ^n(G).$$

So

$$\left|\left(\frac{Z_2^n(G)}{Z^n(G)}\right)^n\right| \leq \prod_{i=1}^{s} \exp([x_i, G^n]).$$

Therefore

$$\begin{aligned}\left|\frac{G}{Z^n(G)}\right| &= \left|\frac{G}{Z_2^n(G)}\right|\left|\frac{Z_2^n(G)}{Z^n(G)}\right| \\ &= \left|\frac{G}{Z_2^n(G)}\right|\left|\left(\frac{Z_2^n(G)}{Z^n(G)}\right)^n\right| \\ &\leq t^{2\log_2 t}\prod_{i=1}^{s} \exp([x_i, G^n]).\end{aligned}$$

□

Lemma 2.5. *Let G be an n-Abelian group and let H be a subgroup of G generated by $h_1, h_2, \ldots, h_t$ and $Z^n(G)$, such that $[h_i, G^n]$ is finite, for $1 \leq i \leq t$. Then $|G/C_G^n(H)| \leq \prod_{i=1}^{t} |[h_i, G^n]|$.*

Proof. Indeed, by Poincare's Lemma

$$|G : C_G^n(H)| \leq \prod_{i=1}^{t} |G : C_G^n(h_i)| = \prod_{i=1}^{t} |h_i^{G^n}|.$$

Since $|G : C_G^n(z)| = 1$, for any $z \in Z^n(G)$ and $|[h_i, G^n]| = |h_i^{G^n}|$, the desired conclusion follows as well. □

Proposition 2.2. *Let G be an n-Abelian group and assume that $G/Z^n(G)$ is finitely generated by $x_1Z^n(G), x_2Z^n(G), \ldots, x_tZ^n(G)$ such that $[x_i, G^n]$ is finite, for $1 \leq i \leq t$. Then*

$$\left|\frac{G}{Z^n(G)}\right| \leq \prod_{i=1}^{t} |[x_i, G^n]|.$$

Proof. Put $H = G$ in previous Lemma. □

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