A NOTE ON THE $p(x)$-CURL-SYSTEMS PROBLEM ARISING IN ELECTROMAGNETISM

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This paper deals with the existence of one non trivial solution for the $p(x)$-Curl systems with sign-changing weight and nonstandard growth conditions. Our main tool is the variational method.

Keywords: Curl $p(x)$-systems, variational method, generalized Sobolev space, variable exponent.

35J20, 35J60, 35G30, 35J35, 46E35.

1. Introduction

The study of partial differential equation with non standard growth conditions is an intriguing content of exploration due to its significant part in many topics and disciplines of mathematics. In fact this type of equations is very active in many fields, we mention e.g.,the filtration of barotropic gas through a porous medium Antontsev-Shmarev [3], image processing Chen-Levine-Rao [5], stationary thermorheological viscous flows of non-Newtonian fluids Rajagopal-Ružička,[21] electrorheological fluids Ružička [22] and elastic mechanics Zhikov [24]. For recent problems involving these kind of operators, the reader can be referred to the papers Ge [12], Hsini-Irzi-Kefi [16], Hou-Ge-Zhang [17], Kefi [18], Kefi-Irzi-Al-Shomrani-Repovš [19] and Hamdani et al [15].

In this paper, we shall show the existence of a non-trivial weak solution for the following problem involving the $p(x)$-curl operator

$$\begin{cases} \nabla\times(|\nabla\times\mathbf{u}|^{p(x)-2}\nabla\times\mathbf{u})+a(x)|\mathbf{u}|^{p(x)-2}\ \mathbf{u}=\lambda V(x)|\mathbf{u}|^{q(x)-2}\mathbf{u}, \quad \nabla\cdot\mathbf{u}=0 & \text{in } \Omega \\ |\nabla\times\mathbf{u}|^{p(x)-2}\nabla\times\mathbf{u}\times\mathbf{n}=0, \quad \mathbf{u}\cdot\mathbf{n}=0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded simply connected domain of $\mathbb{R}^3$ with a $C^{1,1}$ boundary denoted by $\partial\Omega$, $p(x), q(x)\in C(\overline{\Omega})$, $1<p^-\leq p(x)\leq p^+<3$ and $p(x)$ satisfies logarithmic continuity: there exists a function $\gamma:\mathbb{R}_0^+\to\mathbb{R}_0^+$ such that

$$\forall x,y\in\overline{\Omega},\ |x-y|<1,\ |p(x)-p(y)|\leq\gamma(|x-y|),\ \lim_{t\to0^+}\gamma(t)\log\frac{1}{t}=C<\infty.$$

$\mathbf{u}$ is a vector function on Ω. The divergence of $\mathbf{u}=(u_1,u_2,u_3)$ is denoted by

$$\nabla\cdot\mathbf{u}=\partial_{x_1}u_1+\partial_{x_2}u_2+\partial_{x_3}u_3$$

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and the curl of $\mathbf{u}$ is defined by

$$\nabla \times \mathbf{u} = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1).$$

Then $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ satisfy the following identity

$$-\Delta \mathbf{u} = \nabla \times (\nabla \times \mathbf{u}) - \nabla(\nabla \cdot \mathbf{u}),$$

where $\Delta \mathbf{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$ and $\Delta u_i = \nabla \cdot (\nabla u_i), i = 1, 2, 3$.
The $p(x)$-curl operator is a natural generalization of the p-curl operator which appear in many papers in literature, however the constant p has been replaced by the function $p(x)$. Note That due to the fact that the $p(x)$-curl operator is not homogenous, it has more complicated structure than the the p-curl one.

In the recent years many problems involving the $p(x)$- curl operator have been studied in many papers, we refer e.g., to recent works of Afrouzi, Chung-Naghizadeh [1], Ge-Lu [13], Hamdani-Repovš [14] and Xiang, Wang-Zhang [23].

For example and not limited to, we mention the paper of Afrouzi, Chung-Naghizadeh [1] in which the authors consider the problem (1) in the particular case when $a(x) \equiv 0$ and under a suitable condition on the nonlinearity, they proved the existence of solution. Their main tools are essentially based on the mountain pass theorem and fountain theorem. The study of the existence of solutions for $p(x)$-curl systems is a new and interesting topic and only minor results involving these kind of operators are present in literature.

In the hole paper, let

$$C_+(\overline{\Omega}) := \{h \mid h \in C(\overline{\Omega}), h(x) > 1, \text{ for all } x \in \overline{\Omega}\},$$

and for $\eta > 0$, $h \in C_+(\overline{\Omega})$, we set

$$h^- := \inf_{x \in \Omega} h(x), \quad h^+ := \sup_{x \in \Omega} h(x)$$

and

$$[\eta]^h := \sup\{\eta^{h^-}, \eta^{h^+}\}, \quad [\eta]_h := \inf\{\eta^{h^-}, \eta^{h^+}\}.$$

In the sequel, we shall need the following assumptions:
$V : \Omega \to \mathbb{R}$ is a sign-changing function such that $V \in L^\infty(\Omega)$ and

(A) $a \in L^\infty(\Omega)$ and $ess \inf_{x\in\Omega} a(x) = a_0 > 0$.

(V_2) there exist an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $V(x) = 0$ for $x \in \overline{B_R(x_0) \setminus B_r(x_0)}$ and one of the following conditions hold

$$V(x) > 0, \forall x \in B_r(x_0) \text{ and } V(x) < 0, \forall x \in \Omega \setminus \overline{B_R(x_0)} \qquad (V_2^{'})$$

or

$$V(x) < 0, \forall x \in B_r(x_0) \text{ and } V(x) > 0, \forall x \in \Omega \setminus \overline{B_R(x_0)}. \qquad (V_2^{''})$$

Moreover, we assume that

(Q_1) $1 < q(x) < p^*(x) = \frac{3p(x)}{3-p(x)}$ for all $x \in \Omega$.

(Q_2) Either

$$\max_{x\in \overline{B_r(x_0)}} q(x) < p^- \leq p^+ < \min_{x\in \overline{\Omega \setminus B_R(x_0)}} q(x) \qquad (Q_2^{'})$$

or

$$\max_{x\in \overline{\Omega \setminus B_R(x_0)}} q(x) < p^- \leq p^+ < \min_{x\in \overline{B_r(x_0)}} q(x). \qquad (Q_2^{''})$$

Our result can be described as follow.

Theorem 1.1. *Assume that the assertions (A), (V_2) and (Q_1) are fulfilled. Moreover, either the assertions $(V_2^{'}) - (Q_2^{'})$ or the assertions $(V_2^{''}) - (Q_2^{''})$ hold. Then any $\lambda > 0$ is an eigenvalue of problem (1).*

2. **Backgrounds setting**

In this part, let us recall some definitions and results be needed later.

Firstly, we recall some theories of Lebesgue-Sobolev space with variable exponent which are described in details in Diening [15], Edmunds- Rákosnic [7], Fan- Zhang [8], Fan-Zhao [9], Fan- Zhao- Zhao [10], Fan-Zhao [11], Kovacik- Rákosnic [20].

Let

$$\begin{aligned} &C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}, \\ &h^- = \min_{x\in\overline{\Omega}} h(x),\ h^+ = \max_{x\in\overline{\Omega}} h(x) \text{ for any } h \in C_+(\overline{\Omega}). \end{aligned} \tag{2}$$

Obviously, $1 < h^- \leq h^+ < +\infty$.

Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered to be one element of $\mathcal{U}(\Omega)$, when they are equal almost everywhere.

For $p \in C_+(\overline{\Omega})$, define

$$L^{p(x)}(\Omega) = \{u \in \mathcal{U}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\}, \tag{3}$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$, and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\} \tag{4}$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + \|\nabla u\|_{p(x)}$.

Denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{3p(x)}{3-p(x)}, & p(x) < 3, \\ +\infty, & p(x) \geq 3. \end{cases} \tag{5}$$

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Denote by $L^{p'(x)}(\Omega)$ the conjugate Lebesgue space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv| dx \leq (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega) \tag{6}$$

holds. Furthermore, define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx, \tag{7}$$

then the following relations hold

$$\begin{aligned} &|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}, \\ &|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}. \end{aligned} \tag{8}$$

Proposition 2.1. *(See Fan-Zhang[8] If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Let $\mathbf{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}(\Omega)$ and define

$$\mathbf{W}^{p(x)}(\Omega) = \{\mathbf{u} \in \mathbf{L}^{p(x)}(\Omega) : \nabla \times \mathbf{u} \in \mathbf{L}^{p(x)}(\Omega), \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\},$$

where $\mathbf{n}$ denotes the outward unitary normal vector to $\partial\Omega$. Equip $\mathbf{W}^{p(x)}(\Omega)$ with the norm

$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\mathbf{W}^{p(x)}(\Omega)} = |\mathbf{u}|_{\mathbf{L}^{p(x)}(\Omega)} + |\nabla \times \mathbf{u}|_{\mathbf{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, then by Theorem 2.1 of [2], $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}^{1,p(x)}_{\mathbf{n}}(\Omega)$, where

$$\mathbf{W}^{1,p(x)}_{\mathbf{n}}(\Omega) = \{\mathbf{u} \in \mathbf{W}^{1,p(x)}(\Omega) : \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$\mathbf{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega).$$

Thus, we have the following theorem.

Theorem 2.1. *(See S. Antontsev, F. Mirandac* [2, Theorem 2.1]*) Assume that* $1 < p^- \leq p^+ < \infty$ *and* p *satisfies (1.2). Then* $W^{p(x)}(\Omega)$ *is a closed subspace of* $\boldsymbol{W}^{1,p(x)}_{\boldsymbol{n}}(\Omega)$*. Moreover, if* $p^- > \frac{6}{5}$*, then* $|\nabla \times \cdot|_{\boldsymbol{L}^{p(x)}(\Omega)}$ *is a norm in* $\boldsymbol{W}^{p(x)}(\Omega)$ *and there exists* $C = C(N, p^-, p^+) > 0$ *such that*

$$\|\boldsymbol{u}\| \leq C|\nabla \times \boldsymbol{u}|_{\boldsymbol{L}^{p(x)}(\Omega)}$$

Corollary 2.1. *(See Bahrouni-Repovš* [4, Corollary 2.5]*) The embedding* $\boldsymbol{W}^{p(x)}(\Omega) \hookrightarrow \boldsymbol{L}^{q(x)}(\Omega)$ *is compact, with* $1 < p^- \leq p^+ < 3$*,* $q \in C(\overline{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *in* $\overline{\Omega}$*. Moreover,* $(\boldsymbol{W}^{p(x)}(\Omega), \|\boldsymbol{u}\|)$ *is a uniformly convex and reflexive Banach space.*

Let

$$\|\mathbf{u}\|_a = \inf\big\{\mu > 0 : \int_\Omega \big(|\frac{\nabla \times \mathbf{u}}{\mu}|^{p(x)} + a(x)|\frac{\mathbf{u}(x)}{\mu}|^{p(x)}\big)dx \leq 1\big\}$$

for all $\mathbf{u} \in \mathbf{W}^{p(x)}(\Omega)$. Since $a^- > 0$, it's easy to see that $\|\cdot\|_a$ is equivalent to the norm $\|\cdot\|_{\mathbf{W}^{p(x)}(\Omega)}$. In this paper, we shall use for convenience the norm $\|\mathbf{u}\|_a$ on $\mathbf{W}^{p(x)}(\Omega)$.

Proposition 2.2. *(See Hamdani-Repovš* [14]*)*

Let $\Lambda_{p(x),a}(\boldsymbol{u}) = \int_\Omega \big(|\nabla \times \boldsymbol{u}|^{p(x)} + a(x)|\boldsymbol{u}(x)|^{p(x)}\big)dx$ *for all* $\boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega)$.

Then

$$[\| \boldsymbol{u} \|_a]_p \leq \Lambda_{p(x),a}(\boldsymbol{u}) \leq [\| \boldsymbol{u} \|_a]^p.$$

Consider the following function:

$$\Phi(\mathbf{u}) = \int_\Omega \frac{1}{p(x)}|\nabla \times \mathbf{u}|^{p(x)}dx + \int_\Omega \frac{a(x)}{p(x)}|\mathbf{u}|^{p(x)}dx, \mathbf{u} \in \mathbf{W}^{p(x)}(\Omega). \tag{9}$$

We know that(see [23, Lemma 3.1]) $\Phi \in C^1(\mathbf{W}^{p(x)}(\Omega), \mathbb{R})$ and the $p(x)$-curl operator

$\nabla \times (|\nabla \times \mathbf{u}|^{p(x)-2}\nabla \times \mathbf{u})$ is the derivative operator of Φ in the weak sense.

We denote $\xi = \Phi' : \mathbf{W}^{p(x)}(\Omega) \to (\mathbf{W}^{p(x)}(\Omega))^*$, then

$$\begin{aligned}\langle \xi(\mathbf{u}), \mathbf{v}\rangle &= \int_\Omega (|\nabla \times \mathbf{u}(x)|^{p(x)-2}\nabla \times \mathbf{u}(x) \cdot \nabla \times \mathbf{v}(x))dx \\ &\quad + \int_\Omega a(x)|\mathbf{u}(x)|^{p(x)-2}\mathbf{u}(x) \cdot \mathbf{v}(x)dx, \forall \ \mathbf{u}, \mathbf{v} \in \mathbf{W}^{p(x)}(\Omega).\end{aligned} \tag{10}$$

Furthermore, one has

Proposition 2.3. *Set* $X = \boldsymbol{W}^{p(x)}(\Omega)$*,* ξ *is as above, then*

(a) $\xi : X \to X^*$ *is a continuous, bounded and strictly monotone operator.*

(b) $\xi : X \to X^*$ *is a mapping of type* $(S)_+$*, i.e., if* $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ *weakly in* X *and* $\limsup_{n\to\infty}\langle \xi(\boldsymbol{u}_n), \boldsymbol{u}_n - \boldsymbol{u}\rangle \leq 0$*, implies* $\boldsymbol{u}_n \to \boldsymbol{u}$ *in* X.

(c) $\xi : X \to X^*$ *is a homeomorphism.*

In the rest of paper for $\mathbf{u} \in \mathbf{L}^{p(x)}(\Omega)$, we use the notation $|\mathbf{u}|_{p(x)}$ instead of $\|\mathbf{u}\|_{\mathbf{L}^{p(x)}(\Omega)}$.

In order to formulate the variational approach to problem (1), let us recall the definition of a weak solution for our problem.

Definition 2.1. *We say that* $\boldsymbol{u} \in X$ *is a weak solution of problem* (P)*, if*

$$\begin{aligned}\int_\Omega |\nabla \times \boldsymbol{u}|^{p(x)-2} \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} \;\; dx + \int_\Omega a(x)|\boldsymbol{u}|^{p(x)-2}\boldsymbol{u} \cdot \boldsymbol{v} \;\; dx \\ - \lambda \int_\Omega V(x)|\boldsymbol{u}|^{q(x)-2}\boldsymbol{u} \cdot \boldsymbol{v} \;\; dx = 0, \quad \text{for all } \boldsymbol{v} \in X.\end{aligned} \tag{11}$$

3. **Proof of the Main Result**

In order to prove Theorem 1.1 let us define the functions q_1 and q_2 as follows
$q_1 : \overline{B_r(x_0)} \to (1, +\infty), q_1(x) = q(x)$ for any $x \in \overline{B_r(x_0)}$ and
$q_2 : \overline{\Omega \backslash B_R(x_0)} \to (1, +\infty), q_2(x) = q(x)$ for any $x \in \overline{\Omega \backslash B_R(x_0)}$.
We also introduce here the notations

$$q_1^- = \underset{x \in \overline{B_r(x_0)}}{min} q_1(x), \qquad q_1^+ = \underset{x \in \overline{B_r(x_0)}}{max} q_1(x)$$

$$q_2^- = \underset{x \in \overline{\Omega \backslash B_R(x_0)}}{min} q_2(x), \qquad q_2^+ = \underset{x \in \overline{\Omega \backslash B_R(x_0)}}{max} q_2(x).$$

By the conditions (Q_1) and $(Q_2^{'})$

$$1 < q_1^- \leq q_1^+ < p^- \leq p^+ < q_2^- \leq q_2^+ < p^*(x) \text{ for all } x \in \overline{\Omega}, \tag{12}$$

hence, for $i = 1, 2$, X is continuously embedded in $L^{q_i}(\Omega)$, we deduce that there exists a positive constant c_i such that

$$|\mathbf{u}|_{L^{q_i}(\Omega)} \leq c_i ||\mathbf{u}||_a, \quad \text{for all } \; \mathbf{u} \in X \text{ and } i = 1, 2. \tag{13}$$

To begin, let us denote

$$\Psi(\mathbf{u}) := \int_\Omega \frac{V(x)}{q(x)} |\mathbf{u}|^{q(x)} dx.$$

The Euler-Lagrange functional corresponding to problem (1) is then defined by $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(\mathbf{u}) = \Phi(\mathbf{u}) - \lambda \Psi(\mathbf{u}), \text{ for all } u \in X,$$

where

$$\Phi(\mathbf{u}) = \int_\Omega \frac{1}{p(x)} |\nabla \times \mathbf{u}|^{p(x)} dx + \int_\Omega \frac{a(x)}{p(x)} |\mathbf{u}|^{p(x)} dx.$$

By using inequality (8), one has

$$|\Psi(\mathbf{u})| \leq \frac{1}{q^-} \int_\Omega |V(x)||\mathbf{u}|^{q(x)} dx \leq \frac{1}{q^-} |V|_\infty \int_\Omega |\mathbf{u}|^{q(x)} dx \leq \frac{1}{q^-} |V|_\infty [\| |\mathbf{u}\|_a]^q.$$

The following result asserts the existence of a "valley" for Ψ_λ near the origin.

Lemma 3.1. *There exists* $\mathbf{u_0} \in X$*, such that* $\mathbf{u_0} \neq 0$*, and* $I_\lambda(t\mathbf{u_0}) < 0$ *for any* $t > 0$ *small enough.*

Proof. Let $\mathbf{u_0} \in (C_0^\infty(\Omega))^3$, there exist $x_1 \in B_r(x_0)$ and $\epsilon > 0$ such that for any $B_\epsilon(x_1) \subset B_r(x_0)$ we have $|\mathbf{u_0}(x_1)| > 0$. Letting $0 < t < 1$ we then obtain

$$\begin{aligned} I_\lambda(t\mathbf{u}_0) &= (\int_\Omega \frac{1}{p(x)}\Big(|\nabla \times (t\mathbf{u}_0)|^{p(x)} + a(x)|t\mathbf{u}_0|^{p(x)}\Big)dx - \lambda \int_\Omega \frac{V(x)}{q(x)}|t\mathbf{u}_0|^{q(x)}dx \\ &\leq \frac{1}{p^-}\int_\Omega t^{p(x)}\Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx - \lambda \int_{B_r(x_0)} \frac{V(x)}{q_1(x)} t^{q_1(x)}|\mathbf{u}_0|^{q_1(x)}dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega \Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx - \frac{\lambda t^{q_1^+}}{q_1^+}\int_{B_r(x_0)} V(x)|\mathbf{u}_0|^{q_1(x)}dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega \Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx - \frac{\lambda t^{q_1^+}}{q_1^+}\int_{B_\epsilon(x_1)} V(x)|\mathbf{u}_0|^{q_1(x)}dx. \end{aligned}$$

Obviously, we have $I_\lambda(t\mathbf{u}_0) < 0$ for any $0 < t < \delta^{\frac{1}{p^- - q_1^+}}$, where

$$0 < \delta < min\Bigg\{1, \frac{\frac{\lambda p^-}{q_1^+}\int_{B_\epsilon(x_1)} V(x)|\mathbf{u}_0|^{q_1(x)}dx}{\int_\Omega \Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx}\Bigg\}.$$

Finally, we point out that

$$\int_\Omega \Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx > 0.$$

Indeed, supposing the contrary we have $\int_\Omega \Big(|\nabla \times \mathbf{u}_0|^{p(x)} + a(x)|\mathbf{u}_0|^{p(x)}\Big)dx = 0$. By Proposition 2.2, we deduce that $\|\mathbf{u}_0\|_a = 0$ and consequently $\mathbf{u}_0 = 0$ in Ω which is a contradiction. The proof of Lemma 3.1 is complete. □

Proof of Theorem 1.1. We prove Theorem 1.1 in details for the case when the conditions $(V_2') - (Q_2')$ hold, the remaining one can be made by similarly arguments so we omit it. Using Hölder inequality (6) for $\|\mathbf{u}\|_a > 1$ combined with relations (8), it follows for any $\lambda > 0$ and all $\mathbf{u} \in X$ with $\|\mathbf{u}\|_a > 1$,

$$\begin{aligned} I_\lambda(\mathbf{u}) &= \int_\Omega \Big(\frac{1}{p(x)}|\nabla \times \mathbf{u}|^{p(x)}dx + \frac{a(x)}{p(x)}|\mathbf{u}|^{p(x)}\Big)dx - \lambda \int_\Omega \frac{V(x)}{q(x)}|\mathbf{u}|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|\mathbf{u}\|_a^{p^-} - \frac{\lambda}{q^-}\int_\Omega V(x)|\mathbf{u}|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|\mathbf{u}\|_a^{p^-} - \frac{\lambda}{q^-}|V|_\infty \int_{B_r(x_0)} |\mathbf{u}|^{q_1(x)}dx \\ &\geq \frac{1}{p^+}\|\mathbf{u}\|_a^{p^-} - \frac{\lambda}{q^-}|V|_\infty [c_1\|\mathbf{u}\|_a]^q. \end{aligned}$$

By (Q_2'), we have $q_1^+ < p^-$, then $I_\lambda(\mathbf{u}) \to +\infty,$ as $\|u\|_a \to +\infty$. This implies that I_λ is coercive and bounded from below on X. On the other hand, by (Q_1), the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact, so I_λ is weakly lower semicontinuous then it has a global minimizer $\mathbf{w}$. Due to Zeidler [[25] Theorem D.25], $\mathbf{w}$ is weak solution of problem (1). Finally, we point out that due to the Lemma 3.1, this minimizer is nontrivial and thus any $\lambda > 0$ is an eigenvalue of problem (1). Which ends the proof.

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REFERENCES

[1] *G.A. Afrouzi, N.T Chung and Z. Naghizadeh*, Existence of solutions for a class of $p(x)$-curl systems arising in electromagnetism without (A-R) type conditions. Tamkang J. of Math., **51**, 3 (2020), 187-200. https://doi.org/10.5556/j.tkjm.51.2020.2915

[2] *S. Antontsev, F. Mirandac, L. Santos*, em Blow-up and finite time extinction for $p(x,t)$-curl systems arising in electromagnetism, J. Math. Anal. Appl., **440** (2016), 300-322.

[3] *S.N. Antontsev, S.I. Shmarev*, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, Nonlinear Anal., **60** (2005), 515-545.

[4] *A. Bahrouni, D. Repovš*, Existence and nonexistence of solutions for $p(x)$-curl systems arising in electromagnetism, Complex Var. Elliptic Equ., **63** (2018), 292-301.

[5] *Y. Chen, S. Levine and M. Rao*, em Variable exponent, linear growth functionals in image processing, SIAM J. Appl. Math., **66** (2006), 1383-1406.

[6] *L. Diening*, Riesz potential and sobolev embedding on generalized lebesque and sobolev space $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr., **268** (2004), 31-43.

[7] *D.E. Edmunds, J. Rákosnic*, em Sobolev embbeding with variable exponent, II. Math. Nachr., 246-247 (2002), 53-67.

[8] *X.L. Fan, Q.H. Zhang*, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, Nonlinear Anal., **52** (2003), 1843-1852.

[9] *X.L. Fan, D. Zhao*, On the generalized Orlicz-sobolev spaces $W^{k,p(x)}(\Omega)$, J. Gansu Educ. College., **12**, 1 (1998), 1-6.

[10] *X.L. Fan, Y.Z. Zhao, D. Zhao*, Compact imbedding theorems with symmetry of stranss-lions type for the space $W^{1,p(x)}(\Omega)$, J. Math. Anal. Appl., **255** (2001), 333-348.

[11] *X.L. Fan, D. Zhao*, On the space $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl., **263** (2001), 424-446.

[12] *B. Ge* , On superlinear $p(x)$-Laplacian-like problem without Ambrosetti and Rabinowitz condition, Bull. Korean Math. Soc, **51**, 2, (2014), 409-421. .

[13] *B. Ge, J.F. Lu*, Existence and Multiplicity of Solutions for p(x)-Curl Systems Without the Ambrosetti-Rabinowitz Condition. Mediterr. J. Math., **16**, 45 (2019). https://doi.org/10.1007/s00009-019-1312-3

[14] *M.K. Hamdani and D.D. Repovš*, Existence of solutions for systems arising in electromagnetism,J. Math. Anal. Appl., **486**, 2, (2020),123898. https://doi.org/10.1016/j.jmaa.2020.123898.

[15] *M.K. Hamdani, L. Mbarki, M. Allaoui, O. Darhouche and D.D. Repovš*, Existence and multiplicity of solutions involving the $p(x)$-Laplacian equations: On the effect of two nonlocal terms, Discrete Contin. Dyn. Syst. Ser. S, (2022), Doi: 10.3934/dcdss.2022129

[16] *M. Hsini, N. Irzi and K. Kefi*, Existence of solutions for a $p(x)$-biharmonic problem under Neumann boundary conditions, Appl. Anal.,**100**, 10 (2021), 2188-2199.

[17] *GL. Hou, B. Ge,BL. Zhang et al*, em Multiple solutions to a class of electromagnetic p(x)-curl systems, Indian J. Pure Appl. Math. **52**,(2021), 125–137 . https://doi.org/10.1007/s13226-021-00075-y

[18] *K. Kefi*, On the existence of solutions of a nonlocal biharmonic problem, Adv. Pure Appl. Math., **12**, 1,(2021), 50-62 , DOI: 10.21494/ISTE.OP.2020.0581

[19] *K. Kefi, N. Irzi, M. M. Al-Shomrani and D.D. Repovš*, On the fourth-order Leray-Lions problem with indefinite weight and nonstandard growth conditions, Bull. Math. Sci.,https://doi.org/10.1142/S1664360721500089

[20] *O. Kovacik, J. Rákosnic*, On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, Czechoslovak Math. J., **41** (1991), 592-618.

[21] *K. Rajagopal, M. Ružička*, Mathematical modelling of electrorheological fluids, Contin. Mech. Thermodyn., **13** (2001) 59¨C78.

[22] *M. Ružička*, Electrorheological Fluids: Modeling and Mathematical Theory, Lect. Notes Math., 1748, Springer, Berlin (2000).

[23] *M.Q. Xiang, F.L. Wang, B.L. Zhang*, Existence and multiplicity for $p(x)$-curl systems arising in electromagnetism, J. Math. Anal. Appl., **448** (2017), 1600-1617.

[24] *V.V. Zhikov*, Lavrentiev phenomenon and homogenization for some variational problems, C. R. Acad. Sci. Paris Sér. I Math., **316**, 5 (1993), 435-439.

[25] *E. Zeidler*, Nonlinear Functional Analysis and Its Applications II/B, Non-linear Monotone Operators, Springer, Now York, Berlin, Heidelberg, London,Paris, Tokyo, 1990. 1990. E. Zeidler, Nonlinear Functional Analysis and its Applications Vol. II/B: