SOME PRACTICAL INSURANCE PROBLEMS SOLVED BY MATHEMATICAL THEORY AND CREDIBILITY THEORY

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În acest articol introducem modelul original al lui Bühlmann, ce implică doar un singur contract izolat. Vom prezenta cei mai buni estimatori liniari de credibilitate pentru acest model şi vom considera următoarele aplicaţii ale estimatorului optim de credibilitate al lui Bühlmann:

1) câteva probleme practice de asigurări, rezolvate în cazul în care soluţia de credibilitate a lui Bühlmann conţine parametrii cunoscuţi pentru funcţia de structură, ceea ce va face posibilă o cuantificare statistică din observaţii a acestui rezultat; 2) estimarea parametrilor de structură din modelul clasical al lui Bühlman, pentru a putea folosi primele de credibilitate ale acestui model.

In this article we introduce the original Bühlmann model, which involves only one isolated contract. We will present the best linear credibility estimators for this model and we will consider the following applications of the optimal credibility estimator of Bühlmann:

1) some practical insurance problems solved, when the credibility solution of Bühlmann contains known parameters of the structure function, which will make possible a statistic computable from the observations for this result; 2) estimation of the structural parameters in the classical Bühlmann model, to be able to use the credibility premiums of this model.

Key - words: the credibility calculations, the risk premium, Bühlmann's original model, Bühlmann's classical model, the credibility model incorporating risk volumes.

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1. Introduction

In Section **2** we present Bühlmann's original model, which implies only one isolated contract. The original Bühlmann model gives the optimal linear credibility estimate for the risk premium of this case (see Subsection **2.1.**). We end Section **2**, giving **five applications of the optimal credibility estimator of Bühlmann** (see Subsection **2.2.**), solved when **the credibility solution** of **Bühlmann** contains **known parameters** of **the structure function**, which will make possible **a statistic computable** from **the observations** for **this result**. In Section **3** we present Bühlmann's classical model, which consists of a portfolio of contracts satisfying the constraints of the original Bühlmann model. Just as in

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Subsection **2.1.**, we will give the best linearized credibility estimators for this model (see Subsection **3.1.**). To be able to use the credibility premiums of this model, in Subsection **3.2** we give **unbiased estimators** for the structure parameters, such that if the structure parameters in the optimal linearized credibility premium are replaced by these estimators, a homogeneous estimator results. This last estimator can also be shown to be optimal. For proof see [1] of the references, pages 148 to 154.

2. The original credibility model of Bühlmann

In the original credibility model of Bühlmann, we consider one contract with unknown and fixed risk parameter Θ, during a period of t years. The yearly claim amounts are noted by $X_1,\dots,X_t$. The risk parameter Θ is supposed to be taken from some structure distribution $U(\cdot)$. It is assumed that, for given $\Theta = \theta$, the claims are conditionally independent and identically distributed with known common distribution function $F_{X|\Theta}(x,\theta)$. For this model we want to estimate the net premium $\mu(\theta) = E[X_r|\Theta = \theta]$, $r = \overline{1,t}$ as well as X_{t+1} for a contract with risk parameter θ.

2.1. Bühlmann's optimal credibility estimator

Suppose that $X_1,\dots,X_t$ are random variables with finite variation, which are, for given $\Theta = \theta$, conditionally independent and identically distributed with already known common distribution function $F_{X|\Theta}(x, \theta)$. The structure distribution function is $U(\theta) = P[\Theta \le \theta]$. Let D represent the set of non-homogeneous linear combinations $g(\cdot)$ of the observable random variables $X_1, X_2, \dots, X_t$:

$$g(\underline{X}') = c_0 + c_1X_1 + c_2X_2 + \dots + c_tX_t \qquad (2.1).$$

Then the solution of the problem: $\underset{g\in D}{Min}\, E\{[\mu(\Theta) - g(X_1,\dots,X_t)]^2\}$ (2.2)

is: $g(X_1,\dots,X_t) = z\overline{X} + (1-z)m$ (2.3),

where $\underline{X}' = (X_1,\dots,X_t)$ is the vector of observations, $z = at / (s^2+at)$, is the resulting credibility factor, $\overline{X} = \frac{1}{t}\sum_{i=1}^{t} X_i$ is the individual estimator, and a, s^2 and m are the structural parameters as defined in (2.4): $m = E[X_r] = E[\mu(\Theta)]$, $r = \overline{1,t}$, a

= =Var{E[X_r|Θ]} = Var[μ(Θ)], $r = \overline{1,t}$, $\sigma^2(\theta)$ = Var[X_r|Θ $= \theta$], $r = \overline{1,t}$, s^2 = =E{Var[X_r|Θ]} = E[σ^2(Θ)], $r = \overline{1,t}$ (2.4).

For proof see [1] of the references, pages 7 to 20.

2.2. Applications of the optimal credibility estimator of Bühlmann

These applications are solved when **the credibility solution** of **Bühlmann** contains **known parameters** of **the structure function**, which will make possible **a statistic computable** from **the observations** for **this result**.

Application 1

We suppose that the claims are Poisson (θ) distributed, as bellow:

$$dF_{X|\Theta}(x,\theta) = \theta^x e^{-\theta} / x!,\ x = 0, 1 \qquad (2.5),$$

and suppose that the structure distribution of Θ has a Gamma distribution:

$$u(\theta) = \theta^{\beta-1} e^{-\alpha\theta} \alpha^{\beta} / \Gamma(\beta),\ \theta > 0 \qquad (2.6).$$

In this case the best linear credibility estimator for μ(Θ) (see (2.3)) can be written as : $z\overline{X} + (1-z)m = (\sum_{i=1}^{t} X_i + \beta) / (t + \alpha)$ (2.7).

Since here m = E[X] = E{E[X|Θ]} = E[Θ] = β / α, and for the ratio of the structure parameters s^2 and a, that is “s^2 / a“, we have: s^2 / a = E{Var[X|Θ]} / /Var{E[X|Θ]} = E[Θ] / Var[Θ] = (β /α) / (β / α^2) = α, we find z = at / (s^2+at) = at / /{a[(s^2 / a) + t]} = t /(t + α), so the best linear credibility estimator for μ(Θ) can be written under the form (2.7), where α, β > 0 are the parameters of Gamma distribution.

Application 2

We suppose that the claims are Negative Binomial (θ) distributed, so:

$$dF_{X|\Theta}(x,\theta) = \theta^x (1-\theta)^{1-x},\ x \in \{0,1\} \qquad (2.8)$$

and suppose that the structure distribution of Θ has a Beta distribution:

$$u(\theta) = \theta^{\alpha-1}(1-\theta)^{\beta-1} / \beta(\alpha,\beta),\ \theta \in (0,1) \qquad (2.9).$$

In this case the best linear credibility estimator for μ(Θ) (see (2.3)) can be written as: $z\overline{X} + (1-z)m$ = [t /(t + α + β) $\overline{X}$] + [α / (t + α + β)] (2.10).

Since here m = E[X] = E{E[X|Θ]} = E[Θ] = α / (α + β), and for the ratio of the structure parameters s^2 and a, that is “s^2 / a”, we have: s^2 / a = E{Var[X|Θ]} / /Var{E[X|Θ]} = E[Θ(1 - Θ] / Var[Θ] = [E(Θ) - E(Θ^2)] / Var(Θ) = {[α /(α + β)] –

- [α(α + 1) / (α + β) (α + β + 1)]} / {[(αβ) / (α + +β)^2(α + β + 1)]}=[αβ / (α + β)(α + +β + 1)] / [αβ / (α + β)^2 (α + β + 1)] = α + β, we find z = =at/ (s^2 + at) = at / {a[(s^2 / /a) + t]} = t / (t + α + β), so the best linear credibility estimator for μ(Θ) can be written under the form (2.10), where α, β > 0 are the parameters of Beta distribution.

Application 3

We suppose that the claims are Exponential (θ) distributed, so:

$$dF_{X|\Theta}(x,\theta) = \theta e^{-\theta x},\ x>0 \tag{2.11}$$

and suppose that the structure distribution of Θ has a Gamma distribution:

$$u(\theta) = \theta^{\beta-1}e^{-\alpha\theta}\alpha^{\beta} / \Gamma(\beta),\ \theta>0 \tag{2.12}$$

In this case the best linear credibility estimator for μ(Θ) (see (2.3)) can be written as follows: $z\overline{X} + (1-z)m$ = (v + α) / (t + β - 1), if β > 2 (2.13).

Since here m = E[X] = E{E[X|Θ]} = E[1 / Θ] = α / (β - 1), if β > 1, and for the ratio of the structure parameters s^2 and a, that is “s^2 / a “, we have: s^2 / a = =E{Var[X|Θ]} / Var{E[X|Θ]} = E[1 / Θ^2] / Var(1 / Θ) = β – 1, if β > 2, we find z= =at / (s^2 + at) = at / {a[(s^2 / a) + t]} = t / (t + β – 1) , if β > 2. Therefore: $z\overline{X} + (1-z)m$ = [t / (t + β – 1)][$\sum_{i=1}^{t} X_i$ / t] + {1 - [t / (t + β – 1)]}[α / (β - 1)] =.... = =($\sum_{i=1}^{t} X_i$ + α) / [t + β – 1], if β > 2. So the best linear credibility estimator for μ(Θ) can be written under the form (2.13), where α, β > 0 are the parameters of Gamma distribution.

Application 4

We suppose that the claims are Normal (θ, σ^2 > 0) distributed, so:

$$dF_{X|\Theta}(x,\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2},\ x \in \mathbb{R} \tag{2.14}$$

and suppose that the structure distribution of Θ has a Normal (μ$_0$, σ_0^2 > 0)

distribution: $u(\theta) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2},\ \theta \in \mathbb{R}$ (2.15).

In this case the best linear credibility estimator for μ(Θ) (see (2.3)) can be written as follows: $z\overline{X}+(1-z)m = \left[\dfrac{\sum_{i=1}^{t} X_i}{\sigma^2}+\dfrac{\mu_0}{\sigma_0^2}\right] / \left[\dfrac{t}{\sigma^2}+\dfrac{1}{\sigma_0^2}\right]$ (2.16).

Since here m = E[X] = E{E[X|Θ]} = E(Θ) = μ_0 and for the ratio of the structure parameters s^2 and a, that is "s^2 / a", we have: s^2 / a = E{Var[X|Θ]} / /Var{E[X|Θ]} = E(σ^2) / Var(Θ) = σ^2 / σ_0^2, we find z= at / (s^2 + at) = at / {a[(s^2 /a)+ t]} = t / [(σ^2 / $\sigma_0^2) + t$]. Therefore: $z\overline{X}+(1-z)m$ = {t / [(σ^2 / $\sigma_0^2) + t$]} $\dfrac{\sum_{i=1}^{t} X_i}{t}$ + {1 - [t / [(σ^2 / $\sigma_0^2) + t$]]}[μ_0]= = $\left[\dfrac{\sum_{i=1}^{t} X_i}{\sigma^2}+\dfrac{\mu_0}{\sigma_0^2}\right] / \left[\dfrac{t}{\sigma^2}+\dfrac{1}{\sigma_0^2}\right]$. So the best linear credibility estimator for μ(Θ) can be written under the form (2.16).

Application 5 (**Credibility estimator which minimizes the mean squared error for a exponential family with natural parameterization and prior**)

Remark:

The **parameterization** is called **natural** because **the exponent part** is a linear function of θ, and by taking a natural conjugate prior, the posterior distribution is of the same type as the prior distribution. We restrict to x > 0 and θ>0, and suppose furthermore that at the end point of the intervals **the densities** "**f**" and "**u**" **are zero**. These restrictions are not strictly necessary.

We consider a family of exponential distributions with natural **parameterization**: $f_{X|\Theta}(x,\theta) = p(x)e^{-\theta x} / q(\theta)$, x > 0, $\theta > 0$ (2.17)
together with the natural conjugate prior having the density:

$u(\theta) = q(\theta)^{-t_0} e^{-\theta \cdot x_0} / c(t_0, x_0)$, $\theta > 0$ (2.18),

where p(x) is a arbitrary non – negative function, t_0 and x_0 are positive constants, and $c(t_0, x_0)$ is a normalization constant. For this case, the linear credibility estimator (2.3) is: $z\overline{X}+(1-z)m = \left(x_0+\sum_{i=1}^{t} X_i\right)/(t_0+t)$ (2.19),

where m = E[μ(Θ)] = x_0 / t_0, s^2 / a = t_0, z = t / (t +t_0). Indeed: - from the demonstration of the result (2.3) we only have to prove that the optimal estimator:

$E[\mu(\Theta)|\underline{X}']$

(2.20)

is a non-homogeneous linear combination of $X_1, \ldots, X_t$. First we express $E[\mu(\Theta)]$ in the prior parameters x_0 and t_0, and then the result (2.19) follows because of the special form of the posterior distribution. Because $q(\theta)$ is the normalizing constant of the distribution (2.17) we have $\int_0^{+\infty} f_{X|\theta}(x,\theta)dx = 1$, that is

$$\left(\int_0^{+\infty} p(x)e^{-\theta\cdot x}dx\right)/q(\theta) = 1 \text{ and thus: } q(\theta) = \int_0^{+\infty} p(x)e^{-\theta\cdot x}dx \quad (2.21).$$

$$\text{So: } q'(\theta) = -\int_0^{+\infty} xp(x)e^{-\theta\cdot x}dx = -q(\theta)E[X|\Theta=\theta] \quad (2.22),$$

since $E[X|\Theta=\theta] = \int_0^{+\infty} xf_{X|\theta}(x,\theta)dx = \left[\int_0^{+\infty} xp(x)e^{-\theta\cdot x}dx\right]/q(\theta)$. Therefore the risk premium when $\Theta=\theta$ equals is: $\mu(\theta) = E[X|\Theta=\theta] = -q'(\theta)/q(\theta)$ (2.23).

Taking the first derivative of (2.18) with respect to θ and using (2.23), we obtain: $u'(\theta) = [-t_0 q(\theta)^{-t_0-1} q'(\theta) e^{-\theta x_0}] / c(t_0, x_0) + [q(\theta)^{-t_0} e^{-\theta x_0}(-x_0)] / c(t_0, x_0) =$ $= t_0[-q'(\theta) / q(\theta)]\{[q(\theta)^{-t_0} e^{-\theta x_0}] / c(t_0, x_0)\} - x_0\{[q(\theta)^{-t_0} e^{-\theta x_0}] / c(t_0, x_0)\} =$ $= t_0\mu(\theta)u(\theta) - x_0 u(\theta) = [t_0\mu(\theta) - x_0]u(\theta)$. So: $u'(\theta) = [t_0\mu(\theta) - x_0]u(\theta)$ (2.24).

Integrating this derivative over θ gives zero for the left side, since:

$$\int_0^{+\infty} u'(\theta)d\theta = u(+\infty) - u(0) = 0 \quad (2.25),$$

considering that at the end point of the interval the density "$u(\bullet)$" is zero (see the above **remark**). So the right side of (2.24) will be:

$$m = E[\mu(\Theta)] = \int_0^{+\infty} \mu(\theta)u(\theta)d\theta = x_0/t_0 \quad (2.26),$$

as: $(2.24)\wedge(2.25) \Rightarrow \int_0^{+\infty}[t_0\mu(\theta) - x_0]u(\theta)d\theta = 0 \Leftrightarrow t_0E[\mu(\Theta)] - x_0\int_0^{+\infty} u(\theta)d\theta = 0$

$\Leftrightarrow$

$\Leftrightarrow t_0E[\mu(\Theta)] - x_0\cdot 1 = 0 \Leftrightarrow E[\mu(\Theta)] = x_0/t_0$. The conditional density of Θ, given $\underline{X}' = \underline{x}'$ (posterior density) is, apart from a normalizing function of $x_1, \ldots, x_t$ (first apart of $f_{\underline{X}'}(\underline{x}')$, after apart of $\prod_{i=1}^{t} p(x_i)$, and finally apart of $c(t_0, x_0)$):

$f_{\Theta|\underline{X}'}(\theta, \underline{x}') = f_{\underline{X}'|\Theta}(\underline{x}', \theta)u(\theta)/f_{\underline{X}'}(\underline{x}') :: u(\theta)f_{\underline{X}'|\Theta}(\underline{x}', \theta) = u(\theta)\prod_{i=1}^{t} f_{X_i|\Theta}(x_i, \theta) =$

$= u(\theta)\prod_{i=1}^{t} f_{X|\Theta}(x_i, \theta) \overset{(2.17)}{\underset{(2.18)}{=}} \frac{q(\theta)^{-t_0} e^{-\theta \cdot x_0}}{c(t_0, x_0)} \prod_{i=1}^{t} \frac{p(x_i)e^{-\theta \cdot x_i}}{q(\theta)} :: q(\theta)^{-(t_0+t)} e^{-\theta\left(x_0 + \sum_{i=1}^{t} x_i\right)}$,

relations numbered (2.27).

So, the distribution of ($\mu(\Theta)|\underline{X}'$) contains on the first line the values $\mu(\theta)$ for this random variable and on the second line the conditional density of $\mu(\Theta)$ given $\underline{X}' = \underline{x}'$, that is the below function: $f_{\mu(\Theta)|\underline{X}'}(\mu(\theta), \underline{x}') = = f_{\Theta|\underline{X}'}(\theta, \underline{x}')::$

$:: q(\theta)^{-(t_0+t)} e^{-\theta\left(x_0 + \sum_{i=1}^{t} x_i\right)}$. But: $\mu(\Theta): \begin{pmatrix} \mu(\theta) \\ u(\theta) \end{pmatrix} :: \begin{pmatrix} \mu(\theta) \\ q(\theta)^{-t_0} e^{-\theta \cdot x_0} \end{pmatrix}$. Density (2.27) is of the same type as the original structure density (2.18), with x_0 replaced by ($x_0 + \sum_i x_i$) and t_0 by ($t_0 + t$). So by using (2.26), the posterior mean (2.20), which is the mean squared error – optimal estimator for $\mu(\Theta)$, is:

$$E[\mu(\Theta)|X_1, \ldots, X_t] = (x_0 + \sum_{i=1}^{t} X_i) / (t_0 + t) \qquad (2.28),$$

if we consider $E[\mu(\Theta)] \overset{(2.26)}{=} x_0 / t_0$ and the above statements. This (see (2.28)) is indeed a non – homogeneous linear combination of $X_1, \ldots, X_t$. By (2.26) we have $m = x_0 / t_0$, and comparing (2.28) with (2.3) we can observe that $t_0 = s^2 / a$ and $z = t / (t + t_0)$. Indeed: $E[\mu(\Theta)|X_1, \ldots, X_t] = (x_0 + \sum_{i=1}^{t} X_i) / (t_0 + t) = \dfrac{\dfrac{x_0 + \sum_{i=1}^{t} X_i}{t}}{\dfrac{t_0 + t}{t}} =$

$= \dfrac{t}{t_0 + t}\left(\dfrac{x_0}{t} + \overline{X}\right) = z\overline{X} + (1 - z)m$, whit $z = t / (t_0 + t)$ (the ratio of the structure parameters s^2 and a, that is"s^2 / a" from the definition of z "$z = (at) / (s^2 + at) = at / \{a[(s^2 / a) + t]\} = t / [(s^2 / a) + +t]$" is, here, equal to t_0). For this application, the mean squared error – optimal estimator for $\mu(\Theta)$, that is $E[\mu(\Theta)|X_1, \ldots, X_t]$ (which in statistics is called **the posterior Bayes estimator** of $\mu(\Theta)$ with respect to the

quadratic loss distance, and **prior** belief **characterized** by $U(\theta)$, or the **exact credibility result**, or the **exact credibility estimate**) coincides with the **best linear credibility estimator** (2.3) for $\mu(\Theta)$.

Comment on **the solution** of **the linear credibility problem**:

It should be noted that the solution (2.3) of the linear credibility problem only yields a statistic computable from the observations, if the structure parameters m, s^2 and a are known. Generally, however, the structure function $U(\cdot)$ is not known. Then the 'estimator' as it stands is not a statistic. Its interest is merely theoretical, but it will be **the basis for further results on credibility**. In the following section we consider different contracts, each with the same structure parameters a, m and s^2, so we can estimate these quantities using the statistics of the different contracts.

3. The classical credibility model of Bühlmann

In this section we will introduce the classical Bühlmann model, which consists of a portfolio of contracts satisfying the constraints of the original Bühlmann model. The classical credibility model of Bühlmann, presents the best linear credibility estimators for this case. The contract index j is a random structure parameter θ_j and observations $X_{j1},\ldots,X_{jt}$: $(\Theta_j, X_{j1},\ldots,X_{jt}) = (\Theta_j, \underline{X}_j^{'})$. The contracts $j = 1,\ldots,k$ are assumed to be i.i.d (independent and identically distributed). Moreover, for every contract $j = 1,\ldots,k$ and for $\Theta_j = \theta_j$ fixed, the variables $X_{j1},\ldots,X_{jt}$ are conditionally independent and identically distributed. In the classical model of Bühlmann, all contracts have in common the fact that their variances and expectations are represented by the same functions $\sigma^2(\cdot)$ and $\mu(\cdot)$ of the risk parameter. Also: $E[X_{jr}|\Theta_j] = \mu(\Theta_j)$, $r = 1,\ldots,t$. Note that the usual definitions of the structure parameters apply, with Θ_j replacing Θ and X_{jr} replacing X_r, so: $m = E[X_{jr}] = E[\mu(\Theta_j)]$, $a = Var[\mu(\Theta_j)]$, $s^2 = E[\sigma^2(\Theta_j)]$.

3.1. The best linearized credibility estimators for the classical model of Bühlmann

If both assumptions (B_1) and (B_2) exists: (B_1) $E[X_{jr}|\Theta_j] = \mu(\Theta_j)$, $Cov[\underline{X}_j\,|\,\Theta_j] = \sigma^2(\Theta_j)I^{(t,t)}$, $j = 1,\ldots,k$ and: (B_2) the contracts $j = 1,\ldots,k$ are independent, the variables $\Theta_1,\ldots,\Theta_k$ are identically distributed, and the observations X_{jr} have finite variants, then the optimal non-homogeneous linear estimators M_j^a for $\mu(\Theta_j)$, $j = 1,\ldots,k$, in the least squares sense read: $\hat{\mu}(\Theta_j) =$

$= M_j^a = (1 - z)m + zM_j$. Here $M_j = \overline{X}_j = \frac{1}{t}\sum_{s=1}^{t} X_{js}$ represents the individual estimator for $\mu(\Theta_j)$. The resulting credibility factor z which appears in the credibility adjusted estimator M_j^a is found as: $z = at / (s^2 + at)$, with the structural parameters a and s^2 as defined above. For proof see [1] of the references, pages 145 to 147.

3.2. Estimation of the structural parameters in the non-homogeneous linear credibility model

The credibility premium for this classical Bühlmann model involves three parameters a, s^2 and m. Now that we embedded the separate contract j in a group of identical contracts, it is possible to give unbiased estimators of these quantities. For this estimation, we assume that we have a portfolio of k identical and independent policies that have been observed for t (≥ 2) years, and let X_{jr} represent the total claim amount of policy j in year r. Let: $\overline{X}_{j.} = \frac{1}{t}\sum_{s=1}^{t} X_{js}$, $\overline{\overline{X}}_{..} = \frac{1}{k}\sum_{j=1}^{k} \overline{X}_{j.}$. For m we propose the unbiased estimator: $\hat{m} = \overline{\overline{X}}_{..}$. For each policy j, the empirical variance: $\frac{1}{t-1}\sum_{r=1}^{t}\left(X_{jr} - \overline{X}_{j.}\right)^2$ is an **unbiased estimator** of $Var(X_{jr}|\Theta_j)$, and thus: $\hat{s}^2 = \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left(X_{jr} - \overline{X}_{j.}\right)^2$ is an **unbiased estimator** of s^2. The empirical variance: $\frac{1}{k-1}\sum_{j=1}^{k}\left(\overline{X}_{j.} - \overline{\overline{X}}_{..}\right)^2$ is an **unbiased estimator** of $Var(\overline{X}_{j.})$, and as: $Var(\overline{X}j.) = \frac{s^2}{t} + a$, we introduce the **unbiased estimator**: $\hat{a} = \frac{1}{k-1}\sum_{j=1}^{k}\left(\overline{X}_{j.} - \overline{\overline{X}}_{..}\right)^2 - \frac{\hat{s}^2}{t}$ for a. This estimator has the weakness that it may take negative values whereas a is non – negative. Therefore, we replace a by the estimator: $a^* = \max(0, \hat{a})$, thus losing unbiasedness, but gaining admissibility. Note that $\hat{m}$, $\hat{s}^2$ and $\hat{a}$ are **consistent** when $k \to +\infty$.

4. Conclusions

The main results of the paper are: the **five applications of the optimal credibility estimator of Bühlmann** (see Subsection **2.2.**), solved when **the credibility solution** of **Bühlmann** contains **known parameters** of **the structure function**, which will make possible **a statistic computable** from **the observations** for **this result**; to obtain estimations for these structure parameters, for Bühlmann's classical model, we embedded the contract in a group of contracts, all providing independent information about the structure distribution (see Section **3.**); since we embedded the separate contract j in a collective of identical contracts, has been possible to give **useful estimators**, more precisely **unbiased** and **consistent estimators** for **the structural parameters** in the non-homogeneous linear credibility model; to be able to use the credibility premiums of the classical Bühlmann model, in Subsection **3.2** we presented **unbiased estimators** for the structure parameters, such that if the structure parameters in the optimal linearized credibility premium are replaced by these estimators, a homogeneous estimator results; this last estimator can also be shown to be optimal; we also demonstrated that these unbiased estimators are **consistent**, when the number of contracts in the portfolio is very big. So in this paper we gave **some practical insurance problems** solved by **mathematical theory** and **credibility theory**.

R E F E R E N C E S

[1] *V. Atanasiu*, Monograph "Contributions of the credibility theory", Printech Publishing, 2009, pp. 7-20, pp. 145-147, pp. 148-154, ISBN 978-606-521-417-0.

[2] *C. N. Bodea*, *V. Atanasiu*, Applications Of The Hierarchical Structure With Two and Three Levels, Economic Computation and Economic Cybernetics Studies and Research (journal ISI), volume 43, Number 3/2009, 2009, pp. 57-68, ISSN 0424 --267 X.

[3]. *H. Bühlmann*, and *A. Gisler*, Universitext A Course in Credibility Theory and its Applications, Springer-Verlag Berlin Heidelberg 2005, Printed in The Netherlands, pp. 55-74, 77-117, 199-217, 10.1007/3-540-29273-X_8.

[4]. H. *Cossette*, E. *Marceau*, and F. *Toureille*, Risk models based on time series for count random variables, Insurance: Mathematics and Economics, **vol. 48**, issue 1, 2011, pages 19-28.

[5]. *E. Ohlsson*, Combining generalized linear models and credibility models in practice, Scandinavian Actuarial Journal, 2008, **4**, 301-314, Applied Section, Taylor & Francis Group, ISSN 0346-1238 print/ISSN 1651-2030 online.

[6]. *E. Ohlsson*, Credibility estimators in multiplicative models, Mathematical Statistics, Stockholm University, Research Report 2006, **3**. Available online at: http://www.math.su.se/matstat/reports/seriea/

[7]. *G. Pitselis*, Robust regression credibility: The influence function approach, Insurance: Mathematics and Economics, **vol. 42**, issue 1, 2008, pages 288-300.