PARISIAN RUIN PROBABILITY FOR THE CLASSICAL RISK MODEL WITH TWO-STEP PREMIUM

Yifan Zhou[1], Jingmin He[2], Jing Zhao[3]

This paper mainly investigates the Parisian ruin probability for the classical risk model with two-step premium, which arises when the surplus process below 0 longer than a fixed amount of time. By using Strong Markov property, an expression for the Parisian ruin probability is derived. And then the explicit form of Parisian ruin probability is obtained when the claims satisfy exponential distribution. Finally, a numerical example is given to illustrate the Parisian ruin probability.

Keywords: Parisian ruin, Excursion theory, Hitting time, Duration

MSC2000: 91B30.

1. Introduction

The classical risk model $X(t)$ with two-step premium is described by

$$X(t) = u + \int_0^t c(X(s))ds - \sum_{i=1}^{N(t)} Z_i, \tag{1}$$

where u denotes the initial reserve, and $c(x)$ is the rate of premium income per time unit, $c(x) = c_1$ for $x \geq 0$ and $c(x) = c_2 > c_1$ for $x < 0$. For a fixed value $t > 0$, the random variable $N(t)$ denotes the number of claims in the interval $(0, t]$, $\{N(t), t > 0\}$ is a Poisson process with parameter λ, the sequence $\{Z_i, i \geq 1\}$ are claim sizes which are positive, independent and identically distributed random variables with common distribution function $G(x)$ and a finite mean μ. It is assumed that $c_1 > \lambda\mu$ to ensure

$$\lim_{t\to\infty} X(t) = \infty.$$

The classical ruin probability is defined by

$$\Psi(u) = P(\tau_0 < \infty | X(0) = u) = P_u(\tau_0 < \infty),$$

where

$$\tau_0 = \begin{cases} \inf\{t \geq 0 : X(t) < 0\}, \\ \infty, \qquad \text{if } X(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

[1]College of Science, Tianjin University of Technology, Tianjin 300384, P.R. China, e-mail: zhyf_2077@163.com

[2]College of Science, Tianjin University of Technology, Tianjin 300384, P.R. China

[3]College of Science, Tianjin University of Technology, Tianjin 300384, P.R. China

For an arbitrary level $a > u$, let the hitting time τ_a to be

$$\tau_a = \begin{cases} \inf\{t \geq 0 : X(t) = a\}, \\ \infty, \quad \text{if } X(t) < a \text{ for all } t > 0. \end{cases}$$

The deficit at ruin $F(u, y)$ is

$$F(u, y) = P(|X(\tau_0)| < y, \tau_0 < \infty | X(0) = u).$$

The expression for $\Psi(u)$ can be obtained from Rolski et al. (1999). And the hitting time τ_a can be obtained from Gerber (1990), and the expression for $F(u, y)$ can be obtained from Gerber and Shiu(1998).

The concept of Parisian ruin originated from Parisian option (see Dassions and Wu (2008a, 2008b)), which allows the surplus process to stay below level zero for a longer period than $d > 0$ before ruin is recognized. Define the Parisian ruin probability to be

$$\Psi_d(u) = P(\tau^d < \infty | X(0) = u),$$

where

$$\tau^d = \inf\{t > 0 : t - \sup\{s < t : X(s) \geq 0\} \geq d, X(t) < 0\}.$$

In recent years, the Parisian ruin has attracted the attention of many scholars. Czarna and Palmowski (2011) focused on a general spectrally negative Lévy process, and derived an expression for probability. Loeffen et al. (2013) gave a compact formula for the Parisian ruin probability which involves only the scale function of the spectrally negative Lévy process and the distribution of the process at time Parisian ruin time. Wong and Cheung (2015) provided the joint distribution of the number of periods of negative surplus that is of duration more than d and less than d, and the Laplace transform of the occupation time were given when the surplus is negative. Czarna et al. (2016) proposed a new iterative algorithm of calculating the distribution of the Parisian ruin time and the number of claims until Parisian ruin. Krzysztof et al. (2016) investigated the probability of the Parisian ruin, and obtained the tail asymptotic behaviour. Bai and Luo (2017) obtained an approximation of the Parisian ruin probability in the Brownian motion risk model with constant force of interest. Peng and Luo (2017) found that if the time length required by the Parisian ruin tends to zero as the initial reserve goes to infinity, the Parisian ruin probability and the classical one are the same in the precise asymptotic behavior. Lkabous and Renaud (2018) introduced their risk measure which is based on cumulative Parisian ruin, and gave some of its properties. Bladt et al. (2018) provided an method for calculating different kinds of Parisian ruin probabilities, with particular emphasis on variations over Parisian type of ruin. Renaud (2019) considered the De Finetti's control problem in a spectrally negative Lévy process with exponential Parisian ruin, gave necessary and sufficient condition for the barrier strategy at level zero to optimal.

The remainder of the paper is organized as follows. By using strong Markov property, an expression of Parisian ruin probability is derived in section 2. An explicit expression for the Parisian ruin probability in a special case is obtained in section 3. Numerical example is given to illustrate the Parisian ruin probability in section 4.

2. **Parisian ruin probability**

In this section, in the classical risk model with two-step premium rate, the explicit formula for expression of Parisian ruin probability is obtained by using strong Markov property.

Lemma 2.1. *For the classical risk model with premium rate c_2, and any $\delta > 0$, the Laplace transform of the time to hit the level a given that the initial state $u < a$ is given by*

$$E_u[e^{-\delta\tau_a}] = e^{\nu_\delta^+(u-a)},$$

where ν_δ^+ is defined to be the unique positive root of

$$-\delta + c_2\nu + \lambda(\hat{G}(\nu) - 1) = 0,$$

and $\hat{G}(\nu)$ is the Laplace transform of the density function of claims

$$\hat{G}(\nu) = \int_0^\infty e^{-\nu y} dG(y).$$

The proof is omitted. More details can be found in Gerber and Shiu (1998).

The sequence of zero points on the time scale of surplus process is showed as following:

$$\begin{aligned} \gamma_0 &= 0, \\ \gamma_k &= \begin{cases} \inf\{t > \gamma_{k-1} : X(t) = 0\}, \\ \infty, \qquad \text{if the set is empty,} \end{cases} \end{aligned}$$

and

$$\beta_k = \begin{cases} \inf\{t > \gamma_{k-1} : X(t) = 0\}, \\ \infty, \qquad \text{if the set is empty.} \end{cases}$$

If $\beta_k < \infty$ and let

$$T_k = \gamma_k - \beta_k \ , \quad k = 1, 2, 3, 4, \ldots,$$

Then $T_1, T_2, T_3, \ldots$ are the duration of the negative surplus, as shown in Figure. 1.

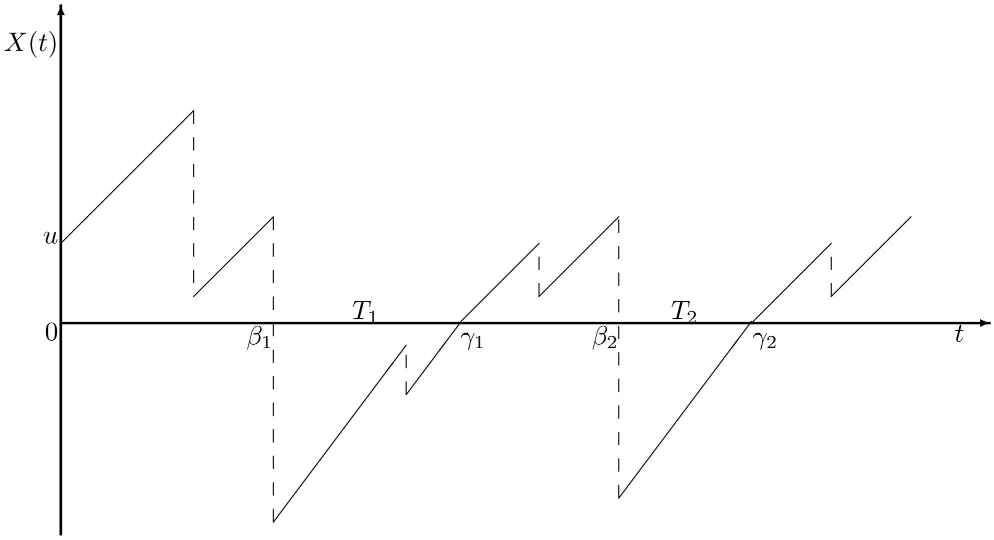

Figure. 1. The surplus process

Lemma 2.2. *For $u \geq 0$, the number N of the negative surplus has the following probability function*

$$P_u(N=n) = P(N=n|X(0)=u) = \begin{cases} 1-\Psi(u), & n=0, \\ \Psi(u)[\Psi(0)]^{n-1}[1-\Psi(0)] & n=1,2,3,\dots. \end{cases}$$

Proof. When $n=0$, it is easy to check that

$$P_u(N=0) = P_u(X(t) \geq 0, \text{ for all } t \geq 0) = 1-\Psi(u).$$

When $n \geq 1$, from definition and the strong Markov property of the surplus process, it follows that

$$\begin{aligned}
& P_u(N=n) \\
= \; & P_u(\gamma_1 < \infty, \gamma_2 < \infty, \cdots, \gamma_n < \infty, X(\gamma_n + t) \geq 0, \text{ for all } t \geq 0) \\
= \; & P_u[\gamma_1 < \infty, \gamma_2 < \infty, \cdots, \gamma_n < \infty, P_u(X(\gamma_n + t) \geq 0, \text{ for all } t \geq 0|\mathscr{F}_{\gamma_n})] \\
= \; & P_u(\gamma_1 < \infty, \gamma_2 < \infty, \cdots, \gamma_n < \infty) P_0(X(t) \geq 0, \text{ for all } t \geq 0) \\
= \; & P_u(\gamma_1 < \infty, \gamma_2 - \gamma_1 < \infty, \cdots, \gamma_n - \gamma_{n-1} < \infty)[1-\Psi(0)] \\
= \; & P_u[\gamma_1 < \infty, \gamma_2 - \gamma_1 < \infty, \cdots, \gamma_{n-1} - \gamma_{n-2} < \infty, P_u(\gamma_n - \gamma_{n-1} < \infty|\mathscr{F}_{\gamma_{n-1}})][1-\Psi(0)] \\
= \; & P_u(\gamma_1 < \infty, \gamma_2 - \gamma_1 < \infty, \cdots, \gamma_{n-1} - \gamma_{n-2} < \infty) P_0(\gamma_n - \gamma_{n-1} < \infty)[1-\Psi(0)] \\
= \; & P_u(\gamma_1 < \infty) P_0(\gamma_2 - \gamma_1 < \infty) \cdots P_0(\gamma_n - \gamma_{n-1} < \infty)[1-\Psi(0)] \\
= \; & P_u(\gamma_1 < \infty)[P_0(\gamma_1 < \infty)]^{n-1}[1-\Psi(0)] \\
= \; & \Psi(u)[\Psi(0)]^{n-1}[1-\Psi(0)].
\end{aligned}$$

This ends the proof. □

Theorem 2.1. *The Parisian ruin probability $\Psi_d(u)$ is given by*

$$\Psi_d(u) = \frac{\Psi(u)}{H_0(d)\Psi(0) - 1}\big(H_u(d)(1-\Psi(0)) + H_0(d)\Psi(0) - 1\big), \tag{2}$$

where $H_u(d)$ is the probability of that excursion less than d with initial surplus u, and $H_0(d)$ is the probability of that excursion less than d without initial surplus

$$\begin{aligned} H_u(d) &= P(T_k < d | X(\gamma_{k-1}) = u), \\ H_0(d) &= P(T_k < d | X(\gamma_{k-1}) = 0). \end{aligned} \tag{3}$$

$h_u(t)$ is density of excursion T_k with initial surplus u, and $h_0(t)$ is density of excursion T_k without initial surplus

$$\begin{aligned} h_u(t) &= \mathscr{L}_\delta^{-1}\left(E[e^{\delta T_k}|X(\gamma_k - 1) = u]\right), \\ h_0(t) &= \mathscr{L}_\delta^{-1}\left(E[e^{\delta T_k}|X(\gamma_k - 1) = 0]\right). \end{aligned}$$

The cumulative distribution function of T_k for $k = 1, 2, 3, \ldots, k$ is given by

$$\begin{aligned} H_u(d) &= \int_0^d h_u(t)dt = \int_0^d \mathscr{L}_\delta^{-1}\left(\int_0^\infty e^{-\nu_\delta^+ y} dF(u,y)\right)dt, \\ H_0(d) &= \int_0^d h_0(t)dt = \int_0^d \mathscr{L}_\delta^{-1}\left(\int_0^\infty e^{-\nu_\delta^+ y} dF(0,y)\right)dt. \end{aligned}$$

Proof. The Laplace transform of T_k for $k = 2, 3, \cdots, k$ is given by

$$\begin{aligned} E[e^{-\delta T_k}|X(\gamma_k - 1) = u] &= E[e^{-\delta(\gamma_k - \beta_k)}|X(\beta_k) = -y, T < \infty] \\ &= \int_0^\infty e^{-\nu_\delta^+ y} dF(u,y). \end{aligned}$$

Likewise,

$$\begin{aligned} E[e^{-\delta T_1}|X(\gamma_k - 1) = 0] &= E[e^{-\delta(\gamma_1 - \beta_1)}|X(0) = 0] \\ &= \int_0^\infty e^{-\nu_\delta^+ y} dF(0,y). \end{aligned}$$

Then

$$\begin{aligned} H_u(d) &= \int_0^d \mathscr{L}_\beta^{-1}\left(\int_0^\infty e^{-\nu_\delta^+ y} dF(u,y)\right)dt, \\ H_0(d) &= \int_0^d \mathscr{L}_\beta^{-1}\left(\int_0^\infty e^{-\nu_\delta^+ y} dF(0,y)\right)dt. \end{aligned} \tag{4}$$

Let L to be the largest ever excursion below zero, such that

$$\begin{aligned} P(L \le d) =& 1 - \Psi(u) + \sum_{i=0}^\infty H_u(d)H_0(d)^i \Psi(u)(1-\Psi(0))\Psi(0)^i \\ =& 1 - \Psi(u) + \frac{H_u(d)\Psi(u)(1-\Psi(0))}{1 - H_0(d)\Psi(0)}. \end{aligned}$$

Hence, the expression of the Parisian ruin probability is found

$$\Psi_d(u) = 1 - P(L \geq d) = \frac{\Psi(u)}{H_0(d)\Psi(0) - 1}\big(H_u(d)(1 - \Psi(0)) + H_0(d)\Psi(0) - 1\big). \quad (5)$$

□

Remark 2.1. *It can be obtained that the classical ruin probability* $\Psi(u)$ *is similar to the limit of the Parisian ruin probability* $\Psi_d(u)$

$$\lim_{d \to 0} \Psi_d(u) = \Psi(u) = \Psi(0)e^{-Ru}.$$

3. Parisian ruin with exponential claims

In this section, claims following the exponential distribution with parameter β are considered. An explicit expression for the Parisian ruin probability is derived by using Excursion theory.

From Rolski(1999) and Gerber and Shiu(1998) it followed the classical ruin probability $\Psi(u)$ and deficit distribution function $F(u, y)$. Assuming that the claim size Z_i has density $\beta e^{-\beta x}$, one obtained

$$\Psi(u) = \frac{\lambda}{\beta c_1} e^{-Ru},$$
$$R = \beta - \frac{\lambda}{c_1},$$
$$F(u, y) = \frac{\lambda}{\beta c_1} e^{-(\beta - \frac{\lambda}{c_1})u}(1 - e^{-\beta y}).$$

The adjustment coefficient ν_δ^+ for the excursion below 0 is obtained from Lundberg's formula

$$-\delta + c_2 \nu_\delta + \lambda(\frac{\beta}{\beta + \nu_\delta} - 1) = 0,$$

which has two roots

$$\nu_\delta^+ = \frac{1}{2c_2}\left(\delta + \lambda - \beta c_2 + \sqrt{\delta^2 + (2\beta c_2 + 2\lambda)\delta + \beta^2 c_2^2 - 2\beta c_2 \lambda + \lambda^2}\right),$$
$$\nu_\delta^- = \frac{1}{2c_2}\left(\delta + \lambda - \beta c_2 - \sqrt{\delta^2 + (2\beta c_2 + 2\lambda)\delta + \beta^2 c_2^2 - 2\beta c_2 \lambda + \lambda^2}\right).$$

The Laplace transform of the length of the excursion below 0 is

$$E[e^{-\delta T_k}] = e^{\nu_\delta^+ x},$$

where x is deficit, and have distribution $F(u, y)$.

Hence

$$
\begin{aligned}
E[e^{-\delta T_k}|X(\gamma_k-1)=u] &= \int_0^\infty e^{-\nu_\delta^+ y} dF(u,y) \\
&= \frac{\lambda}{c_1} e^{-\beta u-\frac{\lambda u}{c_1}} \int_0^\infty e^{-\nu_\delta^+ y} e^{-\beta y} dy \\
&= \frac{\lambda}{c_1(\nu_\delta^+ + \beta)} e^{-\beta u-\frac{\lambda u}{c_1}} \\
&= \frac{2c_2\lambda e^{-\beta u-\frac{\lambda u}{c_1}}}{c_1\left(\delta+\beta c_2+\lambda-\sqrt{(\delta+\beta c_2+\lambda)^2-4\beta\lambda c_2}\right)},
\end{aligned}
$$

$$
\begin{aligned}
E[e^{-\delta T_k}|X(\gamma_k-1)=0] &= \int_0^\infty e^{-\nu_\delta^+ y} dF(0,y) \\
&= \frac{2c_2\lambda}{c_1\left(\delta+\beta c_2+\lambda-\sqrt{(\delta+\beta c_2+\lambda)^2-4\beta\lambda c_2}\right)}.
\end{aligned}
$$

Inverting this Laplace transform with respect to δ gives the transition density $h_0(t)$ and $h_u(t)$

$$
\begin{aligned}
h_u(t) &= \mathscr{L}_\delta^{-1}\left(E[e^{-\delta T_k}|X(\gamma_k-1)=u]\right) \\
&= \sqrt{\frac{c_2\lambda e^{-2\beta u-\frac{2\lambda u}{c_1}}}{\beta c_1^2}} e^{-(\beta c_2+\lambda)t} t^{-1} I_1\left(2t\sqrt{\beta\lambda c_2}\right), \\
h_0(t) &= \mathscr{L}_\delta^{-1}\left(E[e^{-\delta T_k}|X(\gamma_k-1)=0]\right) \\
&= \sqrt{\frac{c_2\lambda}{\beta c_1^2}} e^{-(\beta c_2+\lambda)t} t^{-1} I_1\left(2t\sqrt{\beta\lambda c_2}\right).
\end{aligned}
$$

I_1 is the first kind of Bessel function

$$
I_1(x) \quad = \tfrac{1}{\pi}\textstyle\int_0^\pi e^{x cos(\theta)} d\theta.
$$

The probability that the length of an excursion shorter than d when the initial surplus u or 0 is given by

$$
\begin{aligned}
H_u(d) &= \int_0^d h_u(t)dt, \\
H_0(d) &= \int_0^d h_0(t)dt.
\end{aligned}
$$

The Parisian ruin probability with exponential claims is obtained

$$
\Psi_d(u) = -\frac{\beta c_1 e^{\frac{u(\lambda-\beta c_1)}{c_1}}\big(\lambda+(\beta c_1-\lambda)H_u(d)-\beta c_1 H_0(d)\big)}{\lambda\big(\lambda-\beta c_1 H_0(d)\big)}.
$$

4. **Example**

In this section, figures and numerical examples are presented to illustrate the Parisian ruin probability. It shown that the Parisian ruin probability $\Psi_d(u)$ reacts when the value of its parameters changes. Assuming $c_1 = 1.4, c_2 = 1.6, \lambda = 1.2, G(x) = 1 - e^{-x}, x \geq 0$, one obtains the following graph.

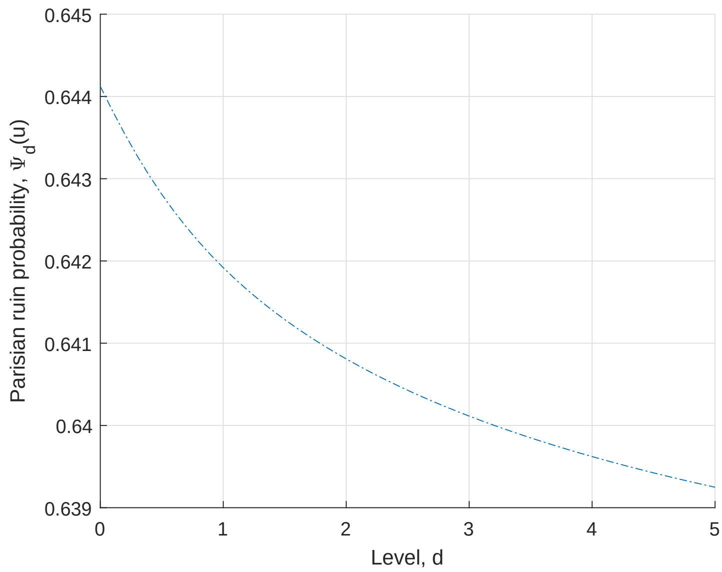

Figure. 2. Parisian ruin probability with level d

The data from Figure 2 show when the level d increase, the Parisian ruin probability decreases. The first derivative of Parisian ruin probability increases with level d. When the level d below 5, the Parisian ruin probability is always greater than 0.639.

Table. 1. Parisian ruin probability with level d

$\Psi(u)$	$\Psi_d(u)$							
	$d = 0$	$d = 0.5$	$d = 1$	$d = 1.5$	$d = 2$	$d = 2.5$	$d = 3$	$d = 3.5$
0.6441	0.6441	0.6428	0.6419	0.6413	0.6408	0.6404	0.6401	0.6398

From Table 1, when $d = 0$, the Parisian ruin probability is equal to the classical ruin probability which is illustrated in Remark 2.4.

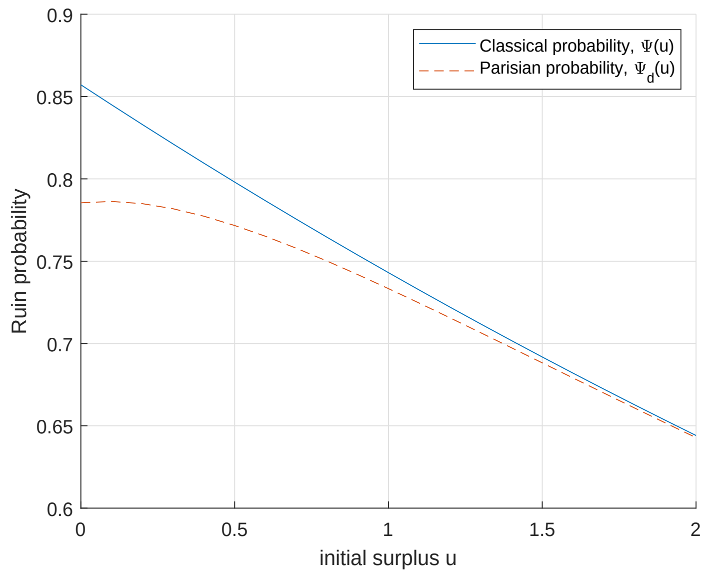

Figure. 3. Ruin probability with initial surplus u

From Figure 3, when initial assets increase, the probability of ruin decreases.

Table. 2. Ruin probability with initial surplus u

	$u=0$	$u=0.2$	$u=0.4$	$u=0.6$	$u=0.8$	$u=1.0$	$u=1.2$	$u=1.4$
$\Psi(u)$	0.8571	0.8330	0.8095	0.7867	0.7646	0.7430	0.7221	0.7018
$\Psi_d(u)$	0.7855	0.7849	0.7773	0.7651	0.7501	0.7333	0.7156	0.6974

From Figure 3 and Table 2, for the certainly initial surplus u, the Parisian ruin probability is lower than the classical ruin probability. When the initial value is small, there is a great difference between the classical ruin probability and the Parisian ruin probability. When the initial value is large, the Parisian ruin probability tends to the classical ruin probability. When the initial surplus of the insurance company is small, the Parisian ruin probability might be a more appropriate measure of risk than the classical ruin probability as it gives the insurance company some time to resolve deficits and resume operations.

Moreover, the result of the study can provide information about the Parisian ruin probability. for the investor, the input option can be selected.

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