U.P.B. Sci. Bull., Series A, Vol. 81, Iss. 1, 2019 ISSN 1223-7027

ON COMMON SOLUTION OF A MONOTONE VARIATIONAL INCLUSION FOR TWO MAPPINGS AND A FIXED POINT PROBLEM

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In this paper, we study the operator $Res^f_{\lambda T} \circ A^f_\lambda$ which is the composition of the resolvent of a maximal monotone operator T and the antiresolvent of a Bregman inverse strongly monotone operator A with respect $\lambda > 0$ and construct an iterative method for approximating a common solution of a monotone inclusion problem and fixed point problem. We further state and prove a strong convergence theorem for obtaining a common solution of a monotone inclusion problem for sum of two operators and a fixed point problem for a Quasi-Bregman strictly pseudocontractive mapping in a reflexive Banach space. Our result extends and compliment related results in the literature.

Keywords: Maximal monotone mapping; Bregman inverse strongly monotone mapping; Quasi-Bregman strictly pseudocontractive; Reflexive Banach space; Strong convergence; Resolvent operator.

MSC2010 *Mathematics Subject Classification*: 47H06, 47H09, 47J05, 47J25.

1. Introduction

Let C be a nonempty, closed and convex subset of a reflexive real Banach space E and let E^* be the topological dual of E. Let the norm and the duality pairing between E and E^* be respectively denoted by $\|.\|$ and $\langle ., . \rangle$ and let $\mathbb{R}$ be the set of real numbers. A functional $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be:

(1) **proper** if its effective domain $D(f) = \{x \in E : f(x) < \infty\} \neq \emptyset$.
(2) **convex** if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall \lambda \in (0,1); x, y \in D(f)$.
(3) **lower semicontinuous** at $x_0 \in D(f)$ if $f(x_0) \leq \liminf_{x \to x_0} f(x)$ and lower semicontinuous on the domain $D(f)$ if it is lower semicontinuous at every point in $D(f)$.

The *Fenchel conjugate* function of f is the convex functional $f^* : E^* \to \mathbb{R}$ defined by

$$f^*(\xi) = sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

It is clear that when f is proper and lower semicontinuous, then so is f^*. The function f is said to be *cofinite* if $dom f^* = E^*$.

Let $f : E \to \mathbb{R}$ be a convex function and $x \in int(dom f)$ where $int(dom f)$ stands for the interior of the domain of f. For any $y \in E$, we define the directional derivative of f at x by

$$f^o(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \quad (1)$$

If the limit as $t \to 0^+$ in (1) exists for each y, then the function f is said to be *Gâteaux differentiable* at x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^o(x, y)$ for all $y \in E$ (see [13]). The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in int(dom f)$. When the limit as $t \to 0$ in (1) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet*

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differentiable at x. In this paper, we will take $f : E \to \mathbb{R}$ to be an admissible function, that is, a proper, lower semicontinuous, convex and Gâteaux differentiable function. Under these conditions we know that f is continuous in $intdomf$ (see [4]).
The function f is said to be *Legendre* if it satisfies the following two conditions.
(L1) $int(domf) \neq \emptyset$, and the subdifferential ∂f is single-valued on its domain.
(L2) $int(domf^*) \neq \emptyset$, and ∂f^* is single-valued on its domain.
Bauschke, Borwein and Combettes in [4] was the first to study the class of Legendre functions in infinite dimensional Banach spaces and their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [4], Theorems 5.4 and 5.6, page 634). In reflexive Banach spaces, it has been established that $\nabla f = (\nabla f^*)^{-1}$ (see [7], page 83). $\nabla f = (\nabla f^*)^{-1}$ together with the conditions (L1) and (L2) gives

$$ran\nabla f = dom\nabla f^* = int(domf)^* \text{and } ran\nabla f^* = dom\nabla f = int(domf).$$

Moreover, f is Legendre if and only if f^* is Legendre (see [4], Corollary 5.5, page 634) and the functions f and f^* are Gateaux differentiable and strictly convex in the interior of their respective domains.

One important and interesting Legendre function is $(\frac{1}{p})\|.\|^p$ with $p \in (1, \infty)$ when E is a smooth and strictly convex Banach space (cf. [4], Lemma 6.2, page 639). In this case the gradient ∇f of f is coincident with the generalized duality mapping of E, i.e., $\nabla f = J_p(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.

Definition 1.1. *The bifunction $D_f : domf \times int(domf) \to [0; +\infty)$, which is defined by*

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \tag{2}$$

is called the Bregman distance (cf. [6, 13]*).*

The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property, which is called the three point identity: for any $x \in domf$ and $y, z \in int(domf)$

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{3}$$

In 1967, Bregman[6] first employed the technique of Bregman distance in the process of designing and analysing feasibility and optimization algorithms . The Bregman distance approach have since been found invaluable in the design and analysis of iterative methods in fixed point theory as it offers an effective way to extend results in Hilbert spaces to reflexive Banach spaces (see, [2, 3] and some the references therein).

According to [9], Section 1.2, page 17 (see also [8]), the modulus of total convexity of f is the bifunction $v_f : int(domf) \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$v_f(x, t) := inf\{D_f(y, x) : y \in domf,\ \|y - x\| = t\}.$$

The function f is said to be *totally convex at a point* $x \in int(domf)$ if $v_f(x, t) > 0$ whenever $t > 0$ and is said to be *totally convex* when it is totally convex at every point $x \in int(domf)$. Examples of totally convex functions can be found in [9, 12].

Butnariu *et al.* [10] established connections between uniform convexity at a point, total convexity at a point, uniform convexity on bounded sets and sequential consistency and used these relations to obtain improved convergence results for the outer Bregman projection algorithm for solving convex feasibility problems and the generalized proximal point algorithm for optimization in Banach spaces. In 2005, Butnariu and Resmerita [12] introduced a Bregman type iterative algorithms and used Bregman type iterative method to solve operator equations. Resmerita [23] studied the existence of totally convex functions in Banach spaces and obtained continuity and stability properties of Bregman projections.

Remark 1.1. *We now make the following observations:*

(1) *The property of totally convexity of f is less stringent than uniform convexity (see* [9]*, section 2.3).*

(2) *The function f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see* [12]*, Theorem 2.10).*

The function f is called sequentially consistent (see [12]) if for any sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in *intdomf* and *domf* respectively, such that $\{x_n\}_{n=0}^{\infty}$ is bounded and

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Let C be a convex subset of *intdomf* and let T be a self-mapping of C. A point $p \in C$ is said to be a fixed point of T if $Tp = p$ and the set of fixed point of a mapping T will be denoted by $F(T)$ in this paper. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 1.2. [5] *Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to intdomf$ is called*

(i) *Bregman Firmly Nonexpansive (BFNE for short) if*

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \qquad \forall x, y \in C.$$

(ii) *Bregman Strongly Nonexpansive (BSNE) with respect to a nonempty $\hat{F}(T)$ if*

$$D_f(p, Tx) \le D_f(p, x)$$

for all $p \in \hat{F}(T)$ and $x \in C$ and if whenever $\{x_n\}_{n=0}^{\infty} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty} \Big(D_f(p, x_n) - D_f(p, Tx_n) \Big) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

iii) *Bregman relatively nonexpansive if $\hat{F}(T) = F(T) \neq \emptyset$ and*

$$D_f(p, Tx) \le D_f(p, x), \forall x \in C, p \in F(T).$$

(iv) *Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and*

$$D_f(p, Tx) \le D_f(p, x), \forall x \in C, p \in F(T).$$

(v) *Bregman quasi-k-strictly pseudocontractive (see* [25]*), if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$; such that*

$$D_f(p, Tx) \le D_f(p, x) + kD_f(x, Tx), \forall x \in C, p \in F(T).$$

(vi) *closed if for any sequence $\{x_n\}_{n=0}^{\infty} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to \infty$, then $Tx = y$.*

Let $B : E \to E^*$ be a single-valued nonlinear mapping and $A : E \to 2^{E^*}$ be a set-valued mapping. Then the Variational Inclusion Problem (VIP) (for the sum of the two mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{4}$$

The solution set of problem (4) is denoted by $VIP(A, B)$.

The VIP (4) has been applied to solving problems arising in mechanics, optimization, nonlinear programming, economics, finance, applied sciences, etc (see for example [1, 14, 15, 24] and the references therein). However, most of the existing iterative schemes for approximating zeros of the sum of two monotone operators are mainly found in the frame work of Hilbert spaces, whereas many important problems related to practical problems are generally defined in Banach spaces. For example, the maximal monotone operator related to elliptic boundary value problem has Sobolev space $W^{1,p}(\Omega)$, $(\Omega \subset \mathbb{R}^n)$ as its natural domain of definition [17]. Thus the need to extend the problem of finding zeros of monotone operators to real Banach spaces.

Let $A : E \to 2^{E^*}$ be a mapping. Then A is said to be a monotone if for any $x, y \in \text{dom}A$, we have

$$\xi \in Ax \text{ and } \eta \in Ay \implies \langle \xi - \eta, x - y\rangle \geq 0, \tag{5}$$

and A is said to be maximal monotone if A is monotone and the graph of A is not properly contained in the graph of any other monotone mapping. Let $A : E \to 2^{E^*}$ be a maximal monotone mapping, then for any $\lambda > 0$ the resolvent of A associated with λ is the operator $\text{Res}^f_{\lambda A} : E \to 2^E$ defined by

$$\text{Res}^f_{\lambda A} = (\nabla f + \lambda A)^{-1} \circ \nabla f. \tag{6}$$

It is known that $\text{Res}^f_{\lambda A}$ is a BFNE operator, single-valued and $\text{F}(\text{Res}^f_{\lambda A}) = A^{-1}(0)$ (see [5]). If $f : E \to \mathbb{R}$ is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then $\text{Res}^f_{\lambda A}$ is BSNE and $\hat{F}(\text{Res}^f_{\lambda A}) = F(\text{Res}^f_{\lambda A})$ (see [19]).

Let C be a nonempty closed and convex subset of a reflexive Banach space E, a mapping $A : E \to E^*$ is called single valued Bregman inverse strongly monotone (BISM) on the set C if

$$C \cap (\text{int dom}f) \neq \emptyset \tag{7}$$

and for any $x, y \in C \cap (\text{int dom}f)$, we have

$$\langle Ax - Ay, \nabla f^*(\nabla f(x) - Ax) - \nabla f^*(\nabla f(y) - Ay)\rangle \geq 0. \tag{8}$$

Remark 1.2. *(see* [16]*). The class of BISM mappings is more general than the class of firmly nonexpansive operators in Hilbert spaces. Indeed, if $f = \frac{1}{2}||.||^2$, then $\nabla f = \nabla f^* = I$, where I is the identity operator and* (8) *becomes*

$$\langle Ax - Ay, x - Ax - (y - Ay)\rangle \geq 0, \tag{9}$$

that is

$$||Ax - Ay||^2 \leq \langle x - y, Ax - Ay\rangle. \tag{10}$$

For more details on BISM see [11, 20] and the references therein. The anti-resolvent $A^f_\lambda :$ $E \to E$ associated with a mapping $A : E \to E^*$ and $\lambda > 0$ is defined by

$$A^f_\lambda := \nabla f^* \circ (\nabla f - \lambda A). \tag{11}$$

If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E; then the anti-resolvent A^f_λ is a single-valued BSNE operator which satisfies $F(A^f_\lambda) = \hat{F}(A^f_\lambda)$ (cf. [19]).

Let E be a real Banach space. A mapping $T : E \to 2^E$ is called accretive if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ (J is the normalised duality mapping), such that

$$\langle j(x - y), u - v\rangle \geq 0, \ \forall \ u \in Tx, v \in Ty.$$

Clearly, the accretive operators and the monotone operators on Banach spaces are two distinct extensions of the monotone operators from Hilbert spaces to Banach spaces but it is the problem of finding the zeros of accretive operators that have been given much attention in the literature. On the other hand, finding the zeros of monotone operators in Banach spaces are still not so popular and even the very few that existed only considered the case for a single operator i.e., $0 \in Bx$ instead of $0 \in (B + A)x$.

Inspired and motivated by the fact that the study of monotone variational inclusion problems for sum of two operators has mainly been restricted to Hilbert spaces, we design an iterative algorithm for approximating a fixed point of Bregman quasi-k-strict pseudo-contraction mappings, which is also a solution of a monotone variational inclusion problem for sum of two mappings A and B where A is a maximal monotone mapping and B is a Bregman inverse strongly monotone mapping on a real reflexive Banach space. Our result extend and compliment some related results in the literature for example [22, 26, 27, 28, 29, 30].

2. Preliminaries

Lemma 2.1. *(see* [22]*, Proposition 8). Let $A : E \to 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Then*

$$D_f\left(u, Res^f_{\lambda A}(x)\right) + D_f\left(Res^f_{\lambda A}(x), x\right) \leq D_f(u, x), \tag{12}$$

for all $\lambda > 0$, $u \in A^{-1}(0^*)$ and $x \in X$.

Lemma 2.2. *(see* [16]*, Proposition 11). Let $A : E \to E^*$ be a BISM mapping such that $A^{-1}(0^*) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a Legendre function which satisfies the range condition, ran $(\nabla f - \lambda A) \subset$ ran (∇f). Then the following hold:*

(i) $A^{-1}(0^*) = F(A^f_\lambda)$,

(ii) *the anti-resolvent A^f_λ is a BFNE operator. In addition,*

$$D_f(u, A^f_\lambda x) + D_f(A^f_\lambda x, x) \leq D_f(u, x)$$

for any $u \in A^{-1}(0^*)$ and for all $x \in$ dom A^f_λ.

Lemma 2.3. *(*[9]*). The function $f : E \to \mathbb{R}$ is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.4. *(*[21]*). If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.5. *(*[16]*). Assume that $f : E \to \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of E. Let C be a nonempty closed and convex subset of E. Let $\{T_i : 1 \leq i \leq N\}$ be BSNE operators which satisfy $\hat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$ and let $T := T_N \circ T_{N-1} \circ \cdots \circ T_1$. If $\cap\{F(T_i) : 1 \leq i \leq N\}$ and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \hat{F}(T)$.*

The Bregman projection (see, [6]) with respect to f of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is defined as the necessarily unique vector $Proj^f_C(x) \in C$, which satisfies

$$D_f(Proj^f_C(x), x) = inf\{D_f(y, x) : y \in C\}. \tag{13}$$

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [12] that $z = Proj^f_C x$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \ \forall y \in C. \tag{14}$$

We also have,

$$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \leq D_f(y, x),\ \forall x \in E, y \in C. \tag{15}$$

Proposition 2.1. *Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. In addition, if $f : E \to (-\infty; +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function (see* [18]*). Hence V_f is convex in the second variable. Thus, for all $z \in E$,*

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i)) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{16}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.6. *(*[21]*). If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.7. *(*[9]*). The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.8. *(*[22]*). Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}_{n=1}^\infty$ is also bounded.*

Lemma 2.9. *(*[25]*). Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a Bregman quasi-strictly pseudocontractive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.10. *(*[25]*). Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a Bregman quasi-strictly pseudocontractive mapping with respect to f. Then, for any $x \in C, p \in F(T)$ and $k \in [0,1)$ the following holds:*

$$D_f(x, Tx) \leq \frac{1}{1-k}\langle \nabla f(x) - \nabla f(Tx), x - p\rangle.$$

3. Main Results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E and Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $B : E \to E^*$ be a BISM mapping and $A : E \to 2^{E^*}$ be a maximal monotone operator. Let $T : C \to C$ be a closed and Bregman quasi-k-strict pseudo-contraction such that $\Gamma = F(T) \cap F(Res_{\lambda A}^f \circ B_\lambda^f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated iteratively as follows.*

$$\begin{cases} x_0 \in C & = & C_0 \\ y_n & = & \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n)]) \\ u_n & = & \nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(Res_{\lambda A}^f \circ B_\lambda^f(y_n))) \\ C_{n+1} & = & \{z \in C_n : D_f(z, y_n) + D_f(x_n, u_n) \\ & \leq & \frac{1+k}{1-k}\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle + \langle \nabla f(Tx_n) - \nabla f(u_n), x_n - z\rangle\} \\ x_{n+1} & = & P_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{17}$$

Let α_n, β_n and γ_n be sequences in $(0,1)$ such that $\liminf_{n\to\infty}(1-\alpha_n)\gamma_n > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then $\{x_n\}$ converges strongly to $x^* = P_\Gamma^f x_0$.

Before we start the proof of the theorem, let us make the following important observation: It is known that $Res^f_{\lambda A}$ and B^f_λ are BSNE operators and since $F(Res^f_{\lambda A}) \cap F(B^f_\lambda) = F(Res^f_{\lambda A} \circ B^f_\lambda) = (A+B)^{-1}0 \neq \emptyset$, it then follows from Lemma 2.5 that $Res^f_{\lambda A} \circ B^f_\lambda$ is BSNE and $F(Res^f_{\lambda A} \circ B^f_\lambda) = \hat{F}(Res^f_{\lambda A} \circ B^f_\lambda)$. Moreover, $F(Res^f_{\lambda A} \circ B^f_\lambda) = (A+B)^{-1}0$.

Proof. We partition the proof into 4 steps.
Step 1. $\Gamma \subset C_n$ for all $n \geq 0$.
It has been shown that $F(T)$ is closed and convex (see Lemma 2.9). Moreover, $Res^f_{\lambda A} \circ B^f_\lambda$ is a Bregman strongly nonexpansive mapping and therefore $F(Res^f_{\lambda A} \circ B^f_\lambda)$ is closed and convex (see, [19]). Hence, we have that Γ is nonempty, closed and convex and that P^f_Γ is well defined.

Next we show that C_n is closed and convex $\forall\ n \geq 0$. From the statement of Theorem 3.1, we have that $C_0 = C$ is closed and convex. Now suppose that C_j is closed and convex for some $j \geq 1$. Let $z \in C_{j+1}$, then

$$\begin{aligned} D_f(z, y_j) + D_f(x_j, u_j) &\leq D_f(z, x_j) + \frac{1+k}{1-k}\langle \nabla f(x_j) - \nabla f(Tx_j), x_j - z\rangle \\ &\quad + \langle \nabla f(Tx_j) - \nabla f(u_j), x_j - z\rangle, \end{aligned} \tag{18}$$

which is the same as

$$\begin{aligned} 2f(x_j) - f(y_j) - f(u_j) &\leq \langle \nabla f(u_j), x_j - u_j\rangle + \langle \nabla f(y_j), z - y_j\rangle - \langle \nabla f(x_j), z - x_j\rangle \\ &\quad + \frac{1+k}{1-k}\langle \nabla f(x_j) - \nabla f(Tx_j), x_j - z\rangle \\ &\quad + \langle \nabla f(Tx_j) - \nabla f(u_j), x_j - z\rangle. \end{aligned} \tag{19}$$

Therefore, C_{j+1} is closed and convex and then we conclude that C_n is closed and convex for all $n \geq 0$.

We now show that $\Gamma \subset C_n\ \forall\ n \geq 0$. Obviously, $\Gamma \subset C_0 = C$. Assume that $\Gamma \subset C_j$ for some $j \in \mathbb{N}$. Let $z \in \Gamma$, then from (17), we have

$$\begin{aligned} D_f(z, u_j) &= D_f(z, \nabla f^*(\beta_j \nabla f(y_j) + (1-\beta_j)\nabla f(Res^f_{\lambda A} \circ B^f_\lambda(y_j))) \\ &\leq \beta_j D_f(z, y_j) + (1-\beta_j) D_f(z, Res^f_{\lambda A} \circ B^f_\lambda(y_j)) \\ &\leq \beta_j D_f(z, y_j) + (1-\beta_j) D_f(z, y_j) \\ &= D_f(z, y_j). \end{aligned} \tag{20}$$

Also,

$$\begin{aligned} D_f(z, y_j) &= D_f(z, \nabla f^*(\alpha_j \nabla f(x_j) + (1-\alpha_j)[(1-\gamma_j)\nabla f(x_j) + \gamma_j \nabla f(Tx_j)])) \\ &\leq \alpha_j D_f(z, x_j) + (1-\alpha_j)(1-\gamma_j) D_f(z, x_j) + (1-\alpha_j)\gamma_j D_f(z, Tx_j) \\ &\leq \alpha_j D_f(z, x_j) + (1-\alpha_j)(1-\gamma_j) D_f(z, x_j) + (1-\alpha_j)\gamma_j D_f(z, x_j) \\ &\quad + (1-\alpha_j)\gamma_j k D_f(x_j, Tx_j) \\ &\leq \alpha_j D_f(z, x_j) + k D_f(x_j, Tx_j) \\ &\leq \alpha_j D_f(z, x_j) + \frac{k}{1-k}\langle \nabla f(x_j) - \nabla f(Tx_j), x_j - z\rangle. \end{aligned} \tag{21}$$

Moreover, from the three point identity (3), we have

$$D_f(z, u_j) = D_f(z, x_j) + D_f(x_j, u_j) + \langle \nabla f(x_j) - \nabla f(u_j), z - x_j\rangle. \tag{22}$$

From (21) and (22), we obtain that

$$\begin{aligned} D_f(x_j,u_j) &= D_f(z,u_j)-D_f(z,x_j)+\langle\nabla f(x_j)-\nabla f(u_j),x_j-z\rangle \\ &\leq D_f(z,y_j)-D_f(z,x_j)+\langle\nabla f(x_j)-\nabla f(u_j),x_j-z\rangle \\ &\leq D_f(z,x_j)-D_f(z,x_j)+\frac{k}{1-k}\langle\nabla f(x_j)-\nabla f(Tx_j),x_j-z\rangle \\ &\quad +\langle\nabla f(x_j)-\nabla f(u_j),x_j-z\rangle \\ &\leq D_f(z,x_j)-D_f(z,x_j)+\frac{1}{1-k}\langle\nabla f(x_j)-\nabla f(Tx_j),x_j-z\rangle \\ &\quad +\langle\nabla f(Tx_j)-\nabla f(u_j),x_j-z\rangle. \end{aligned} \tag{23}$$

Therefore, from (21) and (23), we get that

$$\begin{aligned} D_f(z,y_j)+D_f(x_j,u_j) &\leq D_f(z,x_j)+\frac{1+k}{1-k}\langle\nabla f(x_j)-\nabla f(Tx_j),x_j-z\rangle \\ &\quad +\langle\nabla f(Tx_j)-\nabla f(u_j),x_j-z\rangle. \end{aligned} \tag{24}$$

Hence $z\in C_{j+1}$ and thus $\Gamma\subset C_n\ \forall\ n\geq 0$.

Step 2. $x_n\to x^*$ for some $x^*\in C$.
Since $x_n=P^f_{C_n}x_0$ and $x_{n+1}=P^f_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, then

$$D_f(x_n,x_0)\leq D_f(x_{n+1},x_0),\ n\geq 1. \tag{25}$$

Again from (15), we have

$$\begin{aligned} D_f(x_n,x_0) &= D_f(P^f_{C_n}x_0,x_0) \\ &\leq D_f(z,x_0)-D_f(z,x_n) \\ &\leq D_f(z,x_0). \end{aligned} \tag{26}$$

Combining (25) and (26), we conclude that $\lim_{n\to\infty} D_f(x_n,x_0)$ exists.
Furthermore, since $x_m=P^f_{C_m}x_0\in C_m\subset C_n$ for $m>n$, then from (15), we have

$$\begin{aligned} D_f(x_m,x_n) &= D_f(x_m,P^f_{C_n}x_0) \\ &\leq D_f(x_m,x_0)-D_f(P^f_{C_n}x_0,x_0) \\ &= D_f(x_m,x_0)-D_f(x_n,x_0)\to 0,n,m\to\infty. \end{aligned} \tag{27}$$

Since f is totally convex on bounded subsets of E, then it follows from Lemma 2.7 that f is sequentially consistent. Thus $||x_m-x_n||\to 0,m,n\to\infty$, which means that $\{x_n\}$ is a Cauchy sequence. Since C is a closed subset of a reflexive Banach space E and $\{x_n\}$ is a Cauchy sequence in C, then there exists $x^*\in C$ such that $x_n\to x^*,n\to\infty$.

Step 3. $x^*\in F(T)\cap F(Res^f_{\lambda A}\circ B^f_\lambda)$.
Clearly, since $\{x_n\}$ is a Cauchy sequence, then

$$\lim_{n\to\infty} D_f(x_{n+1},x_n)=0, \tag{28}$$

which implies

$$||x_{n+1}-x_n||\to 0,n\to\infty. \tag{29}$$

Again, from $x_{n+1}=P^f_{C_{n+1}}x_0\in C_{n+1}$, and (17), we have

$$\begin{aligned} D_f(x_{n+1},y_n)+D_f(x_n,u_n) &\leq D_f(x_{n+1},x_n)+\frac{1+k}{1-k}\langle\nabla f(x_n)-\nabla f(Tx_n),x_n-x_{n+1}\rangle \\ &\quad +\langle\nabla f(Tx_n)-\nabla f(u_n),x_n-x_{n+1}\rangle\to 0,n\to\infty. \end{aligned} \tag{30}$$

Thus $D_f(x_{n+1}, y_n) \to 0, n \to \infty$ and $D_f(x_n, u_n) \to 0, n \to \infty$.
Therefore, $||x_{n+1} - y_n|| \to 0$ and $||x_n - u_n|| \to 0$.
Furthermore,

$$||y_n - x_n|| \leq ||y_n - x_{n+1}|| + ||x_{n+1} - x_n|| \to 0, n \to \infty, \tag{31}$$

and

$$||x_{n+1} - u_n|| \leq ||x_{n+1} - x_n|| + ||x_n - u_n|| \to 0, n \to \infty. \tag{32}$$

Also,

$$||y_n - u_n|| \leq ||y_n - x_n|| + ||x_n - u_n|| \to 0, n \to \infty. \tag{33}$$

Since f is uniformly Fréchet differentiable and bounded on bounded subsets of E, we have from Lemma 2.6 that ∇f is uniformly continuous on bounded subsets of E. Thus it follows from (31) and (32) respectively that $||\nabla f(y_n) - \nabla f(x_n)|| \to 0$ and $||\nabla f(u_n) - \nabla f(y_n)|| \to 0$. But from (17), we have

$$\begin{aligned}
& ||\nabla f(y_n) - \nabla f(x_n)|| \\
= \ & ||\alpha_n \nabla f(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n)] - \nabla f(x_n)|| \\
= \ & ||\alpha_n(\nabla f(x_n) - \nabla f(x_n)) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n) - \nabla f(x_n)]|| \\
= \ & (1-\alpha_n)||(1-\gamma_n)(\nabla f(x_n) - \nabla f(x_n)) + \gamma_n(\nabla f(Tx_n) - \nabla f(x_n))|| \\
= \ & (1-\alpha_n)\gamma_n||\nabla f(Tx_n) - \nabla f(x_n))||,
\end{aligned} \tag{34}$$

which gives

$$\begin{aligned}
& ||\nabla f(Tx_n) - \nabla f(x_n))|| \\
= \ & \frac{1}{(1-\alpha_n)\gamma_n}||\nabla f(y_n) - \nabla f(x_n)|| \to 0, n \to \infty.
\end{aligned} \tag{35}$$

Similarly,

$$||\nabla f(u_n) - \nabla f(y_n)|| \ = \ (1-\beta_n)||\nabla f(Res^f_{\lambda A} \circ B^f_\lambda(y_n) - \nabla f(y_n)||, \tag{36}$$

which implies

$$||\nabla f(Res^f_{\lambda A} \circ B^f_\lambda(y_n) - \nabla f(y_n)|| = \frac{1}{(1-\beta_n)}||\nabla f(u_n) - \nabla f(y_n)|| \to 0, n \to \infty. \tag{37}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^*, we have from (35) and (37) respectively that

$$\lim_{n\to\infty} ||Tx_n - x_n|| = 0, \tag{38}$$

and

$$\lim_{n\to\infty} ||Res^f_{\lambda A} \circ B^f_\lambda(y_n) - y_n|| = 0. \tag{39}$$

It follows from (38), the fact that T is closed and $x_n \to x^*$ that $x^* \in F(T)$. Again, since $Res^f_{\lambda A} \circ B^f_\lambda$ is a Bregman strongly nonexpansive mapping such that $\hat{F}(Res^f_{\lambda A} \circ B^f_\lambda) = F(Res^f_{\lambda A} \circ B^f_\lambda)$ and $y_n \to x^*$, then it follows from (39) that $x^* \in F(Res^f_{\lambda A} \circ B^f_\lambda)$. Thus $x^* \in F(T) \cap F(Res^f_{\lambda A} \circ B^f_\lambda)$.

Step 4. $x^* = P^f_\Gamma x_0$.

From $x_n = P^f_{C_n} x_0$ and (14), we have

$$\langle \nabla f(x_0) - \nabla f(x_n), x_n - z\rangle \geq 0, \ \forall z \in C_n.$$

Thus since $\Gamma \subset C_n$, we have

$$\langle \nabla f(x_0) - \nabla f(x_n), x_n - p\rangle \geq 0, \ \forall p \in \Gamma. \tag{40}$$

Taking limit as $n \to \infty$ in (40), we obtain that

$$\langle \nabla f(x_0) - \nabla f(x^*), x^* - p\rangle \geq 0,\ \forall p \in \Gamma, \tag{41}$$

which implies from the characterisation of the Bregman projection that $x^* = P^f_{\Gamma} x_0$. □

Corollary 3.1. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E and Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and let $T : C \to C$ be a closed and Bregman quasi-k-strict pseudo-contraction such that $\Gamma_1 = F(T) \cap F(Res^f_{\lambda A}) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated iteratively as follows.*

$$\begin{cases} x_0 \in C &= C_0 \\ y_n &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n)]) \\ u_n &= \nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(Res^f_{\lambda A}(y_n))) \\ C_{n+1} &= \{z \in C_n : D_f(z, y_n) + D_f(x_n, u_n) \\ &\leq \frac{1+k}{1-k}\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle + \langle \nabla f(Tx_n) - \nabla f(u_n), x_n - z\rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{42}$$

Let α_n, β_n and γ_n be sequences in $(0,1)$ such that $\liminf_{n\to\infty}(1-\alpha_n)\gamma_n > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then $\{x_n\}$ converges strongly to $x^* = P^f_{\Gamma_1} x_0$.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E and Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $B : E \to E^*$ be a BISM mapping and let $T : C \to C$ be a closed and Bregman quasi-k-strict pseudo-contraction such that $\Gamma_2 = F(T) \cap F(B^f_{\lambda}) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated iteratively as follows.*

$$\begin{cases} x_0 \in C &= C_0 \\ y_n &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n)]) \\ u_n &= \nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(B^f_{\lambda}(y_n))) \\ C_{n+1} &= \{z \in C_n : D_f(z, y_n) + D_f(x_n, u_n) \\ &\leq \frac{1+k}{1-k}\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle + \langle \nabla f(Tx_n) - \nabla f(u_n), x_n - z\rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{43}$$

Let α_n, β_n and γ_n be sequences in $(0,1)$ such that $\liminf_{n\to\infty}(1-\alpha_n)\gamma_n > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then $\{x_n\}$ converges strongly to $x^* = P^f_{\Gamma_2} x_0$

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E and Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int(domf)$. Let $B : E \to E^*$ be a Bregman quasi-nonexpansive mapping and $A : E \to 2^{E^*}$ be a maximal monotone operator. Let $T : C \to C$ be a closed and Bregman quasi-nonexpansive such that $\Gamma_3 = F(T) \cap F(Res^f_{\lambda A} \circ B^f_{\lambda}) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated iteratively as follows.*

$$\begin{cases} x_0 \in C &= C_0 \\ y_n &= \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)[(1-\gamma_n)\nabla f(x_n) + \gamma_n \nabla f(Tx_n)]) \\ u_n &= \nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(Res^f_{\lambda A} \circ B^f_{\lambda}(y_n))) \\ C_{n+1} &= \{z \in C_n : D_f(z, y_n) + D_f(x_n, u_n) \\ &\leq \langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle + \langle \nabla f(Tx_n) - \nabla f(u_n), x_n - z\rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{44}$$

Let α_n, β_n *and* γ_n *be sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty}(1-\alpha_n)\gamma_n > 0$ *and* $\liminf_{n\to\infty}(1-\beta_n) > 0$. *Then* $\{x_n\}$ *converges strongly to* $x^* = P^f_{\Gamma_3}x_0$

4. Conclusion

In a real Hilbert space H, A mapping $T : C \to C$, (C is a closed convex subset of H) is said to be $k-$demicontractive, if $\exists k \in [0,1)$ and $F(T) \neq 0$ such that

$$||Tx-p||^2 \leq ||x-p||^2| + k||x-Tx||^2, \forall x \in H, p \in F(T). \tag{45}$$

Moreover, if we take $f = \frac{1}{2}||.||^2$ in a real Hilbert space, we have that $\nabla f = I$ and $D_f(x,y) = \frac{1}{2}||x-y||^2$, where I is the identity operator of H. Thus, it is easy to see that the Bregman-k-strictly pseudocontractive mapping is more general than the k-demicontractive mappings in real Hilbert space. Also, we observe that $Res^f_{\lambda A} = (I+\lambda A)^{-1} = J_{\lambda A}$ and $B^f_\lambda = (I-\lambda B)$, which means that $Res^f_{\lambda A} \circ B^f_\lambda$ becomes the resolvent operator $J_{\lambda A}(I-\lambda A)$ popularly used in approximating VIP (4) for a maximal monotone operator A and an α-inverse strongly monotone operator B in real Hilbert spaces. Hence our result in this paper extends the results on finding common solutions of VIP (4) for two operators and fixed point problems for demicontractive mappings(also quasi nonexpansive mappings) from real Hilbert space to real reflexive Banach space. We note here that the BISM operator is 1-inverse strongly monotone in Hilbert space.

Acknowledgement: This work was done when the author is a Post-doctoral Research Fellow at the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa and the author is supported by the National Research Foundation (NRF) of South Africa(S& F-DST/NRF Innovation Postdoctoral Fellowship; Reference Number: SFP170530234929). Opinions expressed and conclusions arrived are those of the author and are not necessarily to be attributed to the NRF.

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