U.P.B. Sci. Bull., Series A, Vol. 76, Iss. 3, 2014 ISSN 1223-7027

CALCULATION OF THE GRAVITOELECTROMAGNETISM FORCE FOR CYLINDRICALLY SYMMETRIC SPACETIME

Morteza Yavari[1]

Keywords: Threading formalism, cylindrically symmetric spacetime, gravitoelectromagnetism force

Using 1+3 splitting concepts of a spacetime, we first discuss the gravitoelectromagnetism force in terms of the gravitoelectric and gravitomagnetic fields. By applying the Hamilton-Jacobi method, the trajectory of a moving test particle in the cylindrically symmetric spacetime is calculated. Finally, the gravitoelectromagnetism force and the corresponding potential function are determined.

1. Introduction

As is well known [1], theory of general relativity discussed the motion of Mercury perihelion in terms of a relativistic gravitoelectric field correction to the Newtonian gravitational potential of the Sun. Also it contains a gravitomagnetic field due to proper rotation of the Sun and this field influence on planetary orbits, [2-4]. The gravitomagnetic field is much smaller than the gravitoelectric one. Theory of gravitoelectromagnetism[1] that assumes a perfect isomorphism between gravitation and electromagnetism has been established by Heaviside [6] and Jefimenko, [7]. In the same way that a magnetic field is created when a charged object rotates, a gravitomagnetic field is created when a massive body rotates and this effect is too small. The gravitomagnetic field plays an important role in some astrophysical scenarios such as neutron stars [8-10] and rotating black holes. To detect this field, it is necessary to examine a very massive object or build an instrument that is very sensitive. The gravitomagnetic field of Earth can be measured by studying the motion of satellites LAGEOS[2], [11,12]. The LAGEOS measured the frame-dragging of the Earth. The recent results from the LAGEOS and LAGEOS II are available in [13-15]. Theory of general relativity predicts that the rotating bodies drag spacetime around themselves in a phenomenon referred to as frame-dragging. The rotational frame-dragging effect was first derived from the theory of general relativity in 1918

[1]Department of Physics, Islamic Azad University, Kashan Branch, Kashan, Iran, Email: yavari@iaukashan.ac.ir

[1]The gravitoelectric and gravitomagnetic fields are so-called the gravitoelectromagnetism fields, [5]. Gravitoelectromagnetism refers to a set of analogies between Maxwell equations and a reformulation of the Einstein field equations in general relativity.

[2]LAGEOS (Laser Geodynamics Satellites) are a series of satellites designed to provide an orbiting laser ranging benchmark for geodynamical studies of the Earth.

by Josef Lense and Hans Thirring and it is also known as the Lense-Thirring effect, [2,3]. The Lense-Thirring precession of the planetary orbits is too weak, and so is not easily detected. In 2004, Gravity Probe B launched by Stanford physicists to measure the gravitomagnetism on a gyroscope in outer space with much greater precision. The data analysis of the Gravity Probe B mission is still ongoing. The recent observational results from the Gravity Probe B are available in [16,17]. At this time, the measurement of gravitomagnetism via superconducting gyroscopes in a satellite about the Earth is one of the aims of NASA.

A brief review of literature which describe the gravitoelectromagnetism is presented below. Jantzen et al. [18] reviewed the many faces of the gravitoelectromagnetism. Ciufolini and Wheeler [19] described the geodesics, precession and forces on gyroscopes in the gravitoelectromagnetism fields. Mashhoon and Santos [20] discussed the gravitomagnetism in connection with the rotating cylindrical systems. Bini and Jantzen [21] obtained a list of References on spacetime splitting and gravitoelectromagnetism. Ruggiero and Tartaglia [22] reviewed the theory and practice of gravitomagnetism. Barros et al. [23] studied the global properties of the gravitomagnetism by investigating the gravitomagnetic field of a rotating cosmic string. Mashhoon [24] reviewed the theoretical aspects of the gravitoelectromagnetism and some recent developments in this topic. Capozziello et al. [25] discussed the theory of orbits by considering the gravitomagnetic effects in the geodesic motion. Costa and Herdeiro [26] concluded that the actual physical similarities between the gravity and electromagnetism occur only on very special conditions and depends crucially on the reference frame. Schmid [27] have been treated the cosmological gravitoelectromagnetism and Machs principle. Ghose [28] studied the mutual interaction of a relativistic particle and gravitoelectromagnetism both classically and quantum theoretically. Ruggiero and Iorio [29] investigated the effects of a time-varying gravitomagnetic field on the motion of test particles. Tsagas [30] presented the technical and physical aspects of the gravitomagnetic interaction. Costa et al. [31] studied the dynamics of spinning test particles in GR, in the framework of exact gravitoelectromagnetic analogies. Li [32] showed that the gravitational theories with metrics relevant to the gravitoelectromagnetism can be decomposed in terms of two sectors of Abelian gauge field theories. Costa and Natário [33] collected and further developed different gravitoelectromagnetic analogies existing in literature and clarified the connection between them.

2. **Threading formalism and gravitoelectromagnetism force**

The slicing and threading points of view are introduced respectively by Misner, Thorne and Wheeler [34] in 1973 and, Landau and Lifshitz [35] in 1975. Both points of view can be traced back when the Landau and Lifshitz [36] in 1941 introduced the threading point of view splitting of the spacetime metric. After them, Lichnerowicz [37] introduced the beginnings of the slicing point of view. In 1956, Zel'manov [38] discussed the splitting of the Einstein field equations in general case. The slicing point of view is commonly referred as 3+1 or ADM formalism and also term 1+3 formalism has been suggested for the threading point of view. For more details about these formalisms, see reference [39]. In threading point of view, splitting of spacetime is introduced by a family of timelike congruences with unit tangent vector field, may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in the tangent space. Let[3] $(M, \mathrm{g}_{\alpha\beta})$ be a 4-dimensional manifold of a stationary spacetime. We now can construct a 3-dimensional orbit manifold as $\bar{M} = \frac{M}{G}$ with projected metric tensor γ_{ij} by the smooth map $\varrho : M \to \bar{M}$ where $\varrho(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in M$ and G is 1-dimensional group of transformations generated by the timelike Killing vector of the spacetime under consideration, [39, 40]. This splitting of the spacetime leads to the following distance element, [35,40]:

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta = h(dt - \mathrm{g}_i dx^i)^2 - \gamma_{ij} dx^i dx^j, \tag{2.1}$$

where $\gamma_{ij} = -\mathrm{g}_{ij} + h\mathrm{g}_i\mathrm{g}_j$ in which $\mathrm{g}_i = -\dfrac{\mathrm{g}_{0i}}{h}$ and $h = \mathrm{g}_{00}$. We now consider a moving test particle of mass m in a spacetime with the metric tensor (1). The gravitoelectromagnetism force acting on this particle, as measured by the threading observers, is described by the following equation[4], [38, 41]:

$$^*\mathbf{F} = \frac{^*d^*\mathbf{p}}{dt} - \frac{m}{\sqrt{1 - {}^*v^2}} \left\{ ^*\mathbf{E}_g + {}^*v \times {}^*\mathbf{B}_g + \mathbf{f} \right\}, \tag{2.2}$$

here $^*\mathrm{p}^i = \dfrac{m^*v^i}{\sqrt{1-{}^*v^2}}$ such that $^*v^2 = \gamma_{ij}{}^*v^i\,{}^*v^j$ in which $^*v^i = \dfrac{v^i}{\sqrt{h}\,(1 - \mathrm{g}_k v^k)}$ and $v^i = \dfrac{dx^i}{dt}$. Also, the starry total derivative with respect to time is defined as $\dfrac{^*d}{dt} = \dfrac{^*\partial}{\partial t} + {}^*v^i\,{}^*\partial_i$ where $\dfrac{^*\partial}{\partial t} = \dfrac{1}{\sqrt{h}}\dfrac{\partial}{\partial t}$ and $_{*i} = {}^*\partial_i = \partial_i + \mathrm{g}_i \dfrac{\partial}{\partial t}$. In equation (2), the last term is defined as $f^i = -({}^*\lambda^i_{jk}{}^*v^j + 2\mathrm{D}^i_k){}^*v^k$, where 3-dimensional starry Christoffel symbols are defined as $^*\lambda^i_{jk} = \dfrac{1}{2}\gamma^{il}(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l})$. The deformation rates of the reference frame with the respect to observer are represented by tensors $\mathrm{D}_{ij} = \dfrac{1}{2}\dfrac{^*\partial\gamma_{ij}}{\partial t}$ and $\mathrm{D}^{ij} = -\dfrac{1}{2}\dfrac{^*\partial\gamma^{ij}}{\partial t}$. Finally, the time dependent gravitoelectromagnetism fields are defined in terms of gravoelectric potential $\Phi = \ln\sqrt{h}$ and gravomagnetic vector

[3]The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3.

[4]The vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ has components as $\mathrm{C}^i = \dfrac{\varepsilon^{ijk}}{\sqrt{\gamma}}\mathrm{A}_j\mathrm{B}_k$ in which $\gamma = \det(\gamma_{ij})$ and 3-dimensional Levi-Civita tensor ε_{ijk} is skew-symmetric in any exchange of indices while $\varepsilon^{123} = \varepsilon_{123} = 1$, [35].

potential $\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3)$ as follows[5]

$$^*\mathbf{E}_g = -^*\nabla\Phi - \frac{\partial \mathrm{g}}{\partial t} \;;\; ^*\mathrm{E}_{gi} = -\Phi_{*i} - \frac{\partial \mathrm{g}_i}{\partial t}, \tag{2.3}$$

$$\frac{^*\mathbf{B}_g}{\sqrt{h}} = {}^*\nabla \times \mathrm{g} \;;\; \frac{^*\mathrm{B}^i_g}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\,\mathrm{g}_{[k*j]}. \tag{2.4}$$

3. Motion of a test particle in cylindrically symmetric spacetime

We start with the general form of a static metric in cylindrical Weyl coordinates (t, r, ϕ, z) given by, [42]:

$$ds^2 = e^{-2v}dt^2 - e^{-2v}dr^2 - w^2 e^{-2v}d\phi^2 - e^{2v}dz^2, \tag{3.1}$$

where v and w are functions of r. Firstly, we solve the Einstein field equations for this metric and so, the following results are obtained

$$v = a\ln(c_1 r + c_2) + b, \;\; w = c_1 r + c_2, \tag{3.2}$$

where a, b, c_1, c_2 are arbitrary constants. Below we calculate the trajectory of a relativistic test particle with mass m that moving in spacetime (5) by using the Hamilton-Jacobi equation, [43-46]. Therefore, this equation is of the form

$$\left(\frac{\partial \mathrm{S}}{\partial t}\right)^2 - \left(\frac{\partial \mathrm{S}}{\partial r}\right)^2 - \frac{1}{(c_1 r + c_2)^2}\left(\frac{\partial \mathrm{S}}{\partial \phi}\right)^2 - \frac{e^{-4b}}{(c_1 r + c_2)^{4a}}\left(\frac{\partial \mathrm{S}}{\partial z}\right)^2 - \frac{m^2 e^{-2b}}{(c_1 r + c_2)^{2a}} = 0 \tag{3.3}$$

For solving this partial differential equation, we use the method of separation of variables for the Hamilton-Jacobi function as follows

$$\mathrm{S}(t, r, \phi, z) = -Et + \mathrm{R}(r) + p_1\phi + p_2 z, \tag{3.4}$$

here[6] E, p_1 and p_2 are arbitrary constants and can be identified respectively as energy and angular momentum components of particle along ϕ and z-directions. With substituting the last relation into Hamilton-Jacobi equation, the unknown function R is given by

$$\mathrm{R} = \epsilon \int \sqrt{E^2 - \sigma}\, dr, \tag{3.5}$$

where $\sigma = (c_1 r + c_2)^{-2}p_1^2 + e^{-2b}(c_1 r + c_2)^{-2a}m^2 + e^{-4b}(c_1 r + c_2)^{-4a}p_2^2$ and $\epsilon = \pm 1$ stands for the sign changing whenever r passes through a zero of the above integrand, [47]. Let us now obtain the trajectory of particle by considering the following relations, [43-46]:

$$\frac{\partial \mathrm{S}}{\partial E} = \text{constant}, \quad \frac{\partial \mathrm{S}}{\partial p_1} = \text{constant}, \quad \frac{\partial \mathrm{S}}{\partial p_2} = \text{constant}. \tag{3.6}$$

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[5]Here, curl of an arbitrary vector in a 3-space with metric γ_{ij} is defined by $(^*\nabla\times\mathbf{A})^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{A}_{[k*j]}$ while the symbol [ ] represents the anticommutation over indices.

[6]In the next section, we will show that the energy of particle is equal to E.

Consequently, after some calculations, the set of equations (10) respectively are changed to the following relations

$$t = \epsilon E \int \frac{dr}{\sqrt{E^2 - \sigma}}, \tag{3.7}$$

$$\phi = \epsilon p_1 \int \frac{dr}{(c_1 r + c_2)^2 \sqrt{E^2 - \sigma}}, \tag{3.8}$$

$$z = \epsilon p_2 e^{-4b} \int \frac{dr}{(c_1 r + c_2)^{4a} \sqrt{E^2 - \sigma}}, \tag{3.9}$$

we have taken the constants in equations (10) to be zero without any loss of generality. Obviously, it does not seem to be an easy task to find the general solutions for the above integrals. On the other hand, we know that the turning points of trajectory are given by $\frac{dr}{dt} = 0$. As a consequence, the potential curves are

$$E = \pm\sqrt{\sigma}. \tag{3.10}$$

Next, calculations show that we can express the turning points explicity from the above equation only for values[7] $a = \frac{1}{2}, 1$. Here we discuss these two cases as follows:

Case (1) : $a = \frac{1}{2}$

By defining a new variable as[8] $u = \frac{1}{r}$, the equation (12) is transformed to

$$\frac{p_1^2}{(c_1 + c_2 u)^4} \left(\frac{du}{d\phi}\right)^2 + \frac{(p_1^2 + p_2^2 e^{-4b}) u^2}{(c_1 + c_2 u)^2} + \frac{m^2 e^{-2b} u}{c_1 + c_2 u} - E^2 = 0. \tag{3.11}$$

Unfortunately only an integral expression as $\phi = \phi(u)$ can be obtained from this equation. However, the explicit expressions for the solutions of equation (15) can be obtained if we take $c_2 = 0$. For this case, after some calculations, we get

$$u(\phi) = \alpha - \beta \sin\left(\frac{c_1\sqrt{p_1^2 + p_2^2 e^{-4b}}}{p_1}(\phi - \phi_0)\right), \tag{3.12}$$

where ϕ_0 is a constant of integration and

$$\alpha = -\frac{c_1 m^2 e^{-2b}}{2(p_1^2 + p_2^2 e^{-4b})}, \tag{3.13}$$

$$\beta = -\frac{c_1\sqrt{m^4 e^{-4b} + 4E^2(p_1^2 + p_2^2 e^{-4b})}}{2(p_1^2 + p_2^2 e^{-4b})}. \tag{3.14}$$

There are also two constant solutions obtained by setting $\frac{du}{d\phi} = 0$ in equation (15) as

$$r_0 = \frac{1}{\alpha \pm \beta}, \tag{3.15}$$

where r_0 is the radius of the circular orbit. This circular motion occurs in relativity theory just as in classical theory. In relativistic classical mechanics the finite trajectories are, in general, not closed but rather rosette shaped, [35]. From the equation

[7] Most of the calculations were done using Maple software.

[8] See reference [48], for more discussion.

(16), we can see that the trajectory of particle is bounded, i.e. the particle can be trapped by the extended object with the static cylindrically symmetric geometry. In this case, the turning points of the trajectory are[9] $r_{tp} = r_0$. For the trajectories with the equation (16), the period of motion is given by

$$\mathrm{T} = \frac{2\pi p_1}{c_1\sqrt{p_1^2 + p_2^2 e^{-4b}}}, \tag{3.16}$$

which depends explicitly on the angular momentum components of particle. Finally, it can be shown that the exact solution of equation (11) is of the form

$$\begin{aligned} &\alpha \ln\left(\alpha + (\beta^2-\alpha^2)r + \sqrt{\beta^2-\alpha^2}\sqrt{(\beta^2-\alpha^2)r^2 + 2\alpha r - 1}\right) \\ &-\sqrt{\beta^2-\alpha^2}\sqrt{(\beta^2-\alpha^2)r^2 + 2\alpha r - 1} + (\beta^2-\alpha^2)t = 0. \end{aligned} \tag{3.17}$$

Unfortunately, we can not solve exactly the above equation in order to obtain the coordinate r in terms of t. Hence, from the equations (13) and (16), we can not determine the coordinates z and ϕ explicity in terms of t.

Case (2) : $a = 1$

In this case, with the help of variable u, equation (12) converts to

$$\frac{p_1^2}{(c_1+c_2u)^4}\left(\frac{du}{d\phi}\right)^2 + \frac{p_2^2 e^{-4b}u^4}{(c_1+c_2u)^4} + \frac{(p_1^2 + m^2e^{-2b})u^2}{(c_1+c_2u)^2} - E^2 = 0. \tag{3.18}$$

Similarly as in previous case, the above equation can be solved exactly only for $c_2 = 0$, and it can be shown that its solution is[10]

$$u(\phi) = \frac{1}{\zeta}\,\mathrm{JacobiSN}\left(\frac{c_1^2 E\zeta}{p_1}(\phi - \varphi_0)\,, \frac{i\delta}{\sqrt{2}}\right), \tag{3.19}$$

in which $i = \sqrt{-1}$, φ_0 is a constant of integration and

$$\zeta = \frac{\sqrt{p_1^2+m^2e^{-2b}+\sqrt{(p_1^2+m^2e^{-2b})^2+4E^2p_2^2e^{-4b}}}}{c_1E\sqrt{2}}, \tag{3.20}$$

$$\delta = \frac{c_1 E p_2 e^{-2b}}{\sqrt{(p_1^2 + m^2e^{-2b})c_1^2\zeta^2 + p_2^2 e^{-4b}}}. \tag{3.21}$$

Also, if $|E| > \frac{\sqrt{p_1^2+m^2e^{-2b}}}{c_1\zeta}$, the constant solutions of the equation (22) become the circular orbits with the following radius

$$r_0 = \pm\frac{p_2 e^{-2b}}{c_1\sqrt{c_1^2E^2\zeta^2 - p_1^2 - m^2e^{-2b}}}. \tag{3.22}$$

On the other hand, it is easy to show that the following identity is valid

$$\mathrm{JacobiSN}(x, 0) = \sin x, \tag{3.23}$$

here x is an arbitrary variable. By considering the last consequence, it is interesting to rewrite the equation (23) with $\delta = 0$. For doing this, there is only one choice,

[9]The subscript tp in r_{tp} refer to the turning points.

[10]For more details about Jacobi functions, see references [49,50].

i.e. $p_2 = 0$. Therefore, we shall henceforth ignore the motion in the z-direction. By applying this condition, equation (23) is reduced to the following simple relation

$$\frac{1}{r} = \frac{c_1 E}{\sqrt{p_1^2 + m^2 e^{-2b}}} \sin\left(\frac{c_1\sqrt{p_1^2 + m^2 e^{-2b}}}{p_1}(\phi - \varphi_0)\right), \tag{3.24}$$

and so, the period of motion becomes

$$\mathrm{T} = \frac{2\pi p_1}{c_1\sqrt{p_1^2 + m^2 e^{-2b}}}. \tag{3.25}$$

Also, we can check that the trajectories described by equation (28) will be bounded with the turning points as $r_{tp} = \pm\frac{\sqrt{p_1^2+m^2e^{-2b}}}{c_1 E}$. Finally, with the help of equations (11) and (28), we can rewrite the trajectory of test particle in terms of time as follows

$$r = \sqrt{t^2 + \frac{p_1^2+m^2e^{-2b}}{c_1^2E^2}}, \tag{3.26}$$

and

$$\phi = \varphi_0 + \frac{\mathrm{T}}{2\pi}\operatorname{arccot}\left(\frac{c_1 E t}{\sqrt{p_1^2 + m^2 e^{-2b}}}\right). \tag{3.27}$$

4. **Calculation of the gravitoelectromagnetism force**

In this section, we will calculate the gravitoelectromagnetism force acting on a test particle in the static cylindrically symmetric spacetime. But, in case (1), it was shown that the exact determination of coordinates (r, ϕ, z) in terms of t is impossible. Hence, we ignore the study of this case. Next, from the equations (30) and (31) in case (2), we deduce

$$^*v^i = e^b \begin{cases} c_1 t & i = 1, \\ \frac{p_1}{\sqrt{c_1^2E^2t^2+p_1^2+m^2e^{-2b}}} & i = 2, \\ 0 & i = 3. \end{cases} \tag{4.1}$$

With applying this relation, after some work, we lead to

$$\frac{m}{\sqrt{1 - {}^*v^2}} = e^b\sqrt{c_1^2E^2t^2 + p_1^2 + m^2e^{-2b}}. \tag{4.2}$$

Therefore, it follows that

$$^*\mathbf{p} = e^{2b}\left(c_1 t\sqrt{c_1^2E^2t^2 + p_1^2 + m^2e^{-2b}}\,,\, p_1\,,0\right). \tag{4.3}$$

Before continuing, we know that the energy of a particle as measured by the threading observers located at trajectory is given by $\mathcal{E} = \frac{m\sqrt{h}}{\sqrt{1-{}^*v^2}}$ which is a conserved quantity during the motion of particle, [35]. Thus, from equation (33) we can conclude $\mathcal{E} = E$. Next, we can verify that all components of gravitoelectromagnetism fields are vanish except[11]

$$ {}^{*}\mathrm{E}_{g1} = \frac{c_1 E}{\sqrt{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}}}, \tag{4.4} $$

$$ {}^{*}\mathrm{E}_{g2} = -\frac{c_1^2 E t}{p_1}. \tag{4.5} $$

Also, the non-zero components of starry Christoffel symbols are calculated as follows

$$ \begin{aligned} {}^{*}\lambda^1_{11} &= -{}^{*}\lambda^3_{13} = -\frac{c_1 E}{\sqrt{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}}}, \\ {}^{*}\lambda^1_{12} &= -{}^{*}\lambda^3_{23} = \frac{c_1^2 E t}{p_1}, \\ {}^{*}\lambda^1_{33} &= -\frac{c_1 e^{4b}\sqrt{(c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b})^3}}{E^3}, \\ {}^{*}\lambda^2_{11} &= -\frac{c_1^2 E^3 t}{(c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b})p_1}, \\ {}^{*}\lambda^2_{33} &= \frac{c_1^2 e^{4b}(c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b})t}{p_1 E}, \end{aligned} \tag{4.6} $$

and the non-zero components of deformation tensor are

$$ \mathrm{D}_{11} = -\frac{c_1^2 e^{-b} E^3 t}{\sqrt{(c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b})^3}}, \tag{4.7} $$

$$ \mathrm{D}_{33} = \frac{c_1^2 e^{3b} t \sqrt{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}}}{E}. \tag{4.8} $$

From the equations (32), (34) and (37-39), we derive the following expressions

$$ f^i = \frac{1}{p_1}\begin{cases} \frac{2c_1^3 e^{2b} p_1 E t^2}{\sqrt{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}}} & i = 1, \\ \frac{c_1^4 e^{2b} E^3 t^3}{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}} & i = 2, \\ 0 & i = 3. \end{cases} \tag{4.9} $$

and

$$ \frac{{}^{*}d^{*}\mathrm{p}^i}{dt} = \frac{c_1 e^{3b}}{E}\begin{cases} 2c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b} & i = 1, \\ 0 & i \neq 1. \end{cases} \tag{4.10} $$

Finally, from the equations (33-36), (40) and (41), we obtain

$$ {}^{*}\mathbf{F} = c_1^3 e^{3b} E\left(-t^2, \frac{(p_1^2 + m^2 e^{-2b})t}{c_1 p_1 \sqrt{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}}}, 0\right). \tag{4.11} $$

To continue our analysis, we are going to determine the potential function corresponding to the gravitoelectromagnetism force. For doing this, we can prove that in

[11] In our notation $(r, \phi, z) \equiv (1, 2, 3)$.

a 3-space with the metric tensor γ_{ij} the following identity is valid, [51,52]:

$$[{}^*\partial_i\,, {}^*\partial_k] = \sqrt{h}\, \mathrm{g}_{[k*i]} \frac{{}^*\partial}{\partial t}. \tag{4.12}$$

With the help of this identity and equation (4), we can obtain the following identity[12]

$${}^*\nabla \times {}^*\nabla\Psi = \frac{{}^*\partial\Psi}{\partial t}\, {}^*\mathbf{B}_g, \tag{4.13}$$

where Ψ is an arbitrary scalar function. But the gravitomagnetic fields for the metric (5) are zero, so the above relation is changed to

$${}^*\nabla \times {}^*\nabla\Psi = 0. \tag{4.14}$$

On the other hand, by taking curl of the gravitoelectromagnetism force, we get

$$\begin{aligned} ({}^*\nabla \times {}^*\mathbf{F})^i &= \frac{1}{p_1} \begin{cases} \frac{3c_1^3 e^{2b} E^2 (p_1^2 + m^2 e^{-2b}) t}{c_1^2 E^2 t^2 + p_1^2 + m^2 e^{-2b}} & i = 3, \\ 0 & i \neq 3. \end{cases} \\ &\neq 0. \end{aligned} \tag{4.15}$$

Consequently, comparing the equation (45) with the equation (46), we cannot define the potential function (V) corresponding to the gravitoelectromagnetism force with the following classical form

$${}^*\mathbf{F} = -{}^*\nabla \mathrm{V}. \tag{4.16}$$

5. Conclusions

In this paper, the classical behaviour of a test particle in cylindrically symmetric spacetime has been studied. We showed that the particle can be trapped by this gravitational field. The trajectory, turning points and period of motion have been calculated. Also, we showed that the parameters of motion depend on the choice of the angular momentum components of particle. Finally, by determining the gravitoelectromagnetism force, it was shown that the existence of the corresponding potential function with the classical definition is impossible.

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