BENDING OF A EULER-BERNOULLI CRACKED BEAM USING NONLOCAL STRAIN GRADIENT THEORY

Chao FU[1], Xiao YANG[2]

Based on nonlocal strain gradient theory, the bending behaviors of the cracked microbeams are studied. The expression of the higher-order bending moment is established, and the corresponding non-classical boundary conditions are obtained. Then, the general analytical expressions for the bending deformation of a simply-supported Euler-Bernoulli cracked microbeam subjected to a uniform load with two forms of boundary conditions are presented. Numerical results show that the influence of the material length scale parameter on the crack effect of the microbeam bending is of great significance, while that of the nonlocal parameter is not decisive.

Keywords: nonlocal strain gradient theory; flexibility crack model; scale parameter; crack effect; higher-order boundary condition.

1. Introduction

Micro/nano-scale systems and devices are receiving extensive attention and widespread applications in many engineering fields [1,2]. Due to the uncertainty of experimental environment and expensive computational costs of numerical simulation, continuum mechanics theory is more appropriate to predict material behaviors of the small scaled structures [3]. However, local continuum theory has been verified inadequately to capture the size-dependent effect of the material properties, i.e., Young's modulus and bending rigidity [4]. In order to overcome these deficiencies, various non-classical continuum models, such as nonlocal elasticity theory [5] and strain gradient theory [6-8], have been successfully developed and employed [9]. However, the nonlocal elasticity theory can only predict the softening effect, while the strain gradient model generally shows the stiffening effect. By comparing with these two different properties, Lim et al. [3] presented nonlocal strain gradient theory with two scale parameters, named as the nonlocal parameter and material length scale parameter.

With the order increase of the differential governing equations in the non-classical models, it is needed to pay more attention to the corresponding boundary conditions of the non-classical models [10]. Based on nonlocal strain gradient

[1] Dr., College of Architecture and Civil Engineering, Xinyang Normal University, China, e-mail: jerryfc@shu.edu.cn
[2] Prof., Department of Civil Engineering, Shanghai University, China, e-mail: xyang@shu.edu.cn

theory, Li and Hu [11], Li et al. [12], and Lu et al. [13] established the equilibrium equations and corresponding boundary conditions of the beam using Hamilton principle, and analyzed the bending deformation, buckling and free vibration with the higher-order displacement boundary conditions. Xu et al. [10] presented the new variational-consistent boundary conditions by using the weighted residual method and developed the higher-order boundary conditions related to the classical stress resultants.

Up to now, a lot of great achievements of the static and dynamic behaviors of the intact micro/nano-scale beam have been gained. Comparatively, only a few works have focused on the behaviors of the cracked micro/nano-scale beams. The classical analytical approach treats the cracked beam as a system of two intact segments connected by an equivalent massless spring (i.e. rotational spring [14], torsional spring [15], or extensional spring [16]) located at the cracked section. Additionally, the computational costs increase with the number of cracks, and this approach is mainly used to study the vibration of the size-dependent cracked beam. Recently, Donà et al. [17] presented an exact closed-form bending deformations of multi-cracked beams in small scale by employing a mixed stress/strain gradient model and the flexibility crack model with an equivalent rotational spring assumption. Yang et al. [18] derived a similar crack model with considering the discontinuity of the rotation, but this model is limited to the classical elasticity problems. Moreover, these works are mainly focused on the vibration analysis, and the study of the static bending behaviors of the cracked beams due to the size effect is still very limited.

In this paper, the static bending behaviors of a cracked Euler-Bernoulli microbeam are investigated using nonlocal strain gradient theory and the flexibility crack model. The non-classical boundary conditions of the higher-order bending moment are derived first. Then, the general analytical expressions of the bending deformation of a simple-supported cracked microbeam are obtained with two forms of the boundary conditions. By some numerical examples, the influences of the nonlocal parameter and material length scale parameter, and the crack effect on the bending behaviors are examined in detail.

2. Equilibrium equations and boundary conditions

2.1. Nonlocal strain gradient theory

Based on nonlocal strain gradient theory [3], the total stress t_{xx} on the cross-section of the Euler-Bernoulli beam is defined as

$$t_{xx} = \sigma_{xx} - \nabla \sigma_{xx}^{(1)}. \tag{1}$$

where ∇ is the gradient symbol, for one-dimension problem, $\nabla = \mathrm{d}/\mathrm{d}x$. σ_{xx} and $\sigma_{xx}^{(1)}$ denote the classical stress and the higher-order stress, respectively.

Suppose that the proper transform conditions [5] are satisfied, the differential equations are presented as

$$\left[1-(e_0a)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\sigma_{xx}=E\varepsilon_{xx},\ \left[1-(e_1a)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\sigma_{xx}^{(1)}=El^2\frac{\mathrm{d}\varepsilon_{xx}}{\mathrm{d}x}. \tag{2}$$

where E is the Young's modulus, l is the material length scale parameter introduced to determine the effect of higher-order strain gradient. a is the internal characteristic length, such as granular distance and lattice parameter. e_0 and e_1 are the nonlocal parameters related to the classical strain and the higher-order strain gradient, respectively.

By assuming $e_0=e_1=e$, (2) is rewritten as

$$\left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]t_{xx}=E\left(1-l^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\varepsilon_{xx}. \tag{3}$$

Herein, the nonlocal parameter ea and material length scale parameter l are two kinds of scale parameters to account for the size-dependent effect, respectively.

2.2. Governing equations

We consider a micro/nano-scale rectangular elastic beam with length L (x axis), width b (y axis) and thickness h (z axis), subjected to the distributed transverse load $q(x)$. According to the hypothesis of Euler-Bernoulli beam theory, the displacements u_1, u_2 and u_3 along x, y and z directions, respectively, at a reference point (x,y,z) can be expressed as

$$u_1(x,y,z)=-z\frac{\mathrm{d}w(x)}{\mathrm{d}x},\ \ u_2(x,y,z)=0,\ \ u_3(x,y,z)=w(x). \tag{4}$$

where $w(x)$ denotes the transverse deflection of the beam's axial line.

The longitudinal strain $\varepsilon_{xx}(x,y,z)$ of the beam is given as

$$\varepsilon_{xx}(x,y,z)=-z\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}=-z\chi(x). \tag{5}$$

where $\chi(x)=\mathrm{d}^2w(x)/\mathrm{d}x^2$ is the axial line curvature of the deformed beam.

The virtual strain energy of the size-dependent beam is given as

$$\delta U=\int_V\left(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xx}^{(1)}\nabla\delta\varepsilon_{xx}\right)\mathrm{d}V$$
$$=-\left[M^{(1)}(x)\delta\frac{\partial^2w(x)}{\partial x^2}\right]_0^L-\left[M(x)\delta\frac{\partial w(x)}{\partial x}\right]_0^L+\left[\frac{\partial M(x)}{\partial x}\delta w(x)\right]_0^L-\int_0^L\frac{\partial^2M(x)}{\partial x^2}\delta w(x)\mathrm{d}x. \tag{6}$$

where δ is the variational operator; The total bending moment $M(x)$ and higher-order bending moment $M^{(1)}(x)$ are defined as follows

$$M(x)=\int_A zt_{xx}\mathrm{d}A,\ \ M^{(1)}(x)=\int_A z\sigma_{xx}^{(1)}\mathrm{d}A. \tag{7}$$

The virtual work of the external force is obtained as

$$\delta W=\int_0^L q(x)\delta w(x)\mathrm{d}x. \tag{8}$$

According to the principle of virtual work, i.e., $\delta W-\delta U=0$, and by employing (6) and (8), the equilibrium equation and boundary conditions of a Euler-Bernoulli intact beam are given as follows

$$\frac{\mathrm{d}^2M(x)}{\mathrm{d}x^2}+q(x)=0. \tag{9}$$

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x}=0\ \text{ or }\ \delta w(x)=0,\ \ M(x)=0\ \text{ or }\ \delta\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right)=0. \tag{10}$$

$$M^{(1)}(x)=0\ \text{ or }\ \delta\left(\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}\right)=0. \tag{11}$$

where (10) and (11) are viewed as the classical boundary conditions and non-classical boundary conditions, respectively [12,19].

By multiplying variable z on both sides of (3) and utilizing integration with respect to the cross section A, the bending moment with size effect can be obtained as

$$\left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]M(x)=-(EI)_0\left(1-l^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}. \tag{12}$$

where $(EI)_0$ is the flexural rigidity of the intact beam.

2.3. Non-classical boundary condition

Generally, it is difficult to derive the explicit expression of the higher-order bending moment $M^{(1)}(x)$. However, motivated by Li et al. [19], the explicit expression of the higher-order bending moment $M^{(1)}(x)$ and the corresponding non-classical boundary conditions are presented as follows.

Substituting (5) into (2) leads to

$$\left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\sigma_{xx}=-Ez\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2},\ \ \left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]\sigma_{xx}^{(1)}=-El^2z\frac{\mathrm{d}^3w(x)}{\mathrm{d}x^3}. \tag{13}$$

Multiplying variable z on both sides of (1) and (13), respectively, and utilizing integration with respect to the cross section A， it gets

$$M(x)=M^{(0)}(x)-\frac{\mathrm{d}M^{(1)}(x)}{\mathrm{d}x}. \tag{14}$$

$$\left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]M^{(0)}(x)=-(EI)_0\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2},\ \ \left[1-(ea)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]M^{(1)}(x)=-l^2(EI)_0\frac{\mathrm{d}^3w(x)}{\mathrm{d}x^3}. \tag{15}$$

where $M^{(0)}(x)$ is the lower-order bending moment defined as $M^{(0)}(x)=\int_A z\sigma_{xx}\mathrm{d}A$.

From (15), one gets

$$M^{(1)}(x) = l^2 \frac{\mathrm{d}M^{(0)}(x)}{\mathrm{d}x} + A_1 e^{x/ea} + A_2 e^{-x/ea}. \tag{16}$$

Considering $A_1 = A_2 = 0$ as presented by Li et al. [19], (16) can be rewritten as

$$M^{(1)}(x) = l^2 \frac{\mathrm{d}M^{(0)}(x)}{\mathrm{d}x}. \tag{17}$$

Substituting (17) into (14), one obtains

$$\frac{\mathrm{d}^2 M^{(0)}(x)}{\mathrm{d}x^2} = \frac{M^{(0)}(x) - M(x)}{l^2}. \tag{18}$$

Substituting (18) into the first equation of (15), it gets

$$M^{(0)}(x) - (ea)^2 \frac{M^{(0)}(x) - M(x)}{l^2} = -(EI)_0 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}. \tag{19}$$

Combining (9) and (12), the total bending moment $M(x)$ is given as

$$M(x) = -(ea)^2 q(x) - (EI)_0 \left(1 - l^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}. \tag{20}$$

It is verified that the expression of the bending moment derived by Li and Hu [20] is exactly identical to (20) when the kinetic energy and the second-order effect of the longitudinal deformation are neglected.

Combining (17), (19) and (20), the expression of the lower-order bending moment and that of higher-order bending moment are obtained, respectively

$$M^{(0)}(x) = -\frac{(ea)^4 q(x)}{(ea)^2 - l^2} - (EI)_0 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} + \frac{l^2 (ea)^2}{(ea)^2 - l^2} (EI)_0 \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4}. \tag{21}$$

$$M^{(1)}(x) = -l^2 (EI)_0 \frac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3} - \frac{l^2 (ea)^4}{(ea)^2 - l^2} \frac{\mathrm{d}q(x)}{\mathrm{d}x} + \frac{l^4 (ea)^2}{(ea)^2 - l^2} (EI)_0 \frac{\mathrm{d}^5 w(x)}{\mathrm{d}x^5}. \tag{22}$$

By referring to the way of the boundary condition choice as proposed by Li et al. [19] to study the longitudinal vibration of the nanobeam, (22) can be used to solve the beam bending problems as one possible non-classical boundary conditions.

In the case of a simply supported beam with size effect, the boundary conditions at $x = 0$ and $x = L$ can be given in two forms, defined as Case I and Case II, respectively

Case I:

$$w(x) = 0, \quad M(x) = 0, \quad M^{(1)}(x) = 0. \tag{23}$$

Case II:

$$w(x) = 0, \quad M(x) = 0, \quad \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = 0. \tag{24}$$

The former which contains the non-classical boundary conditions (22) by incorporating the classical boundary conditions (10) is proposed in this present paper, while the latter takes another non-classical boundary conditions as shown in the references [10,12,19].

3. Bending deformation of a simple-supported cracked beam

In the following the bending of the cracked beam subjected to a uniform load $q(x)=Q_0$ will be solved. Assuming that cracks exist at the location $x=x_i$ $(i=1,2,\cdots,N)$ and the cracks are always open. The flexibility crack model [18] is employed, and the equivalent flexural rigidity $(EI)_{\mathrm{EQ}}$ of the cracked beam can be expressed as

$$\frac{1}{(EI)_{\mathrm{EQ}}}=\frac{1}{(EI)_0}+\sum_{i=1}^{N}\frac{1}{k_i}\delta(x-x_i). \tag{25}$$

where $\delta(x)$ is the Dirac's delta function, and k_i is the equivalent spring rigidity associated with the crack severity which should be determined by either *ab initio* studies or molecular dynamics calculations [14].

From (9), it gets

$$M(x)=-q^{[2]}(x)-C_1x-C_2. \tag{26}$$

in which C_1 and C_2 are the undetermined constants, and the function $q^{[i]}(x)$ is defined as

$$q^{[i]}(x)=\underbrace{\int_0^{\xi}\cdots\int_0^{\xi}q(x)\mathrm{d}\xi\cdots\mathrm{d}\xi}_{i}. \tag{27}$$

For a cracked beam, (12) should be revised as

$$M(x)-(ea)^2\frac{\mathrm{d}^2M(x)}{\mathrm{d}x^2}=-(EI)_{\mathrm{EQ}}\left[\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}-l^2\frac{\mathrm{d}^4w(x)}{\mathrm{d}x^4}\right]. \tag{28}$$

By employing (25), (26) and (28), one obtains

$$\left(\frac{1}{(EI)_0}+\sum_{\mathrm{i}=1}^{\mathrm{N}}\frac{1}{k_{\mathrm{i}}}\delta(x-x_{\mathrm{i}})\right)\left[(ea)^2q(x)-q^{[2]}(x)-C_1x-C_2\right]=-\left[\frac{\mathrm{d}^2w(x)}{\mathrm{d}x^2}-l^2\frac{\mathrm{d}^4w(x)}{\mathrm{d}x^4}\right]. \tag{29}$$

Introducing the dimensionless variables and parameters, it gets

$$\begin{cases} w^*=\dfrac{w}{L},\quad \xi=\dfrac{x}{L},\quad \xi_i=\dfrac{x_i}{L},\quad h^*=\dfrac{h}{L},\quad A^*=\dfrac{A}{L^2},\quad l_1=\dfrac{l}{L},\quad l_2=\dfrac{ea}{L},\\ M^*=\dfrac{ML}{(EI)_0},\quad M^{(1)*}=\dfrac{M^{(1)}}{(EI)_0},\quad Q^*=\dfrac{Q_0L^3}{(EI)_0},\quad \beta_i=\dfrac{L}{k_i},\quad \chi^*(\xi)=\dfrac{\mathrm{d}^2w^*(\xi)}{\mathrm{d}\xi^2}. \end{cases} \tag{30}$$

(26) and (29) are converted into the dimensionless equations

$$M^*(\xi)=-Q^{*[2]}-C_1\xi-C_2. \tag{31}$$

$$\frac{\mathrm{d}^2\chi^*(\xi)}{\mathrm{d}\xi^2}-\frac{\chi^*(\xi)}{l_1^2}=\left[1+\sum_{i=1}^{N}\beta_i\delta(\xi-\xi_i)\right]\frac{l_2^2Q^*-Q^{*[2]}-C_1\xi-C_2}{l_1^2}. \tag{32}$$

With employing method of variation of constants, the general solution of (32) is derived as

$$\chi^*(\xi) = D_1 e^{\xi/l_1} + D_2 e^{-\xi/l_1} + Q^*\left(l_1^2 - l_2^2 + \frac{\xi^2}{2}\right) + C_2 + C_1\xi$$
$$+\sum_{i=1}^{N}\frac{\alpha_i}{k_i^*}\left(l_2^2 Q^* - Q^*\frac{\xi_i^2}{2} - C_1\xi_i - C_2\right)\frac{1}{l_1}\sinh\left(\frac{\xi-\xi_i}{l_1}\right)H(\xi-\xi_i). \tag{33}$$

where $H(x)$ is the Heaviside function, D_1 and D_2 are the undetermined constants.

Integrating (33) with respect to the dimensionless variable ξ twice, and utilizing the boundary conditions of (23), the dimensionless deflection and the rotation angle of a simple-supported cracked beam with the boundary conditions Case I are given as

$$\phi(\xi) = Q^*\left(\frac{1}{24} - \frac{\xi^2}{4} + \frac{\xi^3}{6}\right) + \sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left[-H(\xi-\xi_i) + (1-\xi_i)\right]$$
$$+Q^*\left(l_2^2 - l_1^2\right)\left\{\frac{1}{2}\left[\coth\left(\frac{1}{2l_1}\right)\sinh\left(\frac{\xi}{l_1}\right) - \cosh\left(\frac{\xi}{l_1}\right)\right] - \left(\xi - \frac{1}{2}\right)\right\}$$
$$+\sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left\{\cosh\left(\frac{1-\xi_i}{l_1}\right)\operatorname{csch}\left(\frac{1}{l_1}\right)\left[l_1\cosh\left(\frac{1}{l_1}\right) - l_1 - \sinh(\frac{\xi}{l_1})\right]\right.$$
$$\left. + \cosh\left(\frac{\xi-\xi_i}{l_1}\right)H(\xi-\xi_i) - l_1\sinh\left(\frac{1-\xi_i}{l_1}\right)\right\}. \tag{34}$$

$$w^*(\xi) = Q^*\left(\frac{\xi}{24} - \frac{\xi^3}{12} + \frac{\xi^4}{24}\right) + \sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left[(1-\xi_i)\xi - (\xi-\xi_i)H(\xi-\xi_i)\right]$$
$$+\frac{Q^*}{2}\left(l_2^2 - l_1^2\right)\left\{l_1\coth\left(\frac{1}{2l_1}\right)\left[\cosh\left(\frac{\xi}{l_1}\right) - 1\right] - l_1\sinh\left(\frac{\xi}{l_1}\right) - \xi(\xi-1)\right\}$$
$$+\sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left\{\cosh\left(\frac{1-\xi_i}{l_1}\right)\operatorname{csch}\left(\frac{1}{l_1}\right)l_1\left[1 - \cosh\left(\frac{\xi}{l_1}\right) + \xi\cosh\left(\frac{1}{l_1}\right) - \xi\right]\right.$$
$$\left. + l_1\sinh\left(\frac{\xi-\xi_i}{l_1}\right)H(\xi-\xi_i) - \xi l_1\sinh\left(\frac{1-\xi_i}{l_1}\right)\right\}. \tag{35}$$

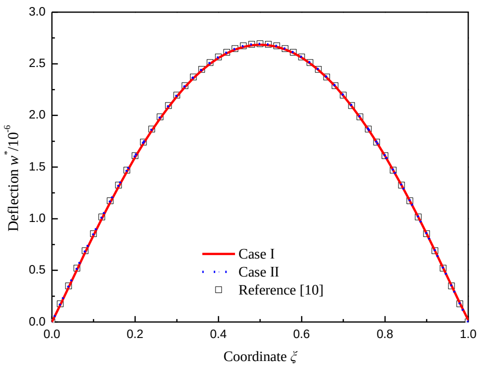
Similarly, the dimensionless deflection and rotation angle of a simple-supported cracked beam with the boundary conditions Case II are presented as

$$\phi(\xi) = Q^*\left(\frac{\xi^3}{6} - \frac{\xi^2}{4} + \frac{1}{24}\right) + \sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left[1 - \xi_i - H(\xi-\xi_i)\right]$$
$$+Q^*\left(l_1^2 - l_2^2\right)\left[\xi - \frac{1}{2} + l_1\sinh\left(\frac{1-2\xi}{2l_1}\right)\operatorname{sech}\left(\frac{1}{2l_1}\right)\right]$$
$$+\sum_{i=1}^{N}\beta_i Q^*\left[l_2^2 + \frac{(1-\xi_i)\xi_i}{2}\right]\left[\cosh\left(\frac{\xi-\xi_i}{l_1}\right)H(\xi-\xi_i) - \cosh\left(\frac{\xi}{l_1}\right)\operatorname{csch}\left(\frac{1}{l_1}\right)\sinh\left(\frac{1-\xi_i}{l_1}\right)\right]. \tag{36}$$

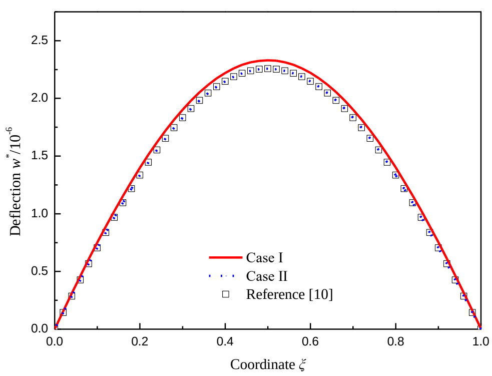
$$w^*(\xi)=Q^*\left(\frac{\xi^4}{24}-\frac{\xi^3}{12}+\frac{\xi}{24}\right)+\sum_{i=1}^{N}\beta_i Q^*\left[l_2^2+\frac{(1-\xi_i)\xi_i}{2}\right]\left[(1-\xi_i)\xi-(\xi-\xi_i)H(\xi-\xi_i)\right]$$
$$+Q^*\left(l_1^2-l_2^2\right)\left[l_1^2-\frac{\xi}{2}+\frac{\xi^2}{2}-l_1^2\cosh\left(\frac{1-2\xi}{2l_1}\right)\mathrm{sech}\left(\frac{1}{2l_1}\right)\right]$$
$$+\sum_{i=1}^{N}\beta_i Q^* l_1\left[l_2^2+\frac{(1-\xi_i)\xi_i}{2}\right]\left[\sinh\left(\frac{\xi-\xi_i}{l_1}\right)H(\xi-\xi_i)-\sinh\left(\frac{\xi}{l_1}\right)\mathrm{csch}\left(\frac{1}{l_1}\right)\sinh\left(\frac{1-\xi_i}{l_1}\right)\right]. \quad (37)$$

4. Numerical examples

The geometric and material parameters of the rectangular microbeam [21] are given as: $h=17.6\ \mu\text{m}$, $b=2h$, and $E=1.44\ \text{GPa}$. The length and two dimensionless scale parameters [3] are taken as: $L/h=20$, $0\le l_1\le 0.2$ and $0\le l_2\le 0.2$. In the following example, a simple-supported microbeam with a single open crack subjected to a uniform loads Q_0 is considered. The crack location, the equivalent spring rigidity associated with the crack severity [14], and the uniform load are given as: $x_1=L/2$, $k_1=2$ and $Q_0=100\ \mu\text{N}$.



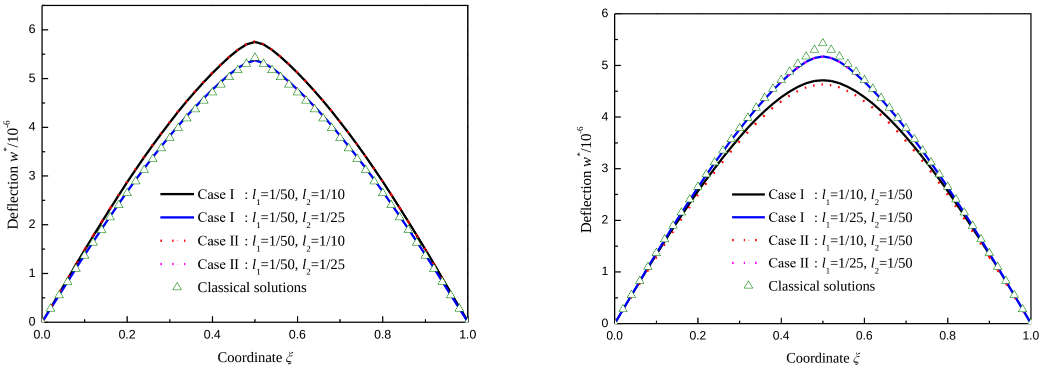
(a) When l_1=1/50, l_2=1/10



(b) When l_1=1/10, l_2=1/50

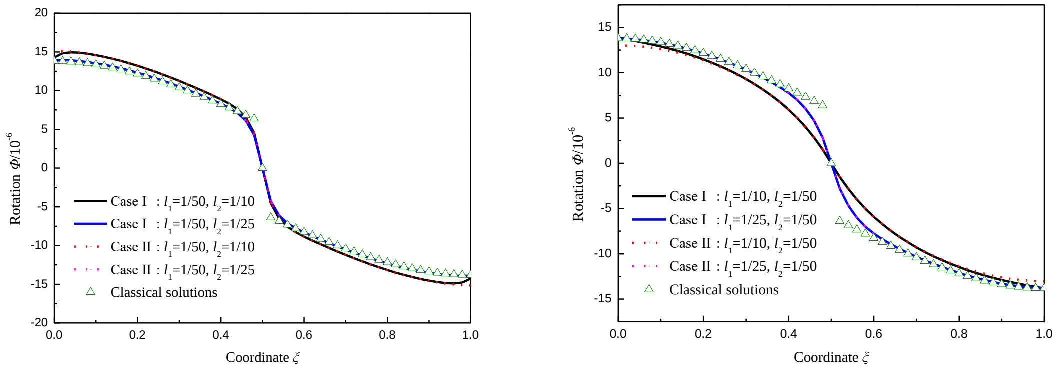
Fig. 1. Distributions of dimensionless deflection $w^*(\xi)$ of the intact beam

In order to verify the numerical results, the bending deflections of an intact beam without crack subjected to the uniform load are given in Figure 1, in which the dimensionless deflections of (35) and (37) with Case I and Case II, respectively, are compared with the results derived with Navier's method [10]. It can be seen that the accuracy of the present solutions is verified to a high degree.



(a) When $l_1 < l_2$ (b) When $l_1 > l_2$

Fig. 2. Distributions of dimensionless deflection $w^*(\xi)$ of the cracked beam

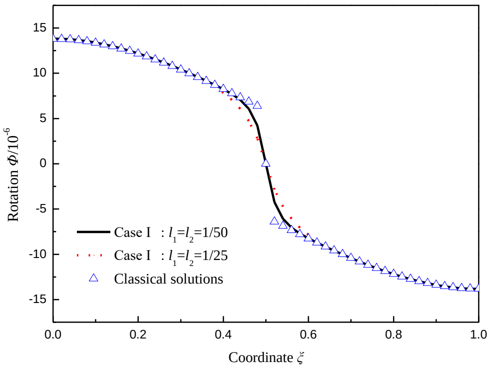


(a) When $l_1 < l_2$ (b) When $l_1 > l_2$

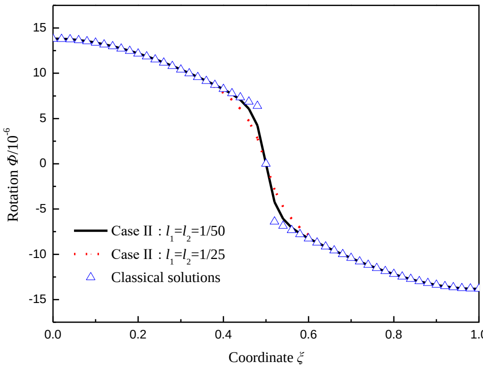
Fig. 3. Distributions of rotation angle $\phi(\xi)$ of the cracked beam

The dimensionless deflections $w^*(\xi)$ and rotation angles $\phi(\xi)$ of the cracked beam with Case I and Case II, respectively, are presented in Figures 2 and 3 for different parameters l_1 and l_2, in which the classical solutions are given by the bending expressions of the classical elasticity cracked beam [18]. It is found that the dimensionless rotation angles with these two different boundary conditions are exactly equal, and the differences of the dimensionless deflection between Case I and Case II is very small, which may be regarded as a proof to verify the correctness of the non-classical boundary conditions derived in this present paper.

Moreover, one may observed that the distributions of the deflection and rotation angle of the cracked beam are rather smooth at the crack location. In other words, the phenomena of deflection cusp and jump of rotation angle existed in the classical elastic theory due to the stiffness varying at the crack location have not appear at all. The possible reasons could be interpreted by taking the rotation angle solutions (36) of Case II as an example: it is clear that the last two terms of (36) not only reveal the influence of the equivalent spring flexibility on the rotation angles of the microbeam, but also play a decisive factor on whether the crack effect does work. When $0 \le l_2 \le 0.2$ and $\xi_1 = 0.5$, it is verified that l_2^2 is much less than $(1-\xi_1)\xi_1/2$, which means that the nonlocal parameter l_2 plays a limited influence on the bending deformation at the crack location. Therefore, ignoring the influence of the nonlocal parameter l_2 (i.e., $l_2=0$), the last two terms of (36) represent the influences of the material length scale parameter l_1 and the classical elasticity theory on the bending deformation at the crack location, respectively. Obviously, if only the influence of the classical elastic theory is considered (i.e., $l_1 = 0$), a jumping phenomenon on the rotation angle curves at the crack location can be observed which indicates the crack effect is significant. While $l_1 \neq 0$, it is proved that the signs of the rotation angle near the crack location from the last two terms of (36) are exactly the opposite, in other words, the material length scale parameter l_1 has a weakening effect on the rotation angle of the classical elastic theory. Therefore, no significant jumping phenomenon is observed. Similar explanations from (37) can also be used to illustrate whether the cusp phenomenon of the deflection curve appears.



(a) Case I



(b) Case II

Fig. 4. Distributions of rotation angle of the cracked beam when $l_1=l_2\neq0$

Fig. 4 shows the rotation angle of the cracked beam with two cases when $l_1 = l_2 \neq 0$, respectively. It is found that in the vicinity of the crack the bending deformations of the nonlocal strain gradient theory are obviously different with those of the classical elastic theory, while away from the crack section, the distributions of these two theories are almost the same. Therefore, it is concluded that the size-dependent effect is more obvious in the vicinity of the crack.

Moreover, when $l_1 = l_2 = 0$, due to (29) is degenerated into the governing equation of the classical elastic cracked beam, the solutions (34)-(35) and (36)-(37) of the nonlocal strain gradient theory with Case I and Case II are degenerated into the same solutions of the classical elastic cracked beam.

5. Conclusions

Based on nonlocal strain gradient theory and the flexibility crack model, the bending deformation of a Euler-Bernoulli micro/nano-scale beam with an open crack is analyzed, and some main findings are listed as follows: (1) The bending behaviors of the nonlocal cracked beams in the vicinity of the crack are only dependent on the material length scale parameter and independent on the nonlocal parameter; (2) When the two scale parameters are equal to a non-zero value, the bending solutions of the cracked beam can't be reduced to those of the classical one, and the differences between them indicate that the size effect on the bending deformation is more obvious in the vicinity of the crack.

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