SIMULATING THE SOLUTION OF THE DISTRIBUTED ORDER FRACTIONAL DIFFERENTIAL EQUATIONS BY BLOCK-PULSE WAVELETS

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In this paper, we introduce methods based on operational matrix of fractional order integration from fixed point (initial value point) for distributed order fractional differential equations (DFDE). We use block-pulse wavelets and hybrid functions matrix of fractional order integration from arbitrary initial point, where a fractional derivative is defined in the Caputo form. By the use of this method we translate a (DFDE) to algebraic linear equations which can be solved then. The proposed method has been tested by some numerical examples.

Keywords: distributed order fractional differential equation, wavelet, block pulse, hybrid function, operational matrices.

1. Introduction

The history of fractional calculus is more than three centuries old; however, only in the last two decades the field has received practical attention and interest; see [1], [2], [3] and [4] for more details on this regard. Fractional calculus is the generalization of calculus, in which the order of derivatives and integrals can be arbitrary numbers. The distributed-order operators can be obtained when we integrate the fractional-order calculus operators with respect to the order variable. The first idea of distributed order differential equation was stated by Caputo in 1969 and later developed by [5] and [6]. These distributed-order differential equations were mainly formed in constitutive equations of dielectric media [7], diffusion equations [8] and the multidimensional random walk models [9]. The interested readers can refer to [10], [11], [12], [13], [14] and [15] for more details. Here and in this paper, we consider the distributed order linear equations of the form

$$\int_a^b \omega(\alpha)\, {}_{t_0}^{c}D_t^{\alpha}\, \mathrm{y}(\mathrm{t})\mathrm{d}\alpha = \mathrm{g(t)},\ t_0 \le t \quad , \tag{1}$$

under initial conditions,

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$$, \quad y^{(i)}(t_0) = y_i, i = 0,1,\dots,\lfloor b \rfloor \tag{2}$$

where ${}_{t_0}^{c}D_t^{\alpha}$ is the α^{th} fractional order derivative of $y(t)$ in Caputo sense from t_0. A recent development of approximation theory is approximation of an arbitrary function by wavelet polynomials. There are different types of wavelet such as block-pulse wavelet, Haar wavelet, Mexican-Hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer's wavelet, and so forth. In this paper, we mainly focus on approximation by block-pulse wavelet and hybrid functions of based on Block-pulse wavelet and Shifted Legendre polynomials. Any time function can be synthesized completely to a tolerable degree of accuracy by using set of orthogonal functions. For such accurate representation of a time function, the orthogonal set should be "complete" [16]. In this paper, we will apply Block-pulse and Hybrid functions based on Block-pulse wavelet and Shifted Legendre polynomials to approximate the solution of (1) under conditions (2). In section 2 we present a number of definitions about fractional calculus, distributed order derivative, block-pulse wavelets, hybrid functions and its properties. In section 4 we will introduce a numerical method based on block-pulse and hybrid operational matrix, and in section 5 we will discuss the convergence of the described method. At the end, we will present some numerical examples.

2. Preliminaries

In this section, we present some basic definitions and properties of fractional calculus, distributed order derivative and wavelets [16].

Definition 2.1. A real function $f(x), x \ge 0$ is said to be in space C_μ, $\mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in [0,\infty)$, and it is said to be in the space C_μ^m if $f^m \in C_\mu$, $m \in N$.

Definition 2.2. The *Riemann-Liouville* fractional integral of order α from t_0 with respect to *t* is ${}_{t_0}I_t^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1} f(\tau)d\tau, \alpha > 0$.

Definition 2.3. The fractional derivative of $f(t)$ by means of Caputo Sense from t_0 is defined as

$${}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, n-1 < \alpha \le n, n \in N, t > t_0, f \in C_{-1}^{n}.$$

The relation between the Riemann-Liouville integral and Caputo derivative operator is given by the following expressions as in [17] and [18]:

$$ {}_{t_0}^{c}D_t^{\alpha}\,{}_{t_0}I_t^{\alpha}\left(f(t)\right)=f(t),\quad {}_{t_0}I_t^{\alpha}\,{}_{t_0}^{c}D_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}\left(t_0^{+}\right)\frac{(t-t_0)^k}{k!},\ t>t_0 \qquad (3)$$

Definition 2.4. The fractional derivative of distributed order in the Caputo sense with respect to order-density function $\omega(\alpha)\geq 0$ from a to b with $0\leq a\leq b$ is as ${}_{a}^{\omega}D_b^{t_0}f(t)=\int_a^b \omega(\alpha)\,{}_{t_0}^{c}D_t^{\alpha}f(t)d\alpha$.

Remark 2.5. We can see that when $\omega(\alpha)$, is Dirac delta function, then fractional derivative of distributed order-density function $\omega(\alpha)$ and fractional derivative of order α are the same.

Definition 2.6. The m-set of block-pulse functions for $i=0,1,2,\dots,m-1$, on $[0,T)$ is defined as $b_i(t)=\begin{cases}1, i\frac{T}{m}\leq t<(i+1)\frac{T}{m},\\ 0, otherwise\ .\end{cases}$

It can be shown that the functions b_i are disjoint and orthogonal [16].

Theorem 2.7. A function $f(t)\in L^2([0,T))$ may be approximated by the block-pulse function as $f(t)\cong\sum_{i=1}^{m_1}f_ib_i(t)=F^TB_m(t)$,

where $F^T=(f_1\cdots f_m)$, $B_m(t)=(b_1(t)\cdots b_m(t))^T$ and $f_i=\frac{1}{h}\int_{(i-1)h}^{ih}f(t)dt$.

Proof: In [16].

Remark 2.8. From above theorem we have,
$\min_{(i-1)h\leq t\leq ih}f(t)\leq f_i\leq\max_{(i-1)h\leq t\leq ih}f(t)$, this shows that if we approximate $f(t)$ by $F^TB_m(t)$; then the function $f(t)-F^TB_m(t)$ has at least one zero in the $[(i-1)h,ih)$.

Now we define the hybrid functions of Block-pulse and shifted Legendre polynomials. Firstly, we recall the shifted Legendre polynomials.

Definition 2.9. The shifted Legendre polynomials are defined on the interval $[0,1]$ and can be determined with the aid of the following recurrence formula

$P_{i+1}(t)=\frac{(2i+1)(2t-1)}{i+1}P_i(t)-\frac{i}{i+1}P_{i-1}(t), i=1,\dots$, where $P_0(t)=1$ and $P_1(t)=2t-1$.

Definition 2.10. Hybrid functions of block-pulse and shifted Legendre polynomials h_{ij}, $i = 0,1,2,...,m-1$ and $j = 0,1,2,...,n-1$ are defined on $[0,T)$ as

$$hy_{ij}(t) = \begin{cases} P_j\left(\frac{m}{T}t - i\right); \frac{iT}{m} \le t < \frac{(i+1)T}{m}, \\ 0; otherwise \end{cases},$$ where P_j is the j^{th} shifted Legendre polynomials on $[0,1]$.

Now, for approximating the function f we can set $f(t) \cong \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} hy_{ij}(t) = H^T Hy_{n,m}(t)$, where $H^T = \left(c_{00} \cdots c_{(m-1)(n-1)}\right)$,
$Hy_{n,m}(t) = \left(hy_{00}(t) \cdots hy_{(m-1)(n-1)}(t)\right)$, and $c_{ij} = \frac{\langle f, hy_{ij}\rangle}{\langle hy_{ij}, hy_{ij}\rangle}$, where $\langle u, v\rangle = \int_0^T u(t)v(t)dt$.

Now we introduce the operational matrix methods based on block-pulse functions. Fractional integration from $t_0 = 0$ of the block-pulse function vector is given as ${}_{t_0}I_t^{\alpha} B_m(t) = F^{(\alpha)} B_m(t)$, where $F^{(\alpha)}$ is the block-pulse operational matrix of the fractional order integration [18] and

$$F^{(\alpha)} = \left(\frac{T}{m}\right)^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}. \quad (4)$$

Now, let $Hy_{n,m}(t) \cong \Phi B_{mn}(t)$ and ${}_0I_t^{\alpha} Hy_{n,m}(t) = Q^{(\alpha)} Hy_{n,m}(t)$; then we can construct operational matrix for Hybrid functions as $Q^{(\alpha)} = \Phi F^{(\alpha)} \Phi^{-1}$. In the following lemmas, we present operational matrix of fractional order integration from arbitrary t_0 for block-pulse wavelets and shifted Legendre hybrid functions.

Lemma 2.11. The operational matrices of the fractional order integration α from t_0 for B_m on $t_0 < t$ are given as $F_{t_0}^{(\alpha)} \cong F^{(\alpha)} - \frac{1}{\Gamma(\alpha+1)} \Psi_{j,\alpha}$, where $q(t) \cong \Psi_{j,\alpha} B_m(t)$ and $q(t) = \begin{cases} \left(t - j\frac{T}{m}\right)^{\alpha} - (t - t_0)^{\alpha}, t_0 < t, \\ 0, otherwise. \end{cases}$

Proof: $\left({}_{t_0}I_t^{\alpha}B_m\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}B_m(\tau)d\tau$

$$=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}B_m(\tau)d\tau-\frac{1}{\Gamma(\alpha)}\int_{0}^{t_0}(t-\tau)^{\alpha-1}B_m(\tau)d\tau$$

$$=F^{(\alpha)}B_m(t)-\frac{1}{\Gamma(\alpha+1)}(0\cdots q(t)\cdots 0)^T_{m\times 1}\cong\left(F^{(\alpha)}-\frac{1}{\Gamma(\alpha+1)}\Psi_{j,\alpha}\right)B_m(t).$$

Remark 2.12. It is clear that if $j=0,(t_0=0)$ then $F_{t_0}^{(\alpha)}=F^{(\alpha)}$.

Lemma 2.13. The operational matrices of the fractional order integration α from t_0 for shifted Legendre hybrid functions vector $Hy_{n,m}$ on $t_0<t$ are given as $Q_{t_0}^{(\alpha)}=\Phi F_{t_0}^{(\alpha)}\Phi^{-1}$.

Proof: Let $\left({}_{t_0}I_t^{\alpha}Hy_{n,m}\right)(t)=Q_{t_0}^{(\alpha)}Hy_{n,m}(t)$, so $\left({}_{t_0}I_t^{\alpha}Hy_{n,m}\right)(t)=Q_{t_0}^{(\alpha)}\Phi B_{mn}(t)$, moreover; $\left({}_{t_0}I_t^{\alpha}Hy_{n,m}\right)(t)=\Phi\left({}_{t_0}I_t^{\alpha}B_{mn}\right)(t)=\Phi F_{t_0}^{(\alpha)}B_{mn}(t)$, that is $Q_{t_0}^{(\alpha)}\Phi=\Phi F_{t_0}^{(\alpha)}$, or $Q_{t_0}^{(\alpha)}=\Phi F_{t_0}^{(\alpha)}\Phi^{-1}$.

In the following section, we will consider block-pulse wavelets and shifted Legendre hybrid functions for solving the distributed order linear equations as (1) under initial conditions (2). By the above descriptions, we can approximate a function f in $L^2([0,T))$ as

$$f(t)\cong w^TW(t). \tag{5}$$

3. Numerical methods

For the sake of the simulation of solution of distributed order fractional equation (1) under condition (2), first, we consider the integration formulas of Newton and Cotes for the integral term in the distributed order equation [19], and next we use block-pulse functions described in the previous section for approximation of fractional derivative by using fractional operational matrices. Then we will have a system of linear equations in which it can be solved by existent methods; for example, we can use modified iterative methods [20] for this matter. Let we consider the following formula for approximation the integral term in the distributed order equation

$$\int_a^b R(\alpha)d\alpha = h\sum_{i=0}^{n}\delta_i R(\alpha_i),\alpha_i = a+ih \tag{6}$$

. By applying (6) to (1) we will have, $\int_a^b \omega(\alpha)\,{}_{t_0}^{c}D_t^{\alpha} y(t)d\alpha \cong h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\,{}_{t_0}^{c}D_t^{\alpha_i} y(t)$

thus, $h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\left({}_{t_0}I_t^{b}\left({}_{t_0}^{c}D_t^{\alpha_i}\right)\right)y(t) \cong \left({}_{t_0}I_t^{b}\right)g(t)$, so we have

$h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\,{}_{t_0}I_t^{b-\alpha_i}\left({}_{t_0}I_t^{\alpha_i}\left({}_{t_0}^{c}D_t^{\alpha_i}\right)\right)y(t) \cong \left({}_{t_0}I_t^{b}\right)g(t)$, from equation (3) we have

$$h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\left({}_{t_0}I_t^{b-\alpha_i}\right)y(t)$$
$$\cong h\sum_{i=0}^{n}\sum_{k=0}^{\lfloor\alpha_i\rfloor}\delta_i\omega(\alpha_i)\frac{y^{(k)}(t_0^+)}{k!}\frac{\Gamma(k+1)}{\Gamma(b-\alpha_i+k+1)}(t-t_0)^{b-\alpha_i+k}+\left({}_{t_0}I_t^{b}\right)g(t).$$

Now by using equation (5) we have

$$hw_y^T\sum_{i=0}^{n}\delta_i\omega(\alpha_i)G^{(b-\alpha_i)}W(t) \cong w_g^T G^{(b)}W(t)+hw_{sum}^T W(t), \tag{7}$$

where $y(t)=w_y^T W(t)$, $g(t)=w_g^T W(t)$, and $W(t)$ is block-pulse or hybrid functions of block-pulse and shifted Legendre vector functions, and *w* is it's corresponding vector coefficients, respectively, so $\sum_{i=0}^{n}\sum_{k=0}^{\lfloor\alpha_i\rfloor}\delta_i\omega(\alpha_i)\frac{y^{(k)}(t_0^+)}{k!}\frac{\Gamma(k+1)}{\Gamma(b-\alpha_i+k+1)}(t-t_0)^{b-\alpha_i+k} = w_{sum}^T W(t)$, with $t_0 \le t$, equation (7) implies that

$$hw_y^T\sum_{i=0}^{n}\delta_i\omega(\alpha_i)G^{(b-\alpha_i)} \cong w_g^T G^{(b)}+hw_{sum}^T. \tag{8}$$

Remark 3.1. Note that, as pointed out in [21], $h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\,{}_{t_0}^{c}D_t^{\alpha_i} y(t)$ can be viewed as the limiting case of $\int_a^b \omega(\alpha)\,{}_{t_0}^{c}D_t^{\alpha} y(t)d\alpha$, where a very large number of terms are considered. On the other hand, from above relations we have

$$
{}_{t_0}I_t^b\left(h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\,{}_{t_0}^{c}D_t^{\alpha_i}y(t)\right)
$$
$$
=h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\left(\left({}_{t_0}I_t^{b-\alpha_i}\right)y(t)-\sum_{k=0}^{\lfloor\alpha_i\rfloor}\frac{y^{(k)}(t_0^+)}{k!}\left({}_{t_0}I_t^{b-\alpha_i}\right)(t-t_0)^k\right),
$$

therefore; the right hand side of the last relation tends to $\left({}_{t_0}I_t^b\right)\int_a^b\omega(\alpha)\,{}_{t_0}^{c}D_t^{\alpha}y(t)d\alpha$ which is equal to $\left({}_{t_0}I_t^b\right)g(t)$, as $n\to\infty$.

If we replace the approximation with equality in the equation (8), we will have a linear algebraic equation which is solvable. By solving that, we can find w_y and then simulate $y(t)$ as $y(t)\cong w_y^TW(t)$.

4. Convergence analysis

In this section we want to investigate the convergence of the method described in the previous section. Let $(C[0,1],\|.\|)$ be the Banach space of all continuous functions with norm $\|f(t)\|=\max_{0\le t\le 1}|f(t)|$.

Theorem 4.1. Let $f(t)$ be an arbitrary real bounded function, which is square integrable in the interval $[0,1)$, and $e_b(t)=f(t)-F^TB_m(t)$; Then $\|e_b(t)\|\le\frac{c}{m}$.

Proof: In [16].

Let $L^2[0,T]$ be the space of square integrable functions on $[0,T]$, and $X=Span\{hy_{ij}(t): i=0,...m-1, j=0,...,n-1\}$. It is clear that hy_{ij} is at most a function of degree *n − 1*. Now, let $f\in L^2[0,T]$. Since X is a finite dimensional vector space, f has the unique best approximation out of X such as $p\in X$; that is $\exists p\in X\forall q\in X:\|f-p\|_2\le\|f-q\|_2$, where $\|f\|_2=\langle f,f\rangle$. Therefore, there exists the unique coefficients such that $f(t)\cong p=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{ij}hy_{ij}(t)$, where c_{ij} are defined in definition 2.10, for more details refer to [22].

Theorem 4.2. Let $f\in L^2[0,T]$ be n times continuously differentiable, and $f^{(n)}(t)<M$ on $[0,T]$. If $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{ij}hy_{ij}(t)=H^T\ Hy_{n,m}(t)$ is the best approximation of f out of X, then we have $\left\|f-H^THy_{n,m}(t)\right\|_2\le\frac{M\sqrt{T^3}}{\sqrt{3}n!m}$.

Proof: Let f_i be the Taylor polynomial of order *n-1* for *f* on $\left[\frac{iT}{m},\frac{(i+1)T}{m}\right)$,therefore; $f_i(t)=\sum_{k=0}^{n-1} f\left(\frac{iT}{m}\right)\frac{\left(t-\frac{iT}{m}\right)^k}{k!}$, moreover; for each *i* there exists $\xi_i \in \left(\frac{iT}{m},\frac{(i+1)T}{m}\right)$ such that $\left|f(t)-f_i(t)\right| \le \left|f^{(n)}(\xi_i)\right|\frac{\left(t-\frac{iT}{m}\right)^n}{n!}$. Since $H^T Hy_{n,m}(t)$ is the best approximation of *f* out of *X*, $f_i \in X$, from the last inequality we have

$$\left\|f-H^T Hy_{n,m}(t)\right\|_2^2 \le \sum_{i=0}^{m-1}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}}\left(\left|f^{(n)}(\xi_i)\right|\frac{\left(t-\frac{iT}{m}\right)^n}{n!}\right)^2 dt \le \frac{M^2T^3}{3(n!)^2 m^2}.$$

Now, we want to show the convergence of the block-pulse wavelets method for (*DFDE*).

Theorem 4.3. If we use, Newton and Cotes for the integral term and the operational matrix of the Block-Pulse functions for fractional term in equation (1), then we can convergence to the exact solution of (*DFDE*).

Proof*:* Let $|\omega(\alpha)| \le \nu$ on $[a,b]$. Now, we show that $E_m = \left\|\left(\int_a^b \omega(\alpha)\,{}_{t_0}^{c}D_t^{\alpha} y(t)d\alpha - g(t)\right)-\left(h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)\,{}_{t_0}^{c}D_t^{\alpha_i}c_y^T B_m(t)-c_g^T B_m(t)\right)\right\|$,

tends to zero when $n,m\to\infty$. By integration of order *b* from t_0 of E_m we have

$${}_{t_0}I_t^b E_m \le \left\|\left(\int_a^b \omega(\alpha)_{t_0} I_t^b y(t)d\alpha - {}_{t_0}I_t^b g(t)\right)-\left(h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)_{t_0} I_t^b c_y^T B_m(t) - {}_{t_0}I_t^b c_g^T B_m(t)\right)\right\|$$

$$\le \left\|\int_a^b \omega(\alpha)_{t_0} I_t^b y(t)d\alpha - h\sum_{i=0}^{n}\delta_i\omega(\alpha_i)_{t_0} I_t^b c_y^T B_m(t)\right\| + \left\|\left({}_{t_0}I_t^b g(t)\right)-\left({}_{t_0}I_t^b c_g^T B_m(t)\right)\right\|,$$ when

$n\to\infty$, we have

$${}_{t_0}I_t^bE_m \le \left\|\int_a^b \omega(\alpha)_{t_0}I_t^b y(t)d\alpha - \int_a^b \omega(\alpha)_{t_0}I_t^b c_y^T B_m(t)d\alpha\right\| + \left\|\left({}_{t_0}I_t^b g(t)\right) - \left({}_{t_0}I_t^b c_g^T B_m(t)\right)\right\|$$

$$= \left\|\int_a^b \omega(\alpha)_{t_0}I_t^b\left(y(t) - c_y^T B_m(t)\right)d\alpha\right\| + \left\|\left({}_{t_0}I_t^b g(t)\right) - \left({}_{t_0}I_t^b c_g^T B_m(t)\right)\right\|$$

$$\le \int_a^b |\omega(\alpha)|_{t_0}I_t^b\left\|y(t) - c_y^T B_m(t)\right\|d\alpha + {}_{t_0}I_t^b\left\|g(t) - c_g^T B_m(t)\right\|$$

thus, ${}_{t_0}I_t^bE_m \le \nu\int_a^b {}_{t_0}I_t^b\left\|y(t) - c_y^T B_m(t)\right\|d\alpha + {}_{t_0}I_t^b\left\|g(t) - c_g^T B_m(t)\right\|$; therefore, from theorem 4.1 and the above referred inequality we can see that ${}_{t_0}I_t^bE_m \le \nu\int_a^b\left({}_{t_0}I_t^b\frac{c_1}{m}\right)d\alpha + {}_{t_0}I_t^b\frac{c_2}{m}$, so ${}_{t_0}I_t^bE_m \le {}_{t_0}I_t^b\left(\nu(b-a)\frac{c_1}{m} + \frac{c_2}{m}\right)$, the last inequality shows that when $m \to \infty$ then $E_m \to 0$; this means that the method described in section 3 is convergent.

A similar theorem can be obtained from theorem 4.3 when we use the operational matrix of the hybrid of block-pulse and shifted Legendre functions.

Theorem 4.4. If we use Newton and Cotes for the integral term and the operational matrix of the hybrid functions of block-pulse and shifted Legendre functions for fractional term in equation (1), then we can have convergence to the exact solution of (*DFDE*).

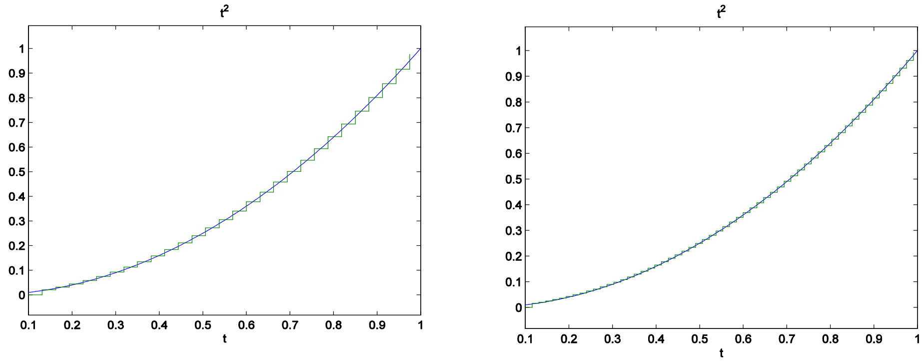
Proof: The proof is similar to the proof of theorem 4.3.

5. Illustrative Examples

In order to show the efficiency of the methods described in section 3 and simulate the exact solution of distributed order equations, we consider some examples that their exact solutions are known.

5.1. Example. Consider $\int_{0.1}^{0.9}\Gamma(3-\alpha)_{0.1}^{\ c}D_t^{\alpha} y(t)d\alpha = 2\frac{t^{1.9} - t^{1.1}}{\ln(t)}, 0 \le t \le 1, y(0.1) = 0.01$.

The exact solution of this example is $y(t) = t^2$. In this manner we use trapezoidal rule [19] for integral term with $h = 0.2$, $B_{32}(t)$ and $B_{64}(t)$ for approximating $y(t)$. Now from equation (8) we can find c_y and then simulate $y(t)$ by $c_y^T B_{32}(t)$, and $c_y^T B_{64}(t)$. Figure 1 shows the numerical results generated by block-pulse vector functions $B_{32}(t)$ and $B_{64}(t)$ for the example 5.1, and from that we will see when m increases from *32* to *64*; the numerical solution tends to exact solution. Table 1 shows the absolute error in some points.



By $B_{32}(t)$ *and* $h = 0.2$

By $B_{64}(t)$ *and* $h = 0.2$

Fig. 1. Numerical and exact solutions of example 5.1 by block-pulse functions.

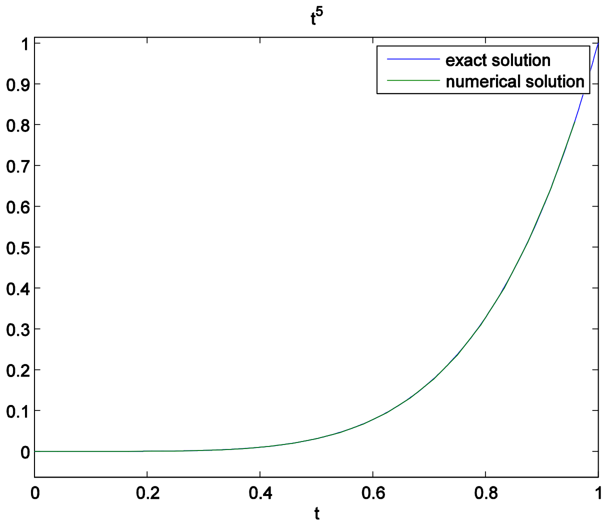


Fig. 2. Numerical and exact solutions of example 5.2 by hybrid functions $Hy_{8,3}$.

5.2. Example. Now consider $\int_0^2 \frac{\Gamma(6-\alpha)}{120} {}_0^cD_t^\alpha y(t) d\alpha = \frac{t^5 - t^3}{\ln(t)}, 0 \le t \le 1$, $y(0) = y'(0) = 0$. The exact solution is $y(t) = t^5$. Similar to example 5.1, in this example we use trapezoidal rule [19] for integral term with $h = 0.2$ and $B_{32}(t)$, $B_{64}(t)$ and $Hy_{8,3}(t)$ for approximating $y(t)$. In figure 2 we present the numerical and exact solutions generated by hybrid functions. Form figure 2 we can see that the numerical solution generated by hybrid functions are so closed to the exact solution. In table 2, we can compare absolute error of solutions generated $B_{32}(t)$, $B_{64}(t)$ and $Hy_{8,3}(t)$ in some points. From table 2 we can see that the errors of hybrid function are less than block-pulse functions.

Table 1

Melting points and elemental analyses

t	$\left\|c_y^T B_{32}(t) - t^5\right\|$	t	$\left\|c_y^T B_{64}(t) - t^2\right\|$
0.144	4.0425×e-6	0.4062	2.0174×e-5
0.242	8.4520×e-5	0.5918	1.9726×e-5

Table 2

Melting points and elemental analyses

t	$\left\|c_y^T B_{32}(t) - t^5\right\|$	$\left\|c_y^T B_{64}(t) - t^2\right\|$	$\left\|Hy_{8,3}(t)\right\| - t^2$
0.1	5.9470×e-6	1.3777×e-7	4.4538×e-7
0.2	1.3073×e-5	5.4498×e-5	1.3319×e-5
0.3	1.9812×e-4	8.4561×e-5	8.6726×e-5
0.4	1.4000×e-3	4.9081×e-4	2.6389×e-4
0.5	4.7000×e-3	1.9000×e-3	5.5607×e-4
0.6	5.4000×e-3	2.2676×e-5	9.2704×e-4
0.7	2.7000×e-3	6.9000×e-3	1.3000×e-3
0.8	7.6000×e-3	8.1000×e-3	1.6000×e-3
0.9	3.1300×e-2	6.8000×e-3	1.6000×e-3

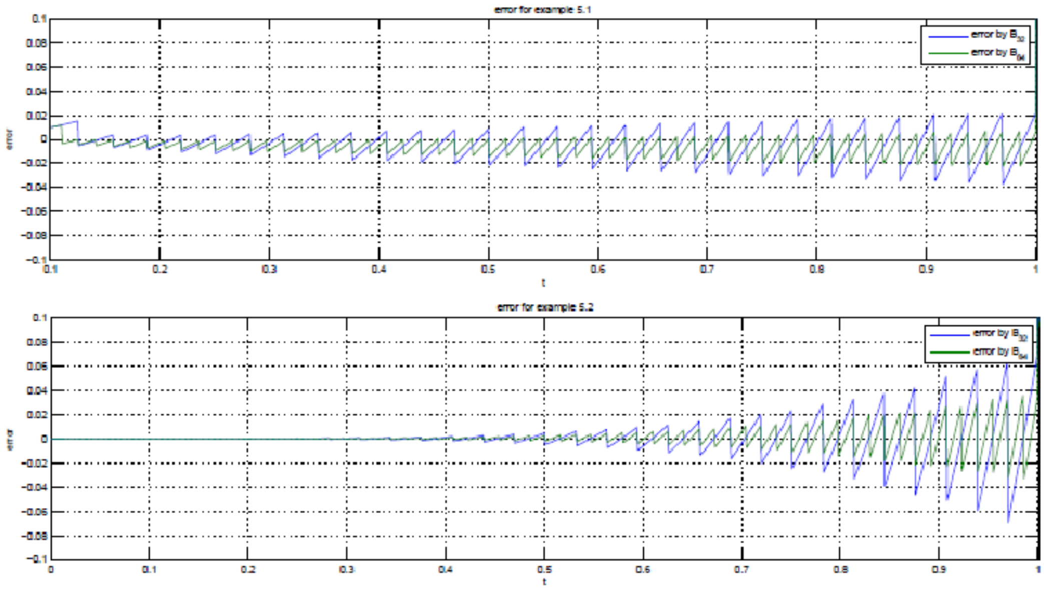


Fig. 3. Error of block-pulse approximations generated by m = 32, 64 for examples 5.1, 5.2.

Also, in figure 3 we present the error for examples 5.1, 5.2 generated by *m = 32, 64*. From figure 3 we see that when we double *m*, the number of zeros of $y(t)-c_y^T B_m(t)$ are double in each interval $\left(\frac{iT}{m}, \frac{(i+1)T}{m}\right]$. This idea was supported in remark 2.8. Notice that in each example we translate a *DFDE* to algebraic linear equations such as $Ax = b$ and then solved these equations.

6. Conclusions

The fractional differential equations play an important role in physics, chemical mixing and biological systems. The distributed-order operators can be obtained when we integrate the fractional-order calculus operators with respect to the order variable. The fundamental goal of this work has been to apply block-pulse and shifted Legendre hybrid functions operational matrix method to simulate the solution of *DFDE* with initial conditions at t_0. This method translates a *DFDE* to an algebraic linear equation which was presented in section 3 and the convergence of this method was demonstrated in section 4, also from section 4 we saw when m increases, we can obtain a good simulation of solution of *DFDE* with initial conditions. Moreover, two numerical examples were given to verify the effectiveness of the proposed schemes to simulation of solutions. Although the proposed numerical algorithms are quite effective in case of deterministic differential equations with smooth solutions, one has to further investigate how pulse wavelets numerically behave in case of stochastic differential equations (which really are sources of fractal signals).

REFERENCES

[1]. *K. B. Oldham and J. Spanier,* The Fractional Calculus, Academic Press, New York, NY, USA, 1974.

[2]. *M. Caputo,* Elasticita e Dissipazione, Zanichelli, Bologna, Italy, 1969.

[3]. *A. Ansari, A. Refahi Sheikhani and H. Saberi Najafi*, "*Solution to system of partial fractional differential equations using the fractional exponential operators*", Math. Meth. Appl. Sci., **vol. 35**, no. 1, Jan. 2012, pp. 119-123.

[4]. *H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh,*"*Sub-equation method for the fractional regularized long-wave equations with conformable fractional derivatives*", Scientia Iranica, **vol. 23,** no. 3, Jan. 2016, pp. 1048-1054.

[5]. *M. Caputo*, "*Distributed order differential equations modelling dielectric induction and diffusion*", Fractional Calculus and Applied Analysis, **vol. 4**, No. 4, Jan. 2001, pp. 421-442.

[6]. *R. L. Bagley and P. J. Torvik*,"*On the existence of the order domain and the solution of distributed order equations I*", International Journal of Applied Mathematics, **vol. 2**, No. 7, Jan. 2000, pp. 865-882.

[7]. *E. Reyes-Melo, J. Martinez-Vega, C. Guerrero-Salazar, and U. Ortiz-Mendez*, "*Application of fractional calculus to the modeling of dielectric relaxation phenomena in polymeric materials*", Journal of Applied Polymer Science, **vol. 98**, no. 2, Oct. 2005, pp. 923-935.

[8]. *N. J. FORD, M. L. MORGADO and AND M. REBELO*, "*An implicit finite difference approximation for the solution of the diffusion equation with distributed order in time*", Electronic Transactions on Numerical Analysis, **vol. 44**, Jun. 2015, pp. 289-315.

[9]. *H. Aminikhah, A. Refahi Sheikhani and H. Rezazadeh*,"*Stability Analysis of Distributed Order Fractional Chen System*",The Scientific World Journal, **vol. 2013**, Oct. 2013, pp. 1-13.

[10]. *H. Aminikhah, A. Refahi Sheikhani and H. Rezazadeh*,"*Stability analysis of linear distributed order system with multiple time delays*", U.P.B. Sci. Bull., **vol. 77**, no. 2, Jan. 2015, pp. 207-218.

[11]. *A. Ansari, A. Refahi Sheikhani and S. Kordrostami*, "*On the generating function ext+yϕ(t) and its fractional calculus*", Cent. Eur. J. Phys., **vol. 11**, no. 10, Oct. 2013, pp. 1457-1462.

[12]. *A. Ansari and A. R. Sheikhani*, "*New identities for the wright and the mittagleffler functions using the laplace transform*"Asian-European Journal of Mathematics, **vol. 7**, no. 3, Sept. 2014, pp. 1-8.

[13]. *H. Saberi Najafi, A. Refahi Sheikhani and A. Ansari*, "*Stability analysis of distributed order fractional differential equations*", Abstract and Applied Analysis, **vol. 2011**, Jul. 2011, pp. 1-12.

[14]. *A. Refahi Sheikhani, H. Saberi Najafi, Alireza Ansari and Farshid Mehrdoust*, " *Analytic study on linear system of distributed order fractional differential equation*", Le Matematiche, **vol. 67**, no. 2, 2012, pp. 3-13.

[15]. *H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh*, "*Approximate analytical solutions of distributed order fractional Riccati differential equation*", Ain Shams Engineering Journal, Article in press.

[16]. *A. Deb and S. Ghosh*, Power electronic Systems: Walsh analysis With Matlabr. CRC Press, Taylor Francis Group, LLC, 2014.
[17]. *Ahmad El-Ajou , Omar Abu Arqub , Zeyad Al Zhour and Shaher Momani*,"*New Results on Fractional Power Series: Theories and Applications*", Entropy, **vol. 15**, no 12, Dec. 2013, pp. 5305-5323.
[18]. *Mingxu Yi, Jun Huang and Jinxia Wei*,"*Block pulse operational matrix method for solving fractional partial differential equation*", Applied Mathematics and Computation, **vol. 221**, Sept. 2013, pp. 121-131.
[19]. *J.Stoer and R.Bulirsch*, Numerical Analysis, Springer-Verlag, New York, Inc, 2002.
[20]. *H. Saberi Najafi, S. A. Edalatpanah and A. H. Refahi Sheikhani*, "*Convergence Analysis of Modified Iterative Method to Solve Linear Systems*", Mediterranean Journal of Mathematics, **vol. 11**, no. 3, Aug. 2014, pp. 1019-1032.
[21]. *K. Diethelm and N.J. Ford*, "*Numerical analysis for distributed order differential equations*", J. Comp. Appl. Math. **Vol. 225**, No. 1, Mar. 2009, pp. 96-104.
[22]. *C. Canuto, A. Quarteroni, M. Y. Hussaini and T.A. Zang*, Spectral Methods, Fundamentals in Single Domains, Springer-Verlag, Berlin, Heidelberg, 2006.