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ON NEW EXTENSIONS OF F-CONTRACTION WITH AN APPLICATION TO INTEGRAL INCLUSIONS

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The purpose of this study is to present some fixed point theorems by combining the contractions of Geraghty and Hardy-Rogers with F-contraction and α-admissible concept in the setting of set-valued mappings under weaker conditions. We derive new fixed point results on a metric space endowed with a partial ordering/graph by using the results obtained herein. We give also an example and an application to support new theory.

Keywords: F-contraction, Geraghty contraction, Hardy-Rogers contraction, α-admissible, semi-lower continuous, integral inclusion.

MSC2010: 47H10, 54H25.

1. Introduction

The multivalued fixed point theory has many and different applications as in integral or differential inclusions, economy, optimization, etc. The contraction principle due to Banach has been generalized in different directions and one of such generalizations is due to Nadler [19], where he used the Pompeiu-Hausdorff metric to establish some fixed point results of multivalued mappings in metric spaces. Later many authors established some results in nonlinear analysis concerning the multivalued fixed point theory and its applications using the Pompeiu-Hausdorff distance, for more details, see [1–3, 6, 8, 13, 14, 17, 25].

Samet et al. [21] introduced a new concept called as α-admissible, they obtained some fixed point results for α-ψ-contractive mappings, later many authors invested such concepts to establish some results, see [5, 12, 14, 18]. Recently, Wardowski [26] introduced a new type of contractions called as F-contraction to show the existence of fixed points for such contraction by more simple method of proof than Banach's one. After that, several authors studied on different variations of F-contraction for single-valued and multivalued mappings, for example, see [1, 3, 4, 7, 9, 13, 16, 17, 20, 22, 25, 27].

In this study, we combine the notion of α-admissible with Wardowski contraction and the contractions of Geraghty and Hardy-Rogers in order to introduce new types of multivalued contractions to establish some fixed point theorems in the setting of complete metric spaces. We derive new fixed point results on a metric space endowed with a partial ordering/graph by using the results obtained herein. Finally, we give an example and an application of the existence of solution for an integral inclusion to illustrate our results.

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2. **Preliminaries**

Here, we recollect some basic definitions, lemmas, notations and some known theorems which are helpful for understanding of this paper. Let (X, d) be a metric space, and let $CB(X)$ be a set of nonempty, closed and bounded subsets of X, the Pompeiu -Hausdorff metric is defined as:

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

for all $A, B \in CB(X)$, where $d(a,B) = \inf\{d(a,b)\colon b \in B\}$. Note that, if $A = \{a\}$ (singleton) and $B = \{b\}$, then $H(A,B) = d(a,b)$. Also, denote the family of nonempty and closed subsets of X by $CL(X)$ and the family of nonempty and compact subsets of X by $K(X)$. Note that $H\colon CL(X) \times CL(X) \to [0,\infty]$ is a generalized Pompeiu-Hausdorff metric, that is, $H(A,B) = \infty$ if $\max\{\sup_{a\in A}\ d(a,B),\ \sup_{b\in B}\ d(b,A)\}$ does not exist in $\mathbb{R}$.

Lemma 2.1. *[19] Let (X, d) be a metric space and $A, B \in CL(X)$ with $H(A,B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a,b) < hH(A,B)$.*

Definition 2.1. *[26] Let $F : (0,+\infty) \to \mathbb{R}$ be a function satisfying:*

(F_1) : *F is strictly increasing,*
(F_2) : *for each sequence $\{\alpha_n\}$ in X, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,*
(F_3) : *there exists $k \in (0,1)$ satisfying $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Denote by $\mathcal{F}$ the set of all functions F satisfying the conditions $(F_1) - (F_3)$.

Example 2.1. *Let $F_i : (0,+\infty) \to \mathbb{R},\ i \in \{1,2,3\}$, defined by*

(1) $F_1(t) = \ln t$,
(2) $F_2(t) = t + \ln t$,
(3) $F_3(t) = -\frac{1}{\sqrt{t}}$.

Then $F_i \in \mathcal{F}$, for each $i \in \{1,2,3\}$.

Theorem 2.1. *[26] Let (X, d) be a complete metric space and $T\colon X \to X$ be an F-contraction, that is, there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F((d(Tx,Ty)) \leq F(d(x,y)), \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point $x^\star$. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ converges to $x^\star$.

Sgroi and Vetro [25] proved the existence of a fixed point for a Hardy-Rogers multivalued contraction as a generalization of Wardowski's theorem in the setting of multivalued case, they showed that must adding another condition on F which is the right continuity. So let $\mathcal{F}_*$ be set of all functions F satisfying $(F_1) - (F_3)$ and
(F_4) : F is right continuous.

Theorem 2.2. *[25] Let (X, d) be a metric space and let $T : X \to CB(X)$ be a multivalued F-contraction of Hardy-Rogers type, that is, there exist $F \in \mathcal{F}_*$, $\tau > 0$ and non-negative real numbers $\alpha, \beta, \gamma, \delta, L$ with $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$ such that*

$$2\tau + F(H(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx)),$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$. Then T has a fixed point in X.

Firstly, Asl *et al.* [5] adapted the notion of α-admissible to multivalued mappings as α_*-admissible. Afterwards, Mohammadi *et al.* [18] introduced the concept of α-admissible for multivalued mappings. On the other side, Iqbal and Hussain [13] introduced the notion of α-lower semi-continuous multivalued mappings.

Let (X, d) be a metric space and $\alpha\colon X \times X \to [0, +\infty)$ be a given mapping. A mapping $T\colon X \to CL(X)$ is an

(1) α_*-admissible, if $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b): a \in Tx, b \in Ty\}$;

(2) α-admissible, if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$

(3) α-lower semi-continuous, if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies

$$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx).$$

One can easily see that each α_*-admissible mapping is also α-admissible, but the converse is not true in general.

Throughout this paper, we will denote by Ω the set of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying $\lim_{n\to\infty} \beta(t_n) = 1$ implies $\lim\limits_{n\to\infty} t_n = 0$.

3. Fixed point results for contractions of Hardy-Rogers type

Definition 3.1. *Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $T : X \to CL(X)$ is called α-F-Geraghty contraction of Hardy-Rogers type if there exist $F \in \mathcal{F}$, $\beta \in \Omega$ and $\tau > 0$ and non-negative real numbers a_1, a_2, a_3, a_4, a_5 with $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$ such that*

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)), \tag{1}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$ where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to K(X)$ be an α-F-Geraghty contraction of Hardy-Rogers type. Assume that the following conditions are satisfied:*

(1) T is an α-admissible;

(2) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(3) T is an α-lower semi-continuous mapping, or X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By the hypothesis (2), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Because of this, assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$, then $d(x_1, Tx_1) > 0$ and hence $H(Tx_0, Tx_1) > 0$. Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. Now, taking (1) into account, we get

$$\begin{aligned} F(d(x_1, x_2)) = F(d(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(\beta(N(x_0, x_1))N(x_0, x_1)) - \tau \end{aligned}$$

Since F is strictly increasing and $\beta(t) < 1$ for all $t \geq 0$, we get

$$\begin{aligned} &F(d(x_1, x_2)) < F(N(x_0, x_1)) - \tau \\ &= F(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Tx_0)) - \tau \\ &\leq F(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2)) - \tau \\ &\leq F((a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2)) - \tau. \end{aligned}$$

Since F is strictly increasing, we get

$$d(x_1, x_2) < (a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2),$$

which implies that

$$(1-a_3-a_4)d(x_1,x_2) < (a_1+a_2+a_4)d(x_0,x_1).$$

Since $a_1+a_2+a_3+2a_4=1$ and $a_3\neq 1$, we infer that $1-a_3-a_4>0$ and so

$$d(x_1,x_2) < \frac{a_1+a_2+a_4}{1-a_3-a_4}d(x_0,x_1)=d(x_0,x_1).$$

Consequently, we obtain

$$F(d(x_1,x_2)) \leq F(d(x_0,x_1))-\tau.$$

Following the previous procedures, we can assume that $x_1\neq x_2$ and $x_2\notin Tx_2$. Then $d(x_2,Tx_2)>0$, and so $H(Tx_1,Tx_2)>0$. Since, $\alpha(x_0,x_1)\geq 1$ and T is an α-admissible multivalued mapping, we derive that $\alpha(x_1,x_2)\geq 1$ for $x_2\in Tx_1$. Also, since Tx_2 is compact, there exists $x_3\in Tx_2$ such that $d(x_2,x_3)=d(x_2,Tx_2)$. Regarding (1), we deduce

$$\begin{aligned}
&F(d(x_2,x_3)) = F(d(x_2,Tx_2)) \leq F(H(Tx_1,Tx_2))\\
&\leq F(\beta(N(x_1,x_2))N(x_1,x_2))-\tau\\
&< F(N(x_1,x_2))-\tau\\
&= F(a_1d(x_1,x_2)+a_2d(x_1,Tx_1)+a_3d(x_2,Tx_2)+a_4d(x_1,Tx_2)+a_5d(x_2,Tx_1))-\tau\\
&\leq F(a_1d(x_1,x_2)+a_2d(x_1,x_2)+a_3d(x_2,x_3)+a_4d(x_1,x_3))-\tau\\
&\leq F((a_1+a_2+a_4)d(x_1,x_2)+(a_3+a_4)d(x_2,x_3))-\tau.
\end{aligned}$$

By calculations similar to the above, we get

$$F(d(x_2,x_3)) \leq F(d(x_1,x_2))-\tau.$$

By continuing in this manner, we can construct a sequence $\{x_n\}$ in X such that such that $x_n\neq x_{n+1}\in Tx_n$, $\alpha(x_n,x_{n+1})\geq 1$ and

$$F(d(x_n,x_{n+1})) \leq F(d(x_{n-1},x_n))-\tau, \quad \text{for all } n\in\mathbb{N}. \tag{2}$$

Let $b_n:=d(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$. Thus, from (2), we have

$$F(b_n)\leq F(b_{n-1})-\tau\leq\cdots\leq F(b_0)-n\tau, \quad \text{for all } n\in\mathbb{N} \tag{3}$$

and so $\lim_{n\to\infty}F(b_n)=-\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} b_n = 0. \tag{4}$$

To prove that $\{x_n\}$ is a Cauchy sequence, let us consider condition $(F3)$. Then, there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty} b_n^kF(b_n)=0. \tag{5}$$

By (3), for all $n\in\mathbb{N}$, we infer that

$$b_n^kF(b_n)-b_n^kF(b_0)\leq -b_n^kn\tau\leq 0. \tag{6}$$

Letting $n\to\infty$ in (6) and using (5), we get

$$\lim_{n\to\infty} nb_n^k=0.$$

By the definition of limit, there exists $n_1\in\mathbb{N}$ such that $nb_n^k\leq 1$ for all $n\geq n_1$, and consequently,

$$b_n\leq \frac{1}{n^{1/k}}, \quad \text{for all } n\geq n_1. \tag{7}$$

Let $m > n \geq n_1$. Then, using the triangular inequality and (7), we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &= \sum_{j=n}^{m-1} b_j \leq \sum_{j=n}^{m-1} \frac{1}{j^{1/k}} \\ &\leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}} < \infty. \end{aligned}$$

Since it is a partial sum of a convergent chain. For $n, m \to \infty$ we get $d(x_n, x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, so $\{x_n\}$ is convergent to some $z \in X$. Now we claim $z \in Tz$, if T is α-lower semi-continuous mapping. Since $d(x_n, Tx_n) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, by (4), we deduce $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From α-lower semi-continuity of T, we obtain

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

Hence $d(z, Tz) = 0$, since Tz is closed, so $z \in Tz$.
If X is α-regular, then $\alpha(x_n, z) \geq 1$. If there exists $p \in \mathbb{N}$ such $d(x_{p+1}, Tz) = 0$, then from the uniqueness of limit, $d(z, Tz) = 0$ and so $z \in Tz$. Otherwise there exists $n_2 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ which gives $H(Tx_n, Tz) > 0$ for all $n > n_2$. Thus, we have

$$\begin{aligned} &F(d(x_{n+1}, Tz)) \leq F(H(Tx_n, Tz)) \\ &\leq F(\beta(N(x_n, z))N(x_n, z)) - \tau \\ &< F(N(x_n, z)) - \tau \\ &= F(a_1 d(x_n, z) + a_2 d(x_n, Tx_n) + a_3 d(z, Tz) + a_4 d(x_n, Tz) + a_5 d(z, Tx_n)) - \tau \\ &\leq F(a_1 d(x_n, z) + a_2 d(x_n, x_{n+1}) + a_3 d(z, Tz) + a_4 d(x_n, Tz) + a_5 d(z, x_{n+1})) - \tau. \end{aligned}$$

Since F is strictly increasing, we get

$$d(x_{n+1}, Tz) < a_1 d(x_n, z) + a_2 d(x_n, x_{n+1}) + a_3 d(z, Tz) + a_4 d(x_n, Tz) + a_5 d(z, x_{n+1}),$$

for all $n > n_2$. Letting $n \to \infty$ in the previous inequality, we obtain

$$d(z, Tz) \leq (a_3 + a_4) d(z, Tz) < d(z, Tz),$$

which gives that $d(z, Tz) = 0$. This completes the proof. □

In the next theorem, we replace $K(X)$ with $CB(X)$ by considering in the setting of $\mathcal{F}_*$.

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be an α-F-Geraghty contraction of Hardy-Rogers type with $F \in \mathcal{F}_*$. Assume that the following conditions are satisfied:*

(1) T is α-admissible.
(2) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$.
(3) T is α-lower semi-continuous or X is α regular.

Then T has a fixed point.

Proof. As in proof of Theorem 3.1, there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point, suppose not, so $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$, using (1), we get

$$F(H(Tx_0, Tx_1))) \leq F(\beta(N(x_0, x_1))N(x_0, x_1)) - \tau < F(N(x_0, x_1)) - \tau.$$

By the property of right continuity of F, there exists a real number $h_1 > 1$ such that

$$F(h_1 H(Tx_0, Tx_1)) \leq F(N(x_0, x_1)) - \tau. \tag{8}$$

From $d(x_1, Tx_1) < h_1 H(Tx_0, Tx_1)$, by Lemma 2.1, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq h_1 H(Tx_0, Tx_1)$. Then, using (F_1) and (8), we deduce

$$\begin{aligned}
&F(d(x_1, x_2)) \leq F(h_1 H(Tx_0, Tx_1))\\
&\leq F(N(x_0, x_1)) - \tau\\
&= F(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Tx_0)) - \tau\\
&\leq F(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2)) - \tau\\
&\leq F((a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2)) - \tau,
\end{aligned}$$

which gives us

$$F(d(x_1, x_2)) \leq F(d(x_0, x_1)) - \tau.$$

In view of the fact that T is α-admissible and $\alpha(x_0, x_1) \geq 1$, we have $\alpha(x_1, x_2) \geq 1$ for $x_2 \in Tx_1$. Assume that $x_2 \notin Tx_2$. Since F is right continuous, there exists $h_2 > 1$ such that

$$F(h_2 H(Tx_1, Tx_2)) \leq F(N(x_1, x_2)) - \tau. \tag{9}$$

From $d(x_2, Tx_2) < h_2 H(Tx_1, Tx_2)$, by Lemma 2.1, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq h_2 H(Tx_1, Tx_2)$. Then, using $(F1)$ and (9), we infer that

$$\begin{aligned}
&F(d(x_2, x_3)) \leq F(h_2 H(Tx_1, Tx_2))\\
&\leq F(N(x_1, x_2)) - \tau\\
&= F(a_1 d(x_1, x_2) + a_2 d(x_1, Tx_1) + a_3 d(x_2, Tx_2) + a_4 d(x_1, Tx_2) + a_5 d(x_2, Tx_1)) - \tau\\
&\leq F(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + a_4 d(x_1, x_3)) - \tau\\
&\leq F((a_1 + a_2 + a_4) d(x_1, x_2) + (a_3 + a_4) d(x_2, x_3)) - \tau,
\end{aligned}$$

which implies that

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau.$$

Continuing in this manner, we build two sequences $\{x_n\} \subset X$ and $\{h_n\} \subset (1, +\infty)$ such that $x_n \neq x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \geq 1$ and

$$F(d(x_n, x_{n+1})) \leq F(h_n H(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \tau, \text{ for all } n \in \mathbb{N}.$$

Hence,

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau, \quad \text{for all } n \in \mathbb{N}.$$

which gives that

$$\lim_{n \to +\infty} F(d(x_n, x_{n+1})) = 1.$$

From (F_2), we obtain

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

The rest of the proof is like in the proof of Theorem 3.1. □

The following example support our theoretical results.

Example 3.1. *Let $X = \{1, 2, 3, 4\}$ and $d(x, y) = |x - y|$. Define $T\colon X \to CB(X)$ and $\alpha : X \times X \to [0, \infty)$ by*

$$Tx = \begin{cases} \{2\}, & x \in \{1, 2\}\\ \{1\}, & x = 3\\ \{3\}, & x = 4 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 0, & (x, y) \in \{(2, 3), (3, 2), (3, 4), (4, 3)\}\\ 1, & \textit{otherwise.} \end{cases}$$

We claim that T is an α-F-Geraghty contraction of Hardy-Rogers type, by taking $F(x) = \ln x$, $\beta(t) = \frac{t}{1+t}$, $\tau = \frac{1}{4}$, $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$. For that, we need to show that

$$H(Tx, Ty) \leq e^{-\frac{1}{4}} \beta(N(x, y))N(x, y), \quad \text{for all } x, y \in X \text{ with } H(Tx, Ty) > 0 \text{ and } \alpha(x, y) \geq 1.$$

Note that $H(Tx, Ty) > 0$ and $\alpha(x, y) \geq 1$ if and only if $(x, y) \notin \{(x, x) : x \in X\} \cup \{(2, 3), (3, 2), (3, 4), (4, 3)\}$. By the symmetry property of the metric, we have the following cases:

(1) For $x = 1$ and $y = 2$, we have

$$H(T1, T2) = 0, \quad N(1, 2) = a_1 d(1, 2) = 1 \quad \text{and} \quad \beta(N(1, 2)) = \frac{1}{2}$$

which implies

$$H(T1, T2) \leq e^{-\frac{1}{4}} \beta(N(1, 2))N(1, 2).$$

(2) For $x = 1$ and $y = 3$, we have

$$H(T1, T3) = 1, \quad N(1, 3) = a_1 d(1, 3) = 2 \quad \text{and} \quad \beta(N(1, 3)) = \frac{2}{3}$$

which implies

$$H(T1, T3) \leq e^{-\frac{1}{4}} \beta(N(1, 3))N(1, 3).$$

(3) For $x = 1$ and $y = 4$, we have

$$H(T1, T4) = 1, \quad N(1, 4) = a_1 d(1, 4) = 3 \quad \text{and} \quad \beta(N(1, 4)) = \frac{3}{4}$$

which implies

$$H(T1, T4) \leq e^{-\frac{1}{4}} \beta(N(1, 4))N(1, 4).$$

(4) For $x = 2$ and $y = 4$, we have

$$H(T2, T4) = 1, \quad N(2, 4) = a_1 d(2, 4) = 2 \quad \text{and} \quad \beta(N(2, 4)) = \frac{2}{3}$$

which implies

$$H(T2, T4) \leq e^{-\frac{1}{4}} \beta(N(2, 4))N(2, 4).$$

Consequently, T is an α-F-Geraghty contraction of Hardy-Rogers type. Moreover, it is easy to see that T is an α-admissible multivalued mapping and there exist $x_0 = 3$ and $x_1 = 1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Also, it is obvious that T is α-lower semi-continuous. But X is not α-regular. Indeed, consider the sequence $\{x_n\} = \{4, 2, 1, 3, 3, \ldots, 3, \ldots\}$ in X. Then $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ and $x_n \to 3$, but $\alpha(x_1, 3) = \alpha(4, 3) = 0$. Consequently, all conditions of Theorem 3.2 (resp. Theorem 3.1) are satisfied. Then T has a fixed point which is 2.

On the other hand, for $x = 3$ and $y = 4$, we have

$$H(T3, T4) = 2, \quad \text{and} \quad N(3, 4) = a_1 d(3, 4) = 1,$$

and hence

$$2\tau + F(H(T3, T4)) > F(N(3, 4)).$$

Therefore, Theorem 2.2 can not applied to this example.

Since each α_*-admissible mapping is also α-admissible, we obtain following result.

Corollary 3.1. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, +\infty)$ be a function and $T : X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following assertions hold:*

(i) T is an α_*-admissible;

(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) T is α-lower semi-continuous or X is α-regular;

(*iv*) *There exist* $F \in \mathcal{F}_*$, $\beta \in \Omega$ *and* $\tau > 0$ *and non-negative real numbers* a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_3 \neq 1$ *such that*

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)),$$

for all $x, y \in X$ *with* $\alpha_*(Tx, Ty) \geq 1$ *and* $H(Tx, Ty) > 0$ *where*

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then T *has a fixed point.*

Corollary 3.2. *Let* (X, d) *be a complete metric space,* $\alpha \colon X \times X \to [0, +\infty)$ *be a function and* $T \colon X \to CB(X)$ *(resp.* $K(X)$*) be an* α*-admissible multivalued mapping and the following assertions hold:*

(*i*) *There exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$*;*
(*ii*) T *is* α*-lower semi-continuous or* X *is* α*-regular;*
(*iii*) *There exist* $F \in \mathcal{F}_*$, $\beta \in \Omega$ *and* $\tau > 0$ *and non-negative real numbers* a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_3 \neq 1$ *such that*

$$x, y \in X, \quad H(Tx, Ty) > 0 \Rightarrow \tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)), \tag{10}$$

where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then T *has a fixed point.*

Proof. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$. Using (F_1) and (10), we have

$$\tau + F(H(Tx, Ty)) \leq \tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)),$$

and hence

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)),$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$. This implies that the inequality (1) holds. Thus, the rest of proof follows from Theorem 3.2 (resp. Theorem 3.1). □

If we take $\alpha(x, y) = 1$ in Corollary 3.2, we obtain a extension of Theorem 2.2 as follows.

Corollary 3.3. *Let* (X, d) *be a complete metric space and* $T \colon X \to CB(X)$ *(resp.* $K(X)$*) be a multivalued mapping satisfying*

$$x, y \in X, \quad H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)),$$

where $F \in \mathcal{F}_*$, $\beta \in \Omega$ *and* $\tau > 0$ *and non-negative real numbers* a_1, a_2, a_3, a_4, a_5 *with* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_3 \neq 1$ *and*

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then T *has a fixed point.*

4. **Some Consequences**

In this section we give new fixed point results on a metric space endowed with a partial ordering/graph, by using the results provided in previous section. Define

$$\alpha \colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following result is a direct consequence of our results.

Theorem 4.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T\colon X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following assertions hold:*

(i) For each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$;
(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;
(iii) T is $\preceq$-lower semi-continuous, that is, for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, implies

$$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$

or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;
(iv) There exist $F \in \mathcal{F}_*$, $\beta \in \Omega$ and $\tau > 0$ and non-negative real numbers a_1, a_2, a_3, a_4, a_5 with $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$ such that

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)), \tag{11}$$

for all $x, y \in X$ with $x \preceq y$ and $H(Tx, Ty) > 0$ where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then T has a fixed point.

Now, we present the existence of fixed point for multivalued mappings from a metric space X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

If we define the function

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

then the following result is a direct consequence of our results.

Theorem 4.2. *Let (X, d) be a complete metric space endowed with a graph G and $T\colon X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following conditions are satisfied:*

(i) For each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$;
(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;
(iii) T is G-lower semi-continuous, that is, for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, implies

$$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$

or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;
(iv) There exist $F \in \mathcal{F}_*$, $\beta \in \Omega$ and $\tau > 0$ and non-negative real numbers a_1, a_2, a_3, a_4, a_5 with $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$ such that

$$\tau + F(H(Tx, Ty)) \leq F(\beta(N(x, y))N(x, y)), \tag{12}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $H(Tx, Ty) > 0$ where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Then T has a fixed point.

5. **An Application**

In this section, we apply our obtained results to prove existence theorem of solution for an integral inclusion of Volterra type in Banach space. For this, let $X := C([a,b],\mathbb{R})$ be the space of all continuous realvalued functions on $[a,b]$. Clearly X with uniform metric $d(x,y)=\sup_{t\in[a,b]}|x(t)-y(t)|$ is a complete metric space.

Consider now the following problem

$$x(t)\in f(t)+\int_a^b Q(t,s,x(s))ds, \qquad t\in J=[a,b]. \tag{13}$$

where $f\in X$ and $Q: J\times J\times\mathbb{R}\to CB(\mathbb{R})$.

Our hypotheses are on the following data :

(A) for each $x\in X$, the multivalued operator $Q_x(t,s):=Q(t,s,x(s))$, $t,s\in J\times J$, is lower semi-continuous;

(B) there exists a continuous mapping $\rho: J\times J\to[0,+\infty)$ such that

$$|Q(t,s,u(s))-Q(t,s,v(s))|\le\rho(t,s)\cdot\ln(|u(s)-v(s)|+1),$$

for all $u,v\in X$ with $(u,v)\in E(G)$ and $u\ne v$ and for each $(t,s)\in J\times J$;

(C) there exists $\tau>0$ such that

$$\sup_{t\in J}\int_a^b\rho(t,s)ds\le e^{-\tau};$$

(D) there exist $x_0\in X$ and $x_1\in Tx_0$ such that $(x_0,x_1)\in E(G)$;

(E) for each $x\in X$ and $y\in Tx$ with $(x,y)\in E(G)$, we have $(y,z)\in E(G)$ for all $z\in Ty$;

(F) for every sequence $\{x_n\}$ in X such that $x_n\to x\in X$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, we have $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$;

Theorem 5.1. *Under assumptions* $(A)-(F)$ *the integral inclusion* (13) *has a solution in* X.

Proof. Consider the set-valued operator $T: X\to CB(X)$ as follows

$$Tx(t)=\left\{y\in X: y\in f(t)+\int_a^b Q(t,s,x(s))ds,\ \ t\in J\right\}.$$

Note that the integral inclusion (13) has a solution if and only if T has a fixed point in X. For the set-valued operator $Q_x(t,s): J\times J\to CB(\mathbb{R})$, it follows from Michaels selection theorem for $x\in X$ there exists a continuous operator $q_x: J\times J\to\mathbb{R}$ such that $q_x(t,s)\in Q_x(t,s)$ for all $t,s\in J\times J$. It follows that $f(t)+\int_a^b q_x(t,s)ds\in Tx$, so Tx is non-empty for all $x\in X$. Since f and Q_x are continuous on $[a,b]$, resp. $[a,b]^2$, their ranges are bounded and hence Tx is bounded, i.e., $T: X\to CB(X)$.

Let $x_1,x_2\in X$ with $(x_1,x_2)\in E(G)$ and $x_1\ne x_2$, and $v_1\in Tx_1$. Then

$$v_1(t)\in f(t)+\int_a^b Q(t,s,x_1(s))ds, \quad t\in J.$$

It follows that

$$v_1(t)=f(t)+\int_a^b q_{x_1}(t,s)ds, \quad (t,s)\in J\times J,$$

where $q_{x_1}(t,s)\in Q_{x_1}(t,s)$.

From (B), we get

$$|Q(t,s,x_1(s))-Q(t,s,x_2(s))|\le\rho(t,s)\cdot\ln(|x_1(s)-x_2(s)|+1),$$

for each $(t,s)\in J\times J$. Hence, there exists $w(t,s)\in Q_{x_2}(t,s)$ such that

$$|q_{x_1}(t,s)-w(t,s)|\le\rho(t,s)\cdot\ln(|x_1(s)-x_2(s)|+1),$$

for all $(t,s) \in J \times J$. Consider the multivalued operator L defined by

$$L(t,s) = Q_{x_2}(t,s) \cap \{z \in \mathbb{R}\colon |q_{x_1}(t,s) - z| \leq \rho(t,s) \cdot \ln(|x_1(s) - x_2(s)| + 1)\},$$

for all $(t,s) \in J \times J$. Since, by (A), L is lower semi-continuous, there exists a continuous function $q_{x_2}(t,s) \in L(t,s)$ for $t,s \in J$. Thus, we have

$$v_2(t) = f(t) + \int_a^b q_{x_2}(t,s)ds \in f(t) + \int_a^b Q(t,s,x_2(s))ds, \quad t \in J$$

and

$$\begin{aligned}
|v_1(t,s) - v_2(t,s)| &\leq \int_a^b |q_{x_1}(t,s) - q_{x_2}(t,s)| ds \\
&\leq \int_a^b \rho(t,s) \cdot \ln(|x_1(s) - x_2(s)| + 1) ds \\
&\leq \ln(d(x_1,x_2) + 1) \int_a^b \rho(t,s) ds \\
&\leq e^{-\tau} \cdot \ln(d(x_1,x_2) + 1) \\
&= e^{-\tau} \cdot \frac{\ln(d(x_1,x_2)+1)}{d(x_1,x_2)} \cdot d(x_1,x_2),
\end{aligned}$$

for each $t \in J$. Hence, we obtain

$$d(v_1,v_2) \leq e^{-\tau} \cdot \frac{\ln(d(x_1,x_2)+1)}{d(x_1,x_2)} \cdot d(x_1,x_2).$$

Interchanging the role of x_1 and x_2, we infer

$$H(Tx_1,Tx_2) \leq e^{-\tau} \cdot \frac{\ln(d(x_1,x_2)+1)}{d(x_1,x_2)} \cdot d(x_1,x_2).$$

Taking logarithm of two sides in above inequality we get

$$\tau + \ln(H(Tx_1,Tx_2)) \leq \ln\left(\frac{\ln(d(x_1,x_2)+1)}{d(x_1,x_2)} \cdot d(x_1,x_2)\right),$$

for all $x_1, x_2 \in X$ with $(x_1,x_2) \in E(G)$ and $x_1 \neq x_2$, Thus, we observe that the operator T satisfies condition (12) with $F(t) = \ln t$, $\beta(t) = \frac{\ln(t+1)}{t}$, $a_1 = 1$ and $a_2 = a_3 = a_4 = a_5 = 0$. All other conditions of Theorem 4.2 immediately follows by the hypothesis. Therefore, T has a fixed point, that is, the Volterra-type integral inclusion (13) has a solution in X. $\square$

6. **Conclusion**

In this paper, we have established some new fixed point theorems in the setting of complete metric spaces by using a new type of F-contractions in which our study gives a more general cases in the study of fixed point theory. An example have been furnished to support of the effectiveness and usability of new theory. We have also derived new fixed point results on a metric space endowed with a partial ordering/graph by means of our main theorems. Finally, we have applied our new theorem to ensure the existence of solutions for Volterra-type integral inclusions under weaker conditions than ones in [4] and [24].

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