THE INCLUSION-EXCLUSION PRINCIPLE AND RECURRENCES FOR PARTITION NUMBERS

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The inclusion-exclusion principle together with Legendre type theorems for number of distinct restricted partitions weighted by the parity of their length are used to give several recurrence relations for restricted partition numbers as well as for overpartition numbers. In the case of overpartitions, POD partitions and 3-color partitions, we give combinatorial proofs for the Legendre type theorems and arrive to combinatorial proofs for the recurrence relations.

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1. Introduction

A *partition* of a non-negative integer n is a non-increasing sequence of positive integers called *parts* that adds up to n. Denote by $\mathcal{P}(n)$ the set of partitions of n and let $p(n) := |\mathcal{P}(n)|$. For example, $p(4) = 5$ because

$$\mathcal{P}(4) = \{(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1)\}.$$

Since $\mathcal{P}(0) = \{\emptyset\}$, we have $p(0) = 1$. A distinct partition is a partitions with distinct parts. See [2] for more on partitions.

Let A be any set of the positive integers. Denote by $\mathcal{P}(n|A)$ (respectively $\mathcal{D}(n|A)$) the set of partitions (respectively distinct partitions) of n with parts in A and set $p(n|A) := |\mathcal{P}(n|A)|$ and $d(n|A) := |\mathcal{D}(n|A)|$. Denote by $\mathcal{D}_e(n|A)$ (respectively $\mathcal{D}_o(n|A)$) the subset of partitions in $\mathcal{D}(n|A)$ with an even (respectively odd) number of parts and set $d_e(n|A) := |\mathcal{D}_e(n|A)|$ and $d_o(n|A) := |\mathcal{D}_o(n|A)|$. If $f(n)$ is the number of partitions of n with certain restrictions and m is not a non-negative integer, set $f(m) := 0$. In general, if the definition of f is unambiguous, we use without further clarification $f_e(n)$ (respectively $f_o(n)$) for the number of partitions counted by $f(n)$ with an even (respectively odd) number of parts and set $f_{eo}(n) := f_e(n) - f_o(n)$ and similarly for $f_{oe}(n)$.

We have the following generating functions

$$\sum_{n\geq 0} p(n|A)q^n = \prod_{i\in A} \frac{1}{1-q^i} \qquad (\mathbf{1}.1)$$

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$$\sum_{n\geq 0} d(n|A)q^n = \prod_{i\in A}(1+q^i) \tag{1.2}$$

and

$$\sum_{n\geq 0} d_{eo}(n|A)q^n = \prod_{i\in A}(1-q^i) \tag{1.3}$$

Multiplying identities (**1**.1) and (**1**.3) we obtain

$$1 = \left(\sum_{n\geq 0} d_{eo}(n|A)q^n\right)\left(\sum_{n\geq 0} p(n|A)\, q^n\right).$$

Equating coefficients in the above identity we have $p(0|A) = 1$ and for $n > 0$,

$$p(n|A) = \sum_{m=1}^{n} d_{oe}(m|A)\, p(n-m|A). \tag{1.4}$$

Identity (**1**.4) can be explained combinatorially using the principle inclusion-exclusion (PIE). To our knowledge, the PIE was first used in to derive partitions identities in [4]. We explain this for the convenience of the reader.

Let $n \geq 1$. For each $1 \leq j \leq n$ let $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ be distinct integers in A. We denote by $\mathcal{P}_{i_1,i_2,\ldots,i_j}(n|A)$ the subset of $\mathcal{P}(n|A)$ with at least one occurrence of each part $i_1, i_2, \ldots, i_j$. By the PIE,

$$|\mathcal{P}(n|A)| = \sum_{j\geq 1}(-1)^{j+1} \sum_{1\leq i_1<i_2<\ldots<i_j\leq n} |\mathcal{P}_{i_1,i_2,\ldots,i_j}(n|A)|.$$

We define a bijection

$$\varphi_{i_1,i_2,\ldots,i_j} : \mathcal{P}_{i_1,i_2,\ldots,i_j}(n|A) \to \mathcal{P}(n-(i_1+i_2+\cdots+i_j)|A)$$

as follows. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{P}_{i_1,i_2,\ldots,i_j}(n|A)$ we define $\varphi_{i_1,i_2,\ldots,i_j}(\lambda)$ to be the partition obtained from λ by removing a single part equal to each $i_1, i_2, \ldots, i_j$. The inverse of this mapping is obtained by inserting a single part equal to each $i_1, i_2, \ldots, i_j$ to a partition in $P(n-(i_1+i_2+\cdots+i_j)|A)$. Hence

$$|\mathcal{P}_{i_1,i_2,\ldots,i_j}(n|A)| = p(n-(i_1+i_2+\cdots+i_j)|A)$$

and

$$p(n|A) = \sum_{j=1}^{n}(-1)^{j+1} \sum_{\substack{1\leq i_1<i_2<\ldots<i_j\leq n\\ i_1,i_2,\ldots,i_j\in A}} p(n-(i_1+i_2+\cdots+i_j)|A). \tag{1.5}$$

Next, we notice that the second sum on the right is indexed by partitions in $\cup_{1\leq m\leq n}\mathcal{D}(m|A)$ with exactly j parts. For $1 \leq j \leq n$, denote by $d_j(m|A)$ the number of partitions in $\mathcal{D}(m|A)$ with exactly j parts. Then identity (**1**.5) becomes

$$p(n|A) = \sum_{j=1}^{n}(-1)^{j+1} \sum_{m=1}^{n} d_j(m|A)\, p(n-m|A).$$

Rearranging, we obtain

$$p(n|A) = \sum_{m=1}^{n} \big(d_o(m|A) - d_e(m|A)\big)\, p(n-m|A) = \sum_{m=1}^{n} d_{oe}(m|A)\, p(n-m|A).$$

If $A = \mathbb{N}$, Franklin's involution [2, pg. 10] gives a combinatorial proof for

$$d_o(m|\mathbb{N}) - d_e(m|\mathbb{N}) = \begin{cases} (-1)^{j+1} & \text{if } m = j(3j-1)/2 \text{ for some } j \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

Combining Franklin's involution with the application of the PIE above, gives a combinatorial proof for Euler's recurrence

$$\sum_{k\in\mathbb{Z}} (-1)^k\, p(n - k(3k-1)/2) = 0.$$

We can expand the use of the PIE to other types of partition identities as follows. A j-colored partition of n is a partition of n whose parts can come in j different colors. For example, a 3-colored partition is a partition in which each part k can come in 3 different colors denoted by subscripts: k_1, k_2, k_3. The parts satisfy the order:

$$1_1 < 1_2 < 1_3 < 2_1 < 2_2 < 2_3 < 3_1 < 3_2 < 3_3 < \dots$$

Let $A = \mathbb{N}_j := \{1_1, 1_2, \dots, 1_j, 2_1, 2_2, \dots, 2_j, 3_1, 3_2, \dots, 3_j, \dots\}$ be the set of integers colored with j colors. Then, the argument above works for j-colored partitions. Similarly, if we consider subsets of $\mathbb{N}_j$ and use the PIE, we obtain identities analogous to (**1**.4) for colored partitions with certain restrictions.

In the next sections we consider applications of this method to obtain combinatorial proofs of recurrence relations. We note that each time, once defined, the set A remains unchanged for the rest of the respective section.

Before we proceed, we introduce some notation. If the parts of a partition λ add up to n, set $|\lambda| := n$. The length of λ, denoted $\ell(\lambda$, is number of parts of λ. Given a partition λ, we sometimes view the partition as a vector of partitions $\lambda = (\lambda^e, \lambda^o)$, where λ^e (respectively λ^o) is the partition consisting of the even (respectively odd) parts of λ.

2. Overpartitions

Recall that an overpartition of a positive integer n is a partition of n in which the first occurrence of a part of each size may be overlined or not. Let $\overline{\mathcal{P}}(n)$ denote the set of overpartitions of an integer n and set $\overline{p}(n) := |\overline{\mathcal{P}}(n)|$. For example, $\overline{p}(4) = 14$ because

$$\begin{aligned} \overline{\mathcal{P}}(4) = \{&(4),\ (\overline{4}),\ (3,1),\ (\overline{3},1),\ (3,\overline{1}),\ (\overline{3},\overline{1}),\ (2,2),\ (\overline{2},2), \\ &(2,1,1), (\overline{2},1,1),\ (2,\overline{1},1),\ (\overline{2},\overline{1},1),\ (1,1,1,1),\ (\overline{1},1,1,1)\}. \end{aligned}$$

Here and throughout, if $\lambda \in \overline{\mathcal{P}}(n)$, we view λ as the vector partition $\lambda = (\overline{\lambda}, \widetilde{\lambda})$ with $\overline{\lambda}$ consisting of the overlined parts of λ (with the overline removed) and $\widetilde{\lambda}$ consisting of the non-overlined parts of λ. Thus, $\overline{\lambda}$ is a distinct partition and $\widetilde{\lambda}$ is an ordinary partition. So the generating function for overpartitions is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)\, q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{(q^2;q^2)_\infty\,(q;q^2)_\infty^2}, \tag{2.1}$$

where we assume that q is a complex number with $|q| < 1$ and the standard q-Pochhammer symbol $(a;q)_\infty$ is defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - a\,q^n).$$

The last equality in (**2**.1) is derived using Euler's identity $(-q;q)_\infty = 1/(q;q^2)_\infty$, which can be proved combinatorially using, for example, a well-known bijection φ_{G} due to Glaisher.

Let $A = \{(2k)_1,\ (2k-1)_1,\ (2k-1)_2 \mid k \in \mathbb{N}\}$ be the subset of $\mathbb{N}_2$ in which odd integers come in two colors while even integers come in one color. Set $\mathcal{CP}(n) := \mathcal{P}(n|A)$, the subset of 2-colored partitions of n with parts in A, and $cp(n) := |\mathcal{CP}(n)|$. Moreover, let $\mathcal{DCP}(n) := \mathcal{D}(n|A)$ and $dcp(n) = |\mathcal{DCP}(n)|$. Define the transformation $\psi : \overline{\mathcal{P}}(n) \to \mathcal{CP}(n)$ as follows. If $\lambda = (\overline{\lambda}, \widetilde{\lambda}) \in \overline{\mathcal{P}}(n)$, let $\varphi_{\mathrm{G}}(\overline{\lambda})$ be the partition with odd parts obtained by applying Glaisher's bijection to $\overline{\lambda}$. Define $\psi(\lambda) \in \mathcal{CP}(n)$ to be the color partition consisting of the parts of $\widetilde{\lambda}$ in color 1 and the parts of $\varphi_{\mathrm{G}}(\overline{\lambda})$ in color 2. The transformation ψ is clearly a bijection. Hence $\overline{p}(n) = cp(n)$.

Using the set A defined above, identity (**1**.4) becomes

Theorem 2.1. *For $n > 0$*

$$cp(n) = \sum_{m=1}^{n} dcp_{oe}(m)\, cp(n-m).$$

Next, we give a combinatorial proof of the following theorem.

Theorem 2.2. *For $n > 0$*

$$dcp_{oe}(n) = \begin{cases} 2\cdot(-1)^{n+1}, & \textit{if } n \textit{ is a square,} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $\overline{p}_e(n)$ (respectively $\overline{p}_o(n)$) denote the number of overpartitions of n into an even (respectively odd) number of parts. In [1], Andrews proved combinatorially that

$$\overline{p}_e(n) - \overline{p}_o(n) = \begin{cases} 2\,(-1)^m & \text{if } n = m^2 \text{ for some } m > 0, \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, to prove the statement of the theorem, it suffices to prove combinatorially that for $n > 0$ we have $dcp_{oe}(n) = \overline{p}_o(n) - \overline{p}_e(n)$.

Let $\mathcal{Q}_{odd}(n)$ be the set of partitions of n into distinct odd parts. Gupta [7] defined an involution φ_{Gu} on $\mathcal{P}(n) \setminus \mathcal{Q}_{odd}(n)$ that reverses the parity of the length of a partition. It shows combinatorially that

$$p_e(n) - p_o(n) = (-1)^n\, Q_{odd}(n).$$

For each partition α with distinct parts and $|\alpha| \le n$ define

$$\overline{\mathcal{P}}_\alpha(n) := \{\lambda = (\overline{\lambda}, \widetilde{\lambda}) \in \overline{\mathcal{P}}(n) \mid \overline{\lambda} = \alpha, \widetilde{\lambda} \notin \mathcal{Q}_{odd}(n - |\alpha|)\}.$$

Then, the transformation

$$(\overline{\lambda}, \widetilde{\lambda}) \mapsto (\overline{\lambda}, \varphi_{\mathrm{Gu}}(\widetilde{\lambda}))$$

shows that the number of overpartitions in $\overline{\mathcal{P}}_\alpha(n)$ with an even number of parts equals the number of overpartitions in $\overline{\mathcal{P}}_\alpha(n)$ with an odd number of parts. The transformation

$$\psi^* : \{\lambda = (\overline{\lambda}, \widetilde{\lambda}) \in \overline{\mathcal{P}}(n) \mid \widetilde{\lambda} \in \mathcal{Q}_{odd}(n - |\overline{\lambda}|)\} \to \mathcal{DCP}(n)$$

that colors $\overline{\lambda}$ in color 1 and $\widetilde{\lambda}$ in color 2 is a bijection that preserves the length of the overpartition. Thus, $\overline{p}_o(n) - \overline{p}_e(n) = dcp_o(n) - dcp_e(n)$. □

Using the bijection ψ, the next corollary is an immediate consequence of Theorems **2**.1 and **2**.2 and our work above gives a combinatorial proof for its statement.

Corollary 2.1. [6, Corollary 4] *For $n > 0$*

$$\overline{p}(n) + 2 \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} (-1)^j \, \overline{p}(n - j^2) = 0.$$

3. POD partitions

Let $\mathcal{POD}(n)$ denote the set of the partitions of n with odd parts distinct and even parts unrestricted and set $pod(n) := |\mathcal{POD}(n)|$. Elementary techniques in the theory of partitions give the following equivalent expressions for the generating function for $pod(n)$:

$$\sum_{n=0}^{\infty} pod(n)\, q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{(q^2; q^4)_\infty}{(q; q)_\infty}. \tag{3.1}$$

The last expression in (**3**.1) is the generating function for $p(n|A)$, where

$$A = \{k \in \mathbb{N} \mid k \not\equiv 2 \pmod 4\}.$$

For a combinatorial proof of $pod(n) = p(n|A)$, see [3].

Set $d_{2,e}(n) := d_e(n|A)$ and $d_{2,o}(n) := d_o(n|A)$. Then, using $pod(n) = p(n|A)$, identity (**1**.4) becomes

Theorem 3.1. *For $n > 0$*

$$pod(n) = \sum_{m=1}^{n} d_{2,oe}(m)\, pod(n - m).$$

Let $\mathcal{PED}(n)$ denote the set of the partitions of n with even parts distinct and odd parts unrestricted and set $ped(n) := |\mathcal{PED}(n)|$. Next, we give analytic and combinatorial proofs of the following theorem.

Theorem 3.2. *For $n \geq 0$*

$$d_{2,eo}(n) = ped_{eo}(n)$$

Analytic proof. For $n, k \geq 0$, we denote by $d_2(n, k)$ the number of partitions in $\mathcal{D}(n|A)$ into exactly k parts. We have the following two-variable generating function for the number of *all* partitions of n into distinct parts incongruent to 2 modulo 4:

$$D_2(z, q) := \sum_{n,k} d_2(n, k)\, z^k q^n = (-zq; q^2)_\infty \, (-zq^4; q^4)_\infty.$$

Thus we deduce that

$$D_2(-1,q)=\sum_{n=0}^{\infty} d_{2,eo}(n)\,q^n=(q;q^2)_\infty\,(q^4;q^4)_\infty=\frac{(q;q)_\infty}{(q^2;q^4)_\infty}=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.$$

Since

$$\sum_{n=0}^{\infty} ped_{eo}(n)\,q^n=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty},$$

this completes the proof. □

Combinatorial proof. Let $\lambda=(\lambda^e,\lambda^o)\in\mathcal{PED}(n)$. Then, $\ell(\lambda)\equiv\ell(\lambda^e)+n \pmod 2$. Let $\mu:=\varphi_{\mathrm{G}}^{-1}(\lambda^o)$ and write $\mu=(\mu^e,\mu^o)$. Let

$$\mathcal{PED}^*(n):=\{\lambda=(\lambda^e,\lambda^o)\in\mathcal{PED}(n)\mid \lambda^e\neq\mu^e\}.$$

We define an involution ζ on $\mathcal{PED}^*(n)$ as follows. Let i be the smallest positive integer such that $\lambda^e_i\neq\mu^e_i$. If $\lambda^e_i>\mu^e_i$ remove part λ^e_i from λ^e and insert it into μ^e, otherwise remove part μ^e_i from μ^e and insert it intro λ^e. Denote the obtained partitions λ^{e*} and μ^{e*}. Write $\mu^*=(\mu^{e*},\mu^o)$ and define $\zeta(\lambda):=(\lambda^{e*},\varphi_{\mathrm{G}}(\mu^*))$. Then ζ is an involution on $\mathcal{PED}^*(n)$ that reverses the parity of $\ell(\lambda)$.

Define

$$\xi:\mathcal{PED}^{**}(n):=\{\lambda=(\lambda^e,\lambda^o)\in\mathcal{PED}(n)\mid\lambda^e=\mu^e\}\to\mathcal{D}(n|A)$$

by $\xi(\lambda)=(2\lambda^e,\mu^o)$, where $2\lambda^e$ is the partition obtained from λ^e by doubling each part. For the inverse, if $\mu=(\mu^e,\mu^o)\in\mathcal{D}(n|A)$, since each part of μ^e is divisible by 4, the partition $\mu^e/2$ obtained from μ^e by dividing each part by 2 is a distinct partition into even parts and $\lambda:=(\mu^e/2,\varphi_{\mathrm{G}}(\mu^e/2,\mu^o))\in\mathcal{PED}^{**}(n)$. Clearly, ξ is an involution that reverses the parity of $\ell(\lambda)$. Hence $ped_{eo}(n)=d_{2,eo}(n)$. □

Corollary 3.1. *For $n>0$*

$$pod(n)=\sum_{m=1}^{n} ped_{oe}(m)\,pod(n-m).$$

Our combinatorial proofs for Theorems **3**.1 and **3**.2 together with Andrews' combinatorial proof [1] for

$$ped_{eo}(n)=\begin{cases}(-1)^n, & \text{if } n=k(k+1)/2\\ 0, & \text{otherwise,}\end{cases}$$

lead to a new combinatorial proof of the following result.

Corollary 3.2. [3, Theorem 1.6] *For $n>0$*

$$\sum_{j=0}^{\infty}(-1)^{j(j+1)/2}\,pod\big(n-j(j+1)/2\big)=0. \qquad (\mathbf{3}.2)$$

4. 3-color partitions

Let $A = \{k_1, k_2, k_3 \mid k \in \mathbb{N}\} = \mathbb{N}_3$. Further let $\mathcal{CP}_3(n) := \mathcal{P}(n|A)$, $\mathcal{DCP}_3(n) := \mathcal{D}(n|A)$ the set of 3-color partitions, and set $cp_3(n) := |\mathcal{CP}_3(n)|$ and $dcp_3(n) := |\mathcal{DCP}_3(n)|$. Set $dcp_{3,e}(n) := d_e(n|A)$ and $dcp_{3,o}(n) := d_o(n|A)$.

Then, (**1**.4) becomes

Theorem 4.1. *For $n \geq 0$,*

$$cp_3(n) = \sum_{j=1}^{n} dcp_{3,oe}(j)\, cp_3(n-j).$$

The generating function for the sequence $d_{3,eo}(n)$ is

$$\sum_{n=0}^{\infty} d_{3,eo}(n)\, q^n = (q;q)_\infty^3.$$

Jacobi [8] gave the following cubic analog of Euler's Pentagonal Number Theorem

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n\,(2n+1)\, q^{n(n+1)/2}. \tag{4.1}$$

For an analytic proof of (**4**.1), see for example [5, Eq. (0.49), p. 17]. Joichi and Stanton [9] gave a combinatorial proof of (**4**.1). Hence, for $n > 0$

$$dcp_{3,eo}(n) = \begin{cases} (-1)^k\,(2k+1), & \text{if } n = k(k+1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

and combining Joichi and Stanton's combinatorial proof [9] with the PIE proof of we obtain a combinatorial proof of Theorem **4**.1, we obtain a combinatorial proof of the following recurrence.

Corollary 4.1. *For $n > 0$*

$$\sum_{j=0}^{\infty} (-1)^j\,(2j+1)\, cp_3\big(n - j(j+1)/2\big) = 0.$$

5. (3,5)-color partitions

A $(3,5)$-color partition is a partition in which each even part can come in 3 different colors and each odd part can come in 5 different colors. Let

$$A = \{(2k)_i, (2k-1)_j \mid k \in \mathbb{N},\ 1 \leq i \leq 3,\ 1 \leq j \leq 5\}.$$

Let $\mathcal{CP}_{3,5}(n) := \mathcal{P}(n|A)$, $\mathcal{DCP}_{3,5}(n) := \mathcal{D}(n|A)$ and set $cp_{3,5}(n) := |\mathcal{CP}_{3,5}(n)|$ and $dcp_{3,5}(n) := |\mathcal{DCP}_{3,5}(n)|$. Set $dcp_{3,5,e}(n) := d_e(n|A)$ and $dcp_{3,5,o}(n) := d_o(n|A)$

Then, (**1**.4) becomes

Theorem 5.1. *For $n \geq 0$,*

$$cp_{3,5}(n) = \sum_{j=1}^{n} dcp_{3,5,oe}(j)\, cp_{3,5}(n-j) = 0.$$

Next, we give an analytic proof of the following theorem.

Theorem 5.2. *For $n \geq 0$,*

$$dcp_{3,5,eo}(n) = \begin{cases} 1-6k, & \text{if } n = k(3k-1)/2,\ k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}.$$

Proof. For $n, k \geq 0$, we denote by $d_{3,5}(n,k)$ the number of 3-color partitions of n into distinct parts, with k parts. The generating function for $d_{3,5}(n,k)$ is given by

$$D_{3,5}(z,q) = \sum_{n,k} d_{3,5}(n,k)\, z^k\, q^n = (-zq;q^2)^3_\infty\, (-zq^2;q^2)^5_\infty.$$

Thus we deduce that

$$D_{3,5}(-1,q) = \sum_{n=0}^{\infty} d_{3,5,eo}(n)\, q^n = (q;q^2)^5_\infty\, (q^2;q^2)^3_\infty = \frac{(q;q)^5_\infty}{(q^2;q^2)^2_\infty}.$$

The generating function for the sequence $d_{3,5,eo}(n)$ can be expressed as

$$\sum_{n=0}^{\infty} dcp_{3,5,eo}(n)\, q^n = (q;q^2)^5_\infty\, (q^2;q^2)^3_\infty = \frac{(q;q)^5_\infty}{(q^2;q^2)^2_\infty}.$$

According to [5, Eq. (0.48), p. 17], this product can be expressed as

$$\frac{(q;q)^5_\infty}{(q^2;q^2)^2_\infty} = \sum_{n=-\infty}^{\infty} (1-6n)\, q^{n(3n-1)/2}. \qquad (\mathbf{5}.1)$$

This concludes the proof. □

Corollary 5.1. *For $n > 0$*

$$\sum_{k\in\mathbb{Z}} (1-6k)\, cp_{3,5}\big(n - k(3k-1)/2\big) = 0.$$

6. (3, 2, 1, 2)-color partitions

A $(3,2,1,2)$-color partition is a partition in which each part congruent to 0 modulo 4 can come in 3 different colors and each part congruent to 1 or 3 modulo 4 can come in 2 different colors Let $\mathcal{P}_{3,2,1,2}(n)$ denote the set of $(3,2,1,2)$-color partitions of n. We set $p_{3,2,1,2}(n) := |\mathcal{P}_{3,2,1,2}(n)|$. So the generating function for $p_{3,2,1,2}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{3,2,1,2}(n)\, q^n = \frac{1}{(q;q^2)^2_\infty\, (q^2;q^4)_\infty\, (q^4;q^4)^3_\infty}.$$

In this context, we remark that

$$\sum_{n=0}^{\infty} (-1)^n\, p_{3,2,1,2}(n)\, q^n = \frac{(q;q)^2_\infty}{(q^2;q^2)^5_\infty}.$$

For $n \geq 0$, we denote by $d_{3,2,1,2,e}(n)$ (respectively $d_{3,2,1,2,o}(n)$) the number of $(3,2,1,2)$-color partitions in $\mathcal{P}_{3,2,1,2}(n)$ with an even (respectively odd) number of distinct parts and set $d_{3,2,1,2,eo}(n) := d_{3,2,1,2,e}(n) - dcp_{3,2,1,2,o}(n)$.

Then, (**1**.4) becomes

Theorem 6.1. *For $n \geq 0$,*

$$p_{3,2,1,2}(n) = \sum_{j=1}^{n} (-1)^{j+1}\, d_{3,2,1,2,eo}(j)\, p_{3,2,1,2}(n-j) = 0.$$

Next, we give an analytic proof of the following theorem.

Theorem 6.2. *For $n > 0$*

$$dcp_{3,2,1,2,eo}(n) = \begin{cases} (-1)^k\,(3k+1), & \text{if } n = k(3k+2),\ k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}.$$

Analytic proof. For $n, k \geq 0$, we denote by $d_{3,2,1,2}(n,k)$ the number of $(3,2,1,2)$-color partitions of n into distinct parts, with k parts. The generating function for $d_{3,2,1,2}(n,k)$ is given by

$$D_{3,2,1,2}(z,q) = \sum_{n,k} d_{3,2,1,2}(n,k)\, z^k\, q^n = (-zq;q^2)^2_\infty\,(-zq^2;q^4)_\infty\,(-zq^4;q^4)^3_\infty.$$

Thus we deduce that

$$D_{3,2,1,2}(-1,q) = \sum_{n=0}^{\infty} d_{3,2,1,2,eo}(n)\, q^n = (q;q^2)^2_\infty\,(q^2;q^4)_\infty\,(q^4;q^4)^3_\infty = \frac{(q^2;q^2)^5_\infty}{(q;q)^2_\infty}.$$

According to [5, Eq. (0.47), p. 17], this product can be expressed as

$$\frac{(q^2;q^2)^5_\infty}{(q;q)^2_\infty} = \sum_{n=-\infty}^{\infty} (-1)^n\,(3n+1)\, q^{n(3n+2)}. \qquad (\mathbf{6}.1)$$

This concludes the proof. □

Corollary 6.1. *For $n > 0$*

$$\sum_{k \in \mathbb{Z}} (-1)^{k(3k-1)}\,(3k+1)\, p_{3,2,1,2}\big(n - k(3k+2)\big) = 0.$$

We conclude the article with an observation. Jacobi's cubic analog of Euler's Pentagonal Number Theorem (**4**.1) can be deduced analytically from the Jacobi triple product identity. However, existing the combinatorial proofs of the Jacobi triple product identity do not reduce to combinatorial proofs of (**4**.1). Using the Involution Principle, Joichi and Stanton [9] argue that the existence of an involution proving combinatorially the Jacobi triple product identity implies the existence of an involution for (**4**.1). They were able to find an explicit involution proving (**4**.1) and did so without the use of the Involution Principle. Similarly, identities (**5**.1) and (**6**.1) can be obtained analytically from the quintuple product identity. It would be interesting to find explicit involutions in the spirit of Joichi and Stanton to prove identities (**5**.1) and (**6**.1) combinatorially.

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