THE COMPLETE TIMOSHENKO FORM OF TORQUE INFLUENCE ON ROTORS LATERAL VIBRATIONS

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The torque carried by a slender rotor has an important influence on the lateral vibration frequencies of the rotor. At this time there in no existing rotordynamics formulation of the torque effect on the lateral vibrations using the more precise Timoshenko beam theory. The present article is proposing to fill this gap and complete the rotordynamics theory with the torque terms containing the shear effect derived from Timoshenko beam theory.

Keywords: rotordynamics, vibration, torque, Timoshenko.

1. Introduction

The first to come with a formulation of the torque influence on the slender rotors lateral vibration was Zorzi and Nelson following the experimental observations of Galomb and Eshleman.[1][2] Earlier studies showed that the torque can produce the failure of slender rotors (shafts) by lateral buckling. This is an extreme case which teach us that even the torque does not achieve the magnitude needed to results in lateral buckling there is an influence, a contribution of this on the lateral displacements of the rotor when subjected to other loads and, in order to obtain correct results for these other cases, this contribution of torque should be precisely accounted for. Zorzi and Nelson solved this problem using the more easy approach of Bernoully Euler beam theory [3] which provide a rather good result but neglects the shear effect, thus rendering results less precise than the Timoshenko beam theory. Today the state of the art solving for the rotordynamics problem involves using the most precise theory which is the Timoshenko beam formulation. Therefore the complete formulation of the torque effect including the shear is required but not available in the present literature. For example a relative recent work on the subject edited by Cambridge University [6] lists just the Euler formulation of the torque effect on the lateral rotordynamic vibrations.

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2. Theoretical aspects

In the effort to derive the relations describing the variation of the fundamental frequency of slender rotors as a function of torque and axial loading the extreme cases are considered. It was shown by previous experiments and demonstrated by Euler [4] and Greenhill [4] that for an amount of compressive axial load and or torque the buckling of the beam will occur.

According to Eshleman and Eubanks [2] the fundamental frequency of a slender rotor (shafts) does not remain constant during load increasing but it is varying and in the case of compressive and torque loads it is decreasing proportional until the buckling phenomena occurs. So the buckling can be redefined as an extreme case of loading when the fundamental frequency of the loaded rotating shaft become zero Hertz. The formula for which the buckling under constant axial load occurs is given by Euler as

$$P = k \cdot \pi^2 \cdot \frac{\alpha}{l^2}, \quad \alpha = E \cdot I \ . \tag{1}$$

The values for k are 1 for short bearings, 4 for long bearings, between 1 and 4 for different combinations of bearings and 0.25 for cantilever rotor.[4]

Regarding the buckling under torque Greenhill gives the following formula

$$M = \pm k \cdot \pi \cdot \frac{\alpha}{l} \tag{2}$$

The values for k are ≥1 for short bearings, 2.861 for long bearings, between 1 and 2.861 for different combinations of bearings and between zero to 1 for cantilever rotor depending of particularities of the application.

A combined case is demonstrated by Ziegler [4] where the torque and tangential compressive load are combining and depending on each other. Using the notations

$$m = \frac{M}{M_0}, \quad p = \frac{P}{P_0}, \tag{3}$$

where with the M_0 is the buckling torque in the absence of compressive load and P_0 is the buckling compressive load in the absence of torque, results the relation between buckling torque M and buckling compressive load P when both are simultaneous acting on the beam is

$$m^2 + p = 1 . \tag{4}$$

Eshleman and Eubanks [2] using as a base the work of Golomb and Rosenberg [4] devised a method to assess the variation of the first fundamental frequency of rotating shaft depending of the amount of torque carried by the shaft. They are providing a rather complex representation of the boundary conditions for a slender rotor in a form of a differential equation accounting for transverse shear, rotating inertia, gyroscopic moments and torque. This is considered in two theoretical cases, for short bearing and for long bearings the intermediate bearings remaining to be considered between the two previous extreme cases.

One important conclusion of Eshleman and Eubanks in their study is that considering Bernoulli-Euler theoretical representation of slender shafts a serious error [2] is introduced in the model and the consideration of shear stresses is mandatory in order to work with exact and safe results.

In order to develop a theoretical tool which can manage the complexity of today aerospace and industrial applications Zorzi and Nelson developed a finite element theoretical extension to Bernoulli Euler beam formulation in order to account for torque and thus to avoid the gross error arising in assessing the natural frequencies for the highly loaded slender shafts.[1]

As the experience showed this theory proved to be quite successful. Nevertheless the more precise Timoshenko beam theory is used in precise calculations of the critical speeds of the shafts. This theory is intrinsically build to account for the shear stresses influence regarding shaft lateral displacements and natural frequencies. As, from the best knowledge of the authors of this article, at this time there in no such extension to the Timoshenko theory accounting for the torque highly loaded slender shafts. Therefore this article main purpose is to fill this theoretical gap and provide a complete Timoshenko theory formulation regarding the finite element calculation of critical speeds of rotating slender shafts and their lateral vibration amplitude A short presentation of the Timoshenko beam theory shows in the figure 1 a rotating element of the considered beam with the sign conventions used for displacements at the both ends.

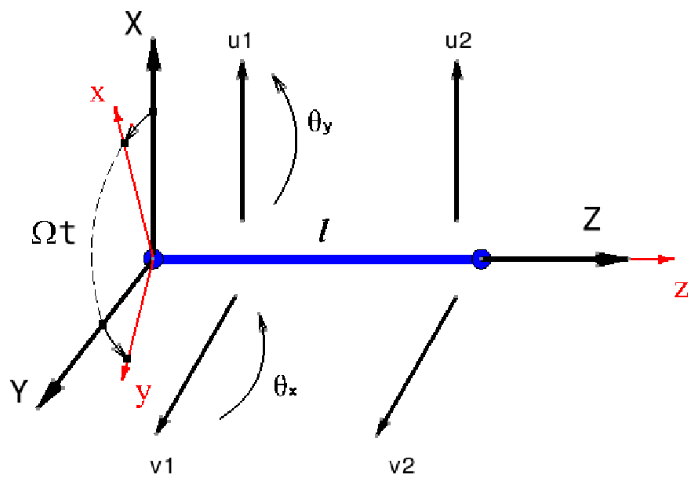

Fig. 1. Timoshenko rotating beam element

The strategy used in finite elements theory is to solve the problem using a finite number of grid points and then extrapolate this results in the rest of the problem domain. Referring to just one finite element this involves calculating displacements first in the element nodes and then find the wanted value for displacement or stress in every point of the element domain. This can be done using the so called shape function. These shape functions are grouped in the shape functions matrix Ψ so that for a considered one-dimensional element presented in the figure 1, for every point on the element there is a value of the s coordinate along the element. Therefore the displacements in every point of the element u(s), v(s), θx(s),θy(s) can be expressed using the shape functions as functions of the displacements at the element grid points (in this case the element two extreme points) u_1,v_1,u_2,v_2,θx1, θy_1, θx_2 and θy_2.

$$\begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = \Psi_T \cdot \begin{bmatrix} u_1 \\ v_1 \\ \theta_{x1} \\ \theta_{y1} \\ u_2 \\ v_2 \\ \theta_{x2} \\ \theta_{y2} \end{bmatrix}. \tag{5}$$

One additional consequence of the Timoshenko beam theory is the particular shape of the stiffness matrix expressed for one beam element as

$$K = \frac{EI}{(1+\Phi)l^3} \cdot \ \dots$$

$$\cdot \begin{bmatrix} 12 & 0 & 0 & 6l & -12 & 0 & 0 & 6l \\ 0 & 12 & -6l & 0 & 0 & -12 & -6l & 0 \\ 0 & -6l & l^2(4+\Phi) & 0 & 0 & 6l & l^2(2-\Phi) & 0 \\ 6l & 0 & 0 & l^2(4+\Phi) & -6l & 0 & 0 & l^2(2-\Phi) \\ -12 & 0 & 0 & -6l & 12 & 0 & 0 & -6l \\ 0 & -12 & 6l & 0 & 0 & 12 & 6l & 0 \\ 0 & -6l & l^2(2-\Phi) & 0 & 0 & 6l & l^2(4+\Phi) & 0 \\ 6l & 0 & 0 & l^2(2-\Phi) & -6l & 0 & 0 & l^2(4+\Phi) \end{bmatrix} \tag{6}$$

where

$$\Phi = \frac{12 \cdot E \cdot I}{k \cdot A \cdot G \cdot l^2}. \tag{7}$$

This stiffness matrix is then integrated in the equation of equilibrium. A general expression of this equation using a noninertial reference system is provided by Vollan and Komzsik in [5] as follows,

$$\begin{aligned}&[M]\{\ddot{g}\}+([D]+2\Omega[C])\{\dot{g}\}+\\&+([K]-\Omega^2[Z]+\Omega^2[K_G]+\Omega[K_D])\{g\}-\{F\}=\{0\}.\end{aligned} \tag{8}$$

The elegance of Vollan approach is that his noninertial theory is built around the classical rotating frame theory by simply adding to the classical stiffness matrix the lines and columns resembling the noninertial character. The objective of this article is not to deal explicitly with these noninertial terms but with the development of the classical subset of the stiffness matrix which is applicable in all the cases, rotating or fully noninertial analysis reference frames.

In order to account for the influence of the torque carried by slender rotors, an addition to the stiffness matrix [K] is needed which will be noted [K_T]. This will be proportional with the torque and will be subtracted from the stiffness matrix such to simulate the softening effect regarding lateral displacements and lateral vibration natural frequencies. So the equilibrium equation will get an additional term [K_T],

$$\begin{aligned}&[M]\{\ddot{g}\}+([D]+2\Omega[C])\{\dot{g}\}+\\&+([K]-[K_T]-\Omega^2[Z]+\Omega^2[K_G]+\Omega[K_D])\{g\}-\{F\}=\{0\}.\end{aligned} \tag{9}$$

This additional stiffness (softening) matrix was determined in the context of Bernoulli Euler beam theory by Zorzi and Nelson and appears in relatively recent books like [6] in this old format.

The purpose of this article is to find the expression of this matrix in the formulation of more exact Timoshenko beam theory with complete consideration of the shear phenomena and shear stresses.

3. Problem solution

The scientist which tackle this difficult issue like Eshleman, Eubanks and Nelson [1][2] are decomposing using the parallelogram rule the torque vector (T) along the two main directions, the direction perpendicular with the beam element section and the direction parallel with the beam section. This can be observed in the figures 2 and 3.

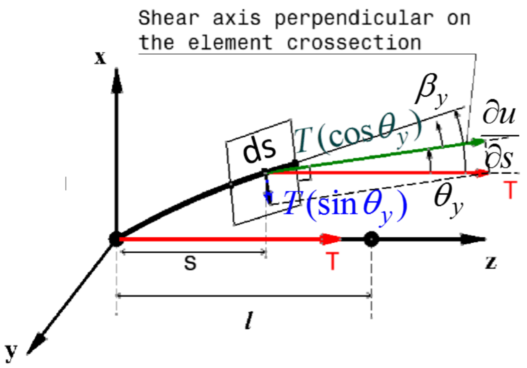

Fig. 2. Torque vector decomposing xOz plane

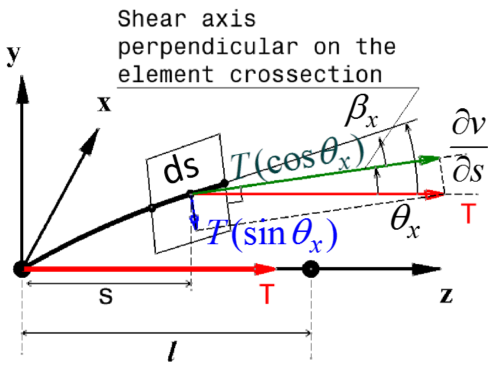

Fig. 3. Torque vector decomposing yOz plane

Considering the approximation associated with small values of θ,

$$T\cdot\sin(\theta_x)\approx T\cdot\theta_x\ ,\quad T\cdot\sin(\theta_y)\approx T\cdot\theta_y \tag{10}$$

The torque decomposing can be further developed at the sections of the extremities of the beam finite element ds such the following figures.

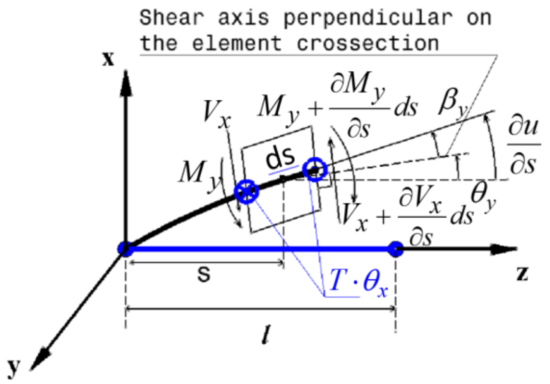

Fig. 4. Force and torque equilibrium in xOz plane

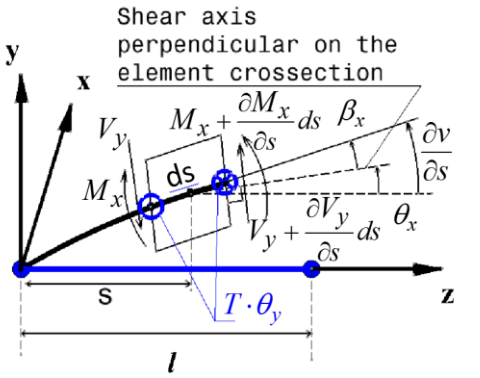

Fig. 5. Force and torque equilibrium in yOz plane

From the figures 4 and 5 one can observe the equilibrium relations.

$$M_x - EI\frac{\partial\theta_x}{\partial s} - T\theta_y = 0, \tag{11}$$

$$M_y + EI\frac{\partial\theta_y}{\partial s} - T\theta_x = 0, \tag{12}$$

and using the sign convention as Eshleman [2]

$$\theta_x = \frac{\partial v}{\partial s} - \beta_x, \tag{13}$$

$$\theta_y = \frac{\partial u}{\partial s} - \beta_y \tag{14}$$

Then using the equation (5.31) from Frishwell [6] we have the expression of shear angle β. Because the sign convention used by Eshleman is opposed comparing with Frishwell the β expressions are used here with changed sign,

$$\beta_x = -\frac{\Phi l^2}{12}\frac{\partial^3 v}{\partial s^3}, \qquad (15) \qquad \beta_y = -\frac{\Phi l^2}{12}\frac{\partial^3 u}{\partial s^3}. \qquad (16)$$

In order to formulate the equilibrium equation the generalized Hamilton principle is used due to the nonconservative nature of the torque [7],

$$\delta\int_{t1}^{t2}(K^e - P^e)dt + \int_{t1}^{t2}\delta W^e dt = 0 . \qquad (17)$$

The first integral of Hamilton principle contains the conservative terms from which are usual obtained using shape function of the finite elements the terms containing the mass matrix and the usual stiffness matrix. The second integral is the place for nonconservative phenomena like the influence of torque on lateral vibrations of slender rotors.

Considering the notations

$$T\cdot\theta_x = M_{tx} , \qquad (18) \qquad T\cdot\theta_y = M_{ty} \qquad (19)$$

the last term, the nonconservative term, from relation (17) becomes

$$\delta W^e = \int_0^l \left(\delta\frac{\partial^2 u}{\partial s^2}M_{tx} - \delta\frac{\partial^2 v}{\partial s^2}M_{ty}\right)ds . \qquad (20)$$

In the relation (20) are inserted the u and v derivatives obtained with the help of relations (13) and (14) where βx and βy are replaced according with the relations (15) and (16). So the equation (20) becomes

$$\delta W^e = \int_0^l \left[\delta\frac{\partial^2 u}{\partial s^2}T\left(\frac{\partial v}{\partial s} + \frac{\Phi l^2}{12}\frac{\partial^3 v}{\partial s^3}\right) - \delta\frac{\partial^2 v}{\partial s^2}T\left(\frac{\partial u}{\partial s} + \frac{\Phi l^2}{12}\frac{\partial^3 u}{\partial s^3}\right)\right]ds. \qquad (21)$$

Expanding further and grouping conveniently the terms, the equation (21) becomes

$$\delta W^e = \int_0^l T\left(\delta\frac{\partial^2 u}{\partial s^2}\frac{\partial v}{\partial s} - \delta\frac{\partial^2 v}{\partial s^2}\frac{\partial u}{\partial s}\right)ds + \int_0^l T\frac{\Phi l^2}{12}\left(\delta\frac{\partial^2 u}{\partial s^2}\frac{\partial^3 v}{\partial s^3} - \delta\frac{\partial^2 v}{\partial s^2}\frac{\partial^3 u}{\partial s^3}\right)ds. \qquad (22)$$

Then further arranging the terms in a matrix format the equation (22) becomes

$$\delta W^e = \int_0^l \left(T \begin{bmatrix} \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial s} \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \dfrac{\partial^2 u}{\partial s^2} \\ \delta \dfrac{\partial^2 v}{\partial s^2} \end{bmatrix} \right) ds + $$
$$+ \int_0^l \left(T \frac{\Phi l^2}{12} \begin{bmatrix} \dfrac{\partial^3 u}{\partial s^3} \\ \dfrac{\partial^3 v}{\partial s^3} \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \dfrac{\partial^2 u}{\partial s^2} \\ \delta \dfrac{\partial^2 v}{\partial s^2} \end{bmatrix} \right) ds. \tag{23}$$

For the equation (20) are used the displacements u and v along the beam element according to the coordinate s. Using the Timoshenko shape functions this can be expressed as function of the element grid points displacements (displacements at the extremities of the beam element). Then with the introduction of the shape functions grouped in the shape functions matrix (5) the relation (23) becomes for the first term

$$\int_0^l \left(T \begin{bmatrix} \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial s} \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \dfrac{\partial^2 u}{\partial s^2} \\ \delta \dfrac{\partial^2 v}{\partial s^2} \end{bmatrix} \right) ds =$$
$$= \{\mathrm{q}^e\}^T \int_0^l \left(T [\Psi_T(s)']^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [\Psi_T(s)''] \right) ds \cdot \delta \{\mathrm{q}^e\}, \tag{24}$$

and the second term of (23) becomes

$$\int_0^l \left(T \frac{\Phi l^2}{12} \begin{bmatrix} \dfrac{\partial^3 u}{\partial s^3} \\ \dfrac{\partial^3 v}{\partial s^3} \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \dfrac{\partial^2 u}{\partial s^2} \\ \delta \dfrac{\partial^2 v}{\partial s^2} \end{bmatrix} \right) ds =$$
$$= \{\mathrm{q}^e\}^T \int_0^l \left(T \frac{\Phi l^2}{12} [\Psi_T(s)''']^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [\Psi_T(s)''] \right) ds \cdot \delta \{\mathrm{q}^e\}. \tag{25}$$

The terms (24) and (25) are further transformed so that the displacements vector $\{q^e\}$ is moved to the right side and then using the notations $K_{T\alpha}$ and $K_{T\beta}$ for the first part. The terms (24) and (25) can be written

$$\{q^e\}^T \int_0^l \left(T[\Psi_T(s)']^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [\Psi_T(s)''] \right) ds \cdot \delta\{q^e\} =$$
$$= \left[\int_0^l \left(T[\Psi_T(s)'']^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\Psi_T(s)'] \right) ds \cdot \{q^e\} \right]^T \cdot \delta\{q^e\} =$$
$$= \left[K_{T\alpha} \cdot \{q^e\} \right]^T \cdot \delta\{q^e\} \qquad (26)$$

and

$$\{q^e\}^T \int_0^l \left(T \frac{\Phi l^2}{12} \left[\Psi_T(s)'''\right]^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left[\Psi_T(s)''\right] \right) ds \cdot \delta\{q^e\} =$$
$$= \left[\int_0^l \left(T \frac{\Phi l^2}{12} \left[\Psi_T(s)''\right]^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left[\Psi_T(s)'''\right] \right) ds \cdot \{q^e\} \right]^T \cdot \delta\{q^e\} =$$
$$= \left[K_{T\beta} \cdot \{q^e\} \right]^T \cdot \delta\{q^e\}. \qquad (27)$$

Using (26) and (27) in the relation (23), this becomes

$$\boxed{\delta W^e = \left[\left(K_{T\alpha} + K_{T\beta} \right) \cdot \{q^e\} \right]^T \cdot \delta\{q^e\}.} \qquad (28)$$

Where $K_T = K_{T\alpha} + K_{T\beta}$ is the stiffness matrix extension which express the influence of the torque transmitted by the analyzed slender rotor according to relation (9).

In order to find the terms of the $K_{T\alpha}$ and $K_{T\beta}$ for one beam element, the matrices, the expressions which they denote are expanded using the actual expressions of the shape functions inside the shape functions matrix Ψ_T. The derivatives and integration are transmitted to the shape functions inside the shape function matrix and after the evaluation of these the following matrices are obtained

$$K_{T\alpha} = T\,\frac{1}{(1+\Phi)^2}\cdot\ \ldots$$

$$\ldots\cdot\begin{bmatrix}
0 & 0 & \frac{\Phi}{l}+\frac{1}{l} & 0 & 0 & 0 & -\frac{\Phi}{l}-\frac{1}{l} & 0 \\
0 & 0 & 0 & \frac{\Phi}{l}+\frac{1}{l} & 0 & 0 & 0 & -\frac{\Phi}{l}-\frac{1}{l} \\
\frac{\Phi^2}{l}+\frac{2\cdot\Phi}{l}+\frac{1}{l} & 0 & 0 & -\frac{\Phi}{2}-\frac{1}{2} & -\frac{\Phi^2}{l}-\frac{2\cdot\Phi}{l}-\frac{1}{l} & 0 & 0 & \frac{\Phi}{2}+\frac{1}{2} \\
0 & \frac{\Phi^2}{l}+\frac{2\cdot\Phi}{l}+\frac{1}{l} & \frac{\Phi}{2}+\frac{1}{2} & 0 & 0 & -\frac{\Phi^2}{l}-\frac{2\cdot\Phi}{l}-\frac{1}{l} & -\frac{\Phi}{2}-\frac{1}{2} & 0 \\
0 & 0 & -\frac{\Phi}{l}-\frac{1}{l} & 0 & 0 & 0 & \frac{\Phi}{l}+\frac{1}{l} & 0 \\
0 & 0 & 0 & -\frac{\Phi}{l}-\frac{1}{l} & 0 & 0 & 0 & \frac{\Phi}{l}+\frac{1}{l} \\
-\frac{\Phi^2}{l}-\frac{2\cdot\Phi}{l}-\frac{1}{l} & 0 & 0 & -\frac{\Phi}{2}-\frac{1}{2} & \frac{\Phi^2}{l}+\frac{2\cdot\Phi}{l}+\frac{1}{l} & 0 & 0 & \frac{\Phi}{2}+\frac{1}{2} \\
0 & -\frac{\Phi^2}{l}-\frac{2\cdot\Phi}{l}-\frac{1}{l} & \frac{\Phi}{2}+\frac{1}{2} & 0 & 0 & \frac{\Phi^2}{l}+\frac{2\cdot\Phi}{l}+\frac{1}{l} & -\frac{\Phi}{2}-\frac{1}{2} & 0
\end{bmatrix}. \quad .(29)$$

$$K_{T\beta} = T\frac{\Phi}{(1+\Phi)^2}\cdot\ \ldots$$

$$\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\Phi}{l}-\frac{1}{l} & 0 & 0 & -\frac{\Phi}{2}-\frac{1}{2} & \frac{\Phi}{l}+\frac{1}{l} & 0 & 0 & -\frac{\Phi}{2}-\frac{1}{2} \\ 0 & -\frac{\Phi}{l}-\frac{1}{l} & \frac{\Phi}{2}+\frac{1}{2} & 0 & 0 & \frac{\Phi}{l}+\frac{1}{l} & \frac{\Phi}{2}+\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\Phi}{l}+\frac{1}{l} & 0 & 0 & \frac{\Phi}{2}+\frac{1}{2} & -\frac{\Phi}{l}-\frac{1}{l} & 0 & 0 & \frac{\Phi}{2}+\frac{1}{2} \\ 0 & \frac{\Phi}{l}+\frac{1}{l} & -\frac{\Phi}{2}-\frac{1}{2} & 0 & 0 & -\frac{\Phi}{l}-\frac{1}{l} & -\frac{\Phi}{2}-\frac{1}{2} & 0 \end{bmatrix}. \quad (30)$$

One can observe that the obtained stiffness matrices expressing the influence of the torque are complying with the rule which state that making the term Φ=0 the Timoshenko beam theory is reduced to the Bernoulli Euler theory. Indeed the $K_{T\beta}$ vanishes being multiplied by zero and the $K_{T\alpha}$ take exactly the shape deduced by Zorzi and Nelson in [1] according the Bernoulli Euler beam theory.

4. Conclusions

The important contribution of this article to the present state of the art in the field of both inertial and noninertial frame rotating machines vibration analysis is the full expression of the stiffness matrix in the Timoshenko beam theory including the contribution of the torque to the evaluation of lateral displacement and vibration frequencies. This is realized by the mean of complete formulation of the equilibrium equation in the Timoshenko theory, equilibrium equation which is the basis to further calculate the system natural frequencies, displacements and tensions. This is especially important as the Timoshenko theory is considerable more accurate than Bernoulli Euler because considers additionally the influence of shear stress and shear displacements.

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