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AN APPROXIMATE SOLUTION OF THE MHD FLOW OVER A NON-LINEAR STRETCHING SHEET BY RATIONAL CHEBYSHEV COLLOCATION METHOD

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The problem of the boundary layer flow of an incompressible viscous fluid over a non-linear stretching sheet is considered. A spectral collocation method is performed in order to find an analytical solution of the governing nonlinear differential equations. The obtained results are finally compared through the illustrative graphs and tables with the exact solution and some well-known results obtained by other researchers. The comparison shows that the obtained results with the rational Chebyshev collocation method are more accurate.

Keywords: Boundary-layer; Non-linear stretching; Rational Chebyshev polynomials; Collocation method.

AMS Classification: 65L60

1. Introduction

In recent years, interest in non-Newtonian fluids has increased due to their several applications in industry and technology. Many materials such as polymer solutions or melts, drilling muds, clastomers, certain oils and greases and many other emulsions are classified as non-Newtonian fluids. It is well known that the governing equations for the non-Newtonian fluids are more non-linear and of higher order than the Navier-Stokes equations [1]. Thus, to find the analytic solutions of such equations is not an easy task. Recently, many problems dealing with non-Newtonian fluids have been solved by analytical methods, such as, the homotopy analysis method (HAM) [2, 3, 4] and see the references therein.

Spectral methods, in the context of numerical schemes for differential equations, generically belong to the family of weighted residual methods. Spectral methods represent a particular group of approximation techniques, in which the residuals (or errors) are minimized in a certain way and thereby leading to specific methods including the Galerkin, Petrov-Galerkin, collocation and Tau formulations. In many papers, various spectral methods are discussed for

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problems in bounded intervals or with special boundary conditions [5, 6, 7, 8, 9, 10]. There are, however, many problems in science and engineering arising in unbounded domains. Several spectral methods for treating unbounded domains have been proposed by different researchers. The some options for unbounded domains fall into three broad categories:

1. For problems posed on a semi-infinite interval $[0,\infty)$, it is natural to consider the usual Laguerre polynomials $L_n(\tau)$ which form a complete orthogonal system in $L^2_\omega(0,\infty)$ with $\omega(\tau)=e^{-\tau}$ [5, 11, 12].
2. When a solution $f(\tau)$ decays rapidly in the direction or directions for which the computational interval is unbounded, then the exact solution can be calculated by solving the differential equation on a large but finite interval (approximation of $\tau\in[0,\infty)$ by $[0,L]$). This strategy for unbounded domains is called domain truncation [5].
3. Another effective direct approach for solving such problems is based on rational approximations [5, 7].

Recently, Parand et al. applied a spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational Tau and spectral methods [13, 14, 15, 16, 17, 18].

2. MHD flow over a non-linear stretching sheet

Let us consider the Magnetohydrodynamic (MHD) flow of an incompressible viscous fluid over a stretching sheet at $y=0$. The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the stretching sheet. The induced magnetic field is neglected. The resulting boundary layer equations are as follows [19]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B^2(x)}{\rho}u, \tag{2.2}$$

where u and v are the velocity components in the x- and y-directions respectively, ν is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity of the fluid. In Eq. (2.2), the external electric field and the polarization effects are negligible and following Chiam [20] we assume that the magnetic field B takes the form

$$B(x)=B_0x^{(n-1)/2}.$$

The boundary conditions corresponding to the non-linear stretching of the sheet are

$$u(x,0)=cx^n,\qquad v(x,0)=0,$$

$$u(x,y)\to 0 \quad as \quad y\to\infty,$$

where c and n are constants. Upon making use of the following substitutions:

$$\tau=\sqrt{\frac{c(n+1)}{2\nu}}x^{(n-1)/2}y, \qquad u=cx^n f'(\tau),$$

$$v=-\sqrt{\frac{c\nu(n+1)}{2}}x^{(n-1)/2}\left[f(\tau)+\frac{n-1}{n+1}\tau\ f'(\tau)\right],$$

the resulting non-linear differential system is of the following form:

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\tau^3}+f\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2}-\beta\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^2-M\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)=0, \quad (2.3)$$

$$f(0)=0, \quad f'(0)=1, \quad \lim_{\tau\to\infty} f'(\tau)=0, \quad (2.4)$$

where

$$\beta=\frac{2n}{1+n}, \quad M=\frac{2\,\sigma\,B_0^2}{\rho c(1+n)}.$$

In [19], Hayat et al. employed the modified Adomian decomposition method with the Padé approximant and developed the series solution of the governing non-linear problem (2.3)-(2.4). Rashidi in [21] used the differential transform method with the Padé approximant and obtained analytical solutions for this problem. Recently, authors in [22, 23] employed the HAM in order to obtain an analytical solution of the governing nonlinear differential equations. For the special case of $\beta=1$, the exact analytical solution of (2.3)-(2.4) as given in [24] is

$$f(\tau)=\frac{1-\exp(-\sqrt{1+M}\,\tau)}{\sqrt{1+M}}. \quad (2.5)$$

The purpose of this paper is to employ an important type of spectral methods called the rational Chebyshev collocation method, that has already been successfully applied to some nonlinear problems, for solving the problem (2.3)-(2.4).

3. Rational Chebyshev polynomials

A commonly used sets of orthogonal polynomials are the rational Chebyshev polynomials. In this section, we will present some of their basic properties.

The well-known Chebyshev polynomial $T_l(x)$ is the lth normalized eigenfunction of the singular Sturm-Liouville problem:

$$\sqrt{1-x^2}[\sqrt{1-x^2}T_l'(x)]'+l^2T_l(x)=0, \quad x\in(-1,1).$$

Also the Chebyshev polynomials satisfy the following three-term recurrence relation:

$$T_0(x)=1,\quad T_1(x)=x,$$
$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\quad n\geq 1,$$

and are orthogonal in the interval $[-1,1]$ with respect to the weight function $\omega(x)=1/\sqrt{1-x^2}$,

$$\int_{-1}^{1}T_i(x)T_j(x)\omega(x)dx=\frac{c_i\pi}{2}\delta_{ij},$$

where $c_0=2$, $c_i=1$ for $i\geq 1$ and δ_{ij} is the Kronecker function. From the above relations it is evident that the well-known Chebyshev polynomials are valid only for $x\in[-1,1]$, but for problems with semi-infinite domain, by using a transformation that maps a semi-infinite interval into a finite domain, it is possible to generate a great variety of new basis sets for the semi-infinite interval that are the images under the change-of-coordinate of Chebyshev polynomials. For this purpose, Boyd [5, 25, 26, 27] presented algebraic maps in the following form

$$\tau=L\frac{1+x}{1-x}\quad\leftrightarrow\quad x=\frac{\tau-L}{\tau+L},$$

where L is a constant parameter. The presented algebraic maps for every fixed L, map the semi-infinite interval $[0,\infty)$ into $[-1,1]$, and

$$R_l(\tau)=T_l\left(\frac{\tau-L}{\tau+L}\right)=\cos(lt),\qquad t=2\cot^{-1}\left(\sqrt{\frac{\tau}{L}}\right),\quad t\in[0,\pi]. \tag{3.1}$$

So the rational Chebyshev polynomials can be defined as the following three-term recurrence relation:

$$R_0(\tau)=1,\quad R_1(\tau)=\frac{\tau-L}{\tau+L},$$
$$R_{n+1}(\tau)=2\left(\frac{\tau-L}{\tau+L}\right)R_n(\tau)-R_{n-1}(\tau),\quad n\geq 1.$$

It can be shown that $R_l(\tau)$ is the lth eigenfunction of the singular Sturm-Liouville problem

$$(\tau+L)\frac{\sqrt{\tau}}{L}[(\tau+L)\sqrt{\tau}R_l'(\tau)]'+l^2R_l(\tau)=0,\quad \tau\in(0,\infty),$$

and rational Chebyshev polynomials are orthogonal with respect to the weight function $\omega(\tau)=\dfrac{\sqrt{L}}{\sqrt{\tau}(\tau+L)}$ in the interval $[0,\infty)$, with the orthogonality property:

$$\int_0^{\infty}R_i(\tau)R_j(\tau)\omega(\tau)d\tau=\frac{c_i\pi}{2}\delta_{ij}, \tag{3.2}$$

where $c_0 = 2$, $c_i = 1$ for $i \geq 1$.

Let $I = [0, \infty)$ and $\omega(\tau) = \dfrac{\sqrt{L}}{\sqrt{\tau}(\tau + L)}$ be a weight function over the interval I. We define

$$L^2_\omega(I) = \{v \mid v\, is\, measurable\, on\, I\, and \| v \|_\omega < \infty\},$$

where

$$\| v \|_\omega = \left(\int_0^\infty | v(\tau) |^2 \ \omega(\tau) d\tau \right)^{\frac{1}{2}}.$$

We denote by $\langle u, v \rangle_\omega$ the inner product of the space $L^2_\omega(I)$, i.e.

$$\langle u, v \rangle_\omega = \int_0^\infty v(\tau) u(\tau) \omega(\tau) d\tau.$$

Hence, from the orthogonality relation of rational Chebyshev polynomials (3.2) and the fact that the rational Chebyshev polynomials $R_l(\tau)$ form a set of orthogonal basis for $L^2_\omega(I)$, for any function $f \in L^2_\omega(I)$ the following expansion holds

$$f(\tau) = \sum_{i=0}^{\infty} f_i R_i(\tau), \tag{3.3}$$

with

$$f_i = \frac{\langle f, R_i \rangle_\omega}{\| R_i \|_\omega^2},$$

the f_i's are the expansion coefficients associated with the family $\{R_i\}_{i \geq 0}$.

4. Rational Chebyshev collocation method

Let N be any positive integer, and $\Re_N = span\{R_0, R_1, \ldots, R_N\}$. Then the spectral approximation is of the form

$$f_N(\tau) = \sum_{k=0}^{N} f_k R_k(\tau). \tag{4.1}$$

The collocation approximation is to find the coefficients f_k such that the residual function equals to zero at the interior collocation points $\{\tau_j\}_{j=0}^N$. In the rational Chebyshev collocation method for solving problem (2.3) with boundary conditions (2.4), we use the $N+1$ rational Chebyshev-Gauss-Radau points as the collocation points in the following form

$$\tau_j = L \frac{1 + x_j}{1 - x_j}, \quad j = 0, 1, \ldots, N, \tag{4.2}$$

where x_j 's are the $N+1$ Chebyshev-Gauss-Radau points;

$$x_j = \cos(\frac{2j\pi}{2N+1}), \quad j = 0,1,\dots,N.$$

Now we apply the rational Chebyshev collocation method presented above for solving problem (2.3) with boundary conditions (2.4). For this purpose we define the following residual function for any $f_N \in \Re_N$ as an approximate solution of the problem

$$Res_N(\tau) = \frac{\mathrm{d}^3 f_N}{\mathrm{d}\tau^3} + f_N \frac{\mathrm{d}^2 f_N}{\mathrm{d}\tau^2} - \beta(\frac{\mathrm{d}f_N}{\mathrm{d}\tau})^2 - M(\frac{\mathrm{d}f_N}{\mathrm{d}\tau}). \qquad (4.3)$$

So from the collocation method we have

$$Res_N(\tau_j) = 0, \quad j = 1,2,\dots,N-1,$$
$$f_N(0) = 0, \; f'_N(0) = 1 \qquad (4.4)$$

where τ_j 's are the rational Chebyshev-Gauss-Radau points presented in (4.2). Taking into account $R'_i(\infty) = 0$, for $i = 0,1,\dots,N$ the infinity boundary condition $f'_N(\infty) = 0$ is already satisfied. System (4.4) contains $N+1$ nonlinear equations, it can be solved for $N+1$ unknowns f_k (the expansion coefficients of $f_N(\tau)$ in term of the polynomials orthogonal with $\omega(\tau)$). This nonlinear system can be solved by Newton's method.

To obtain the order of convergence of rational Chebyshev approximation, we need to investigate several orthogonal projections. From Eq. (4.1), it is evident that f_N is the orthogonal projection of f upon $\Re_N$ with respect to the weighted inner product $\langle .,. \rangle_\omega$. In general we define the $L^2_\omega(I)$-orthogonal projection $P_N : L^2_\omega(I) \rightarrow \Re_N$ by

$$\langle P_N f - f, \phi \rangle_\omega = 0, \quad \forall \phi \in \Re_N,$$

where $P_N f(\tau) = f_N(\tau)$. In order to estimate $\|P_N f - f\|_\omega$, we define the normed space

$$H^r_\omega(I) = \{ v \mid v \textit{ is measurable on } I \textit{ and } \|v\|_{r,\omega} < \infty \},$$

where the norm is induced by

$$\|v\|_{r,\omega} = \left(\sum_{k=0}^{r} \left\| (\tau+1)^{\frac{r}{2}+k} \frac{d^k}{d\tau} v \right\|_w^2 \right)^{\frac{1}{2}}.$$

We have the following convergence theorem: For any $f \in H^r_\omega(I)$ and $r \geq 0$,

$$\| P_N f - f \|_\omega \le cN^{-r} \| f \|_{r,\omega} .$$

proof: see [28].

This theorem shows that the rational Chebyshev approximation has exponential convergence. In the next section we present some results obtained by the rational Chebyshev collocation method for problem (2.3).

5. Numerical results

In this section the rational Chebyshev collocation method is applied to obtain the approximate solution of the problem (2.3) with some typical values of parameters. In the application of the rational Chebyshev collocation method for problems with semi-infinite domain $[0,\infty)$, the difficulty is in choosing the optimal map parameter L. Boyd in [25] offered guidelines for optimizing the map parameter L.

For the special case $\beta = 1$, the exact analytical solution for problem (2.3) is available (2.5). In this paper, the residual function on the domain $(\| Res \|_2 = (\int_0^\infty | Res |^2 \, d\tau)^{\frac{1}{2}})$ and the maximum norm of error function on the domain $\| Err \|_\infty = \{\max | f_N(\tau) - f(\tau) | : \tau \in [0,\infty)\}$ are employed for checking the accuracy of the presented method. The approximations of $f''(0)$ for the problem (2.3) with $\beta = 1$ and $M = 50$ computed by the present method with suitable L and their relative errors are shown in Table 1.

Table 1

Numerical results for the $f''(0)$ and the maximum norm of error function for $\beta = 1$, $M = 50$ and several values of N.

N	L	$f''(0)$	$\| Err \|_\infty$
10	0.819	-7.1404682462	4.500×10^{-6}
15	1.451	-7.1414357227	5.267×10^{-8}
20	2.111	-7.1414283589	5.163×10^{-10}
25	2.339	-7.1414284276	5.884×10^{-12}
30	2.584	-7.1414284285	7.145×10^{-14}
exact	-	-7.1414284285	-

Obviously, this method is convergent by increasing the number of points and obtaining a suitable L and also it is evident that the presented method can compute the unknown value $f''(0)$ with high accuracy. The comparison between the exact and approximation solution of problem (2.3) with $\beta = 1$ and for several values of magnetic parameter (M) have been shown in Fig.1.

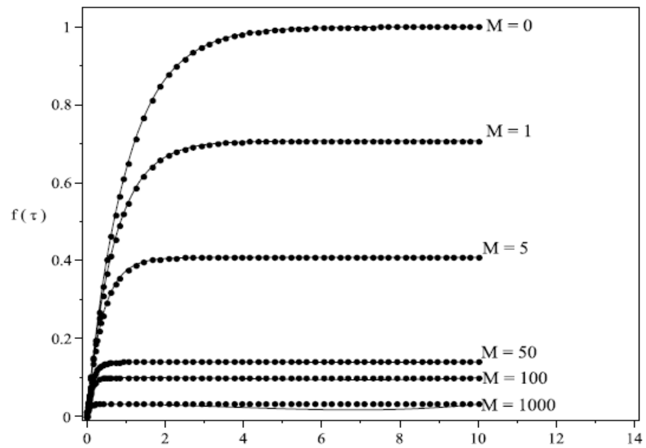

Fig. 1: Comparison of the solution obtained by the rational Chebyshev collocation (circle) and the exact solution (line) with $\beta = 1$, $N = 20$ and several values of M.

A very good agreement was illustrated between the results obtained by the rational Chebyshev collocation method and the exact values for all values of τ.

For $\beta \neq 1$, there are no explicit exact solutions found for problem (2.3), but some semi-analytical methods have been applied for solving (2.3). For two cases of problem's parameters $\beta = -1,\ M = 1$ and $\beta = 5,\ M = 10$, the approximation values of $f''(0)$ and also the norm-2 of residual function obtained by the proposed approach with some numbers of collocation point and a suitable L are given in Tables 2 and 3.

Table 2

Numerical results for the $f''(0)$ and norm-2 of residual function for $\beta = -1,\ M = 1$ and several values of N.

N	L	$f''(0)$	$\Vert Res \Vert_2$
10	2.811	-0.8511182003	3.960×10^{-4}
15	3.436	-0.8511091034	1.890×10^{-5}
20	3.511	-0.8511095789	4.421×10^{-7}
25	3.750	-0.8511095738	3.764×10^{-8}
30	4.071	-0.8511095740	6.093×10^{-10}
[19]	-	-0.8511	-

Table 3

Numerical results for the $f''(0)$ and norm-2 of residual function for $\beta = 5$, $M = 10$ and several values of N.

N	L	$f''(0)$	$\Vert Res \Vert_2$
10	2.726	-3.6956084166	6.146×10^{-4}
15	3.021	-3.6956556846	8.795×10^{-6}
20	2.639	-3.6956559955	1.894×10^{-7}
25	3.201	-3.6956559936	2.906×10^{-9}
[19]	-	-3.6956	-

It is evident that this method is convergent (decreasing norm-2 of residual function) by increasing the number of points and obtaining a suitable L. In these cases there are no exact values of $f''(0)$ available for comparison, but we believe that the results given in Tables 2 and 3 by choosing $N = 30$ and $N = 25$, respectively, are high estimates for $f''(0)$ and are accurate to the last decimal positions.

In Figs. 2-4 the variations of $f'(\tau)$ and $f(\tau)$ approximated by the presented method for some typical problem's parameters are plotted that agree with boundary conditions (2.4).

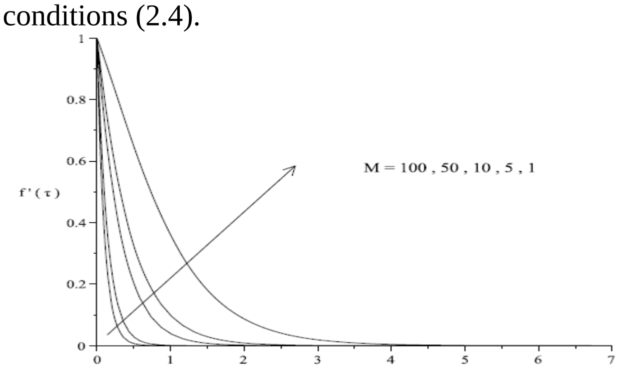

Fig. 2: Effect of M on $f'(\tau)$ obtained by rational Chebyshev collocation method with $N = 20$ when $\beta = -1.5$.

Finally, logarithmic graphs of the absolute coefficients $|f_i|$ of the rational Chebyshev functions in the approximate solutions for $\beta = -1, M = 1$ and $\beta = 5, M = 10$ with a suitable L are shown in Figs. 5 and 6, respectively. The graphs illustrate that the method has an appropriate convergence rate.

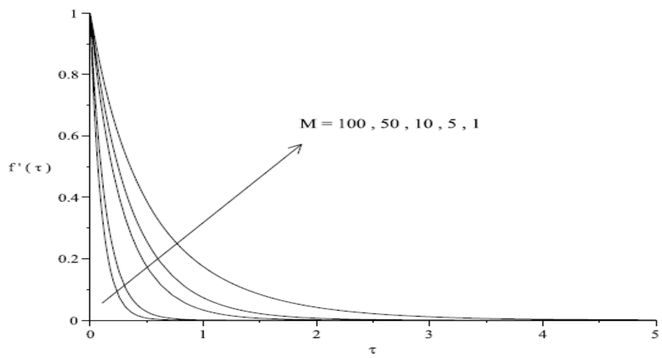

Fig. 3: Effect of M on $f'(\tau)$ obtained by rational Chebyshev collocation method with $N = 20$ when $\beta = 5$.

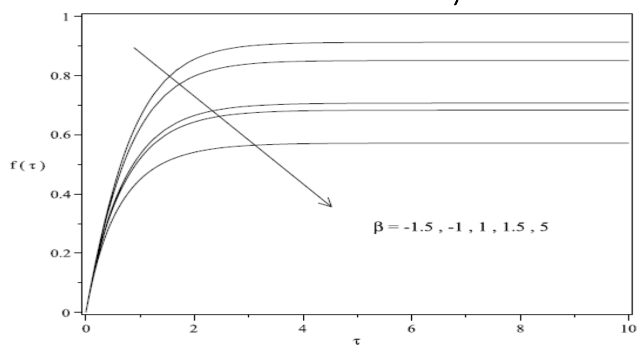

Fig. 4: Effects of parameter β on analytical solution of $f(\tau)$ obtained by the rational Chebyshev collocation method with $N = 20$ when $M = 1$.

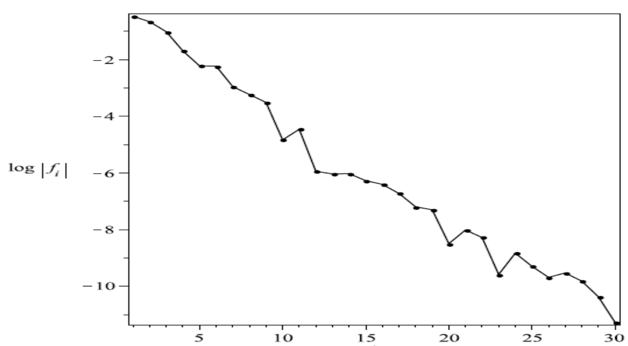

Fig.5: Logarithmic graph of absolute coefficients $|f_i|$ of Rational Chebyshev functions in the approximate solution for $\beta = -1$, $M = 1$ and $L = 4.071$.

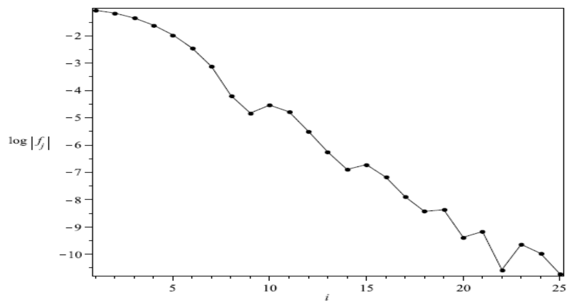

Fig. 6: Logarithmic graph of absolute coefficients $|f_i|$ of Rational Chebyshev functions in the approximate solution for $\beta = 5$, $M = 10$ and $L = 3.201$.

6. Conclusions

In the present work, an efficient and accurate numerical method based on orthogonal functions is successfully applied to get analytical solution of the boundary layer flow of an incompressible viscous fluid over a non-linear stretching sheet. The numerical solutions are given for different values of the problem's parameters by using the collocation method with choosing the rational Chebyshev polynomials as the basis functions, which these basis functions have some advantages: easy to compute, rapid convergence and completeness, which means that any solution can be represented, and very efficient for problems with semi-infinite interval. Comparing the computed results by this method with the other methods shows that this method provides more accurate and numerically stable solutions than those obtained by other methods.

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