QUALITY EVALUATION OF NON-MARKETABLE ACTIVITIES BASED ON A NEW OUTPUT DIRECTIONAL MEASURE FUNCTION

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În această lucrare sunt aduse completari unei teorii actuale de mare interes pentru domeniul aplicativ, anume acela al cuantificării activităţilor sociale caracterizabile preponderent sau în totalitate prin rezultate de natură calitativă. Foarte exact, în această lucrare noi propunem o alternativă nouă pentru funcţia Shephard şi pentru funcţia direcţionala Malmquist–Luenberger de măsurare a "cantităţii" de "calitate" rezultate în urma unei activităţi non-marketabile cum ar fi o activitate de instruire profesională sau ştiinţifică, iar pe baza ei redefinim atât indecşii Caves-Christensen-Diewert, cât şi indexul Malmquist-Luenberger de măsurare a productivităţii unor astfel de activităţi.

In this paper there are made additions to a current theory from the practical field, namely that field which quantifies social activities mainly or totally characterized by qualitative results. More precisely, within this paper we propose a new alternative to the Shephard function and to the directional Malmquist – Luenberger function to measure the "quantity" of "quality" which results out of a non-marketable professional training or scientific activity, and based on it we redefine the Caves-Christensen-Diewert indices, as well as the Malmquist-Luenberger index which measures the productivity of such activities.

Keywords: Non-marketable activities, measure of quality, Shephard output measure, directional output measure, training processes productivity, Malmquist, respectively, Malmquist – Luenberger directional productivity index.

1. Introduction

The mathematical modelling of the notion of "productivity of educational, training, scientific or professional development", in a nutshell the mathematical modelling of "non-marketable"[3] processes, has for a long time been a subject of

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[3] The notion of "non-marketable services" was developed by R. Färe, S. Grosskopf, F. R. Forsund, K. Hayes, A. Heshmati, and introduced in [4].

interest for the researchers concerned with the study of phenomena lying at the cross-road between various fields.

The interest in the subject is justified by at least two reasons: the objective perspective is given by the necessity to quantify non-marketable activities, while the subjective perspective relies on the difficulty to elaborate means which quantitatively evaluate, preponderantly, or even exclusively qualitative notions, such as the ones describing the processes chosen as a research object.

To this subject we have brought our own contributions through several papers (see, for example, [10] and [11]), wherein we have tried to extend fundamental results presented in pioneering works for this field, such as [1, 2, 3, 4, 5].

In this paper we shall present our own results meant to accomplish the results obtained so far within the literature in the field. The first result refers to a new index which measures the quality of outcomes generated by non-marketable activities. The new index proposed by us aims at generalizing the classic Shepard measure, as well as improving the properties of the Malmquist-Luenberger directional measure based on which Shepard's successors tried to generalize his ideas. The second result is a consequence of the first one. Namely, by applying a new way to express the "quantity" of quality resulted from a non-marketable activity we shall give a new meaning to the notions used in order to quantify the process dynamics, by using an extension of some productivity indices such as the Caves-Christensen-Diewert or Malmquist-Luenberger.

The content of this paper is structured as it follows:

Sections 2 and 3 describe the components of technologies with quality outputs, as well as the way these are built up. Due to the relationship between the topic developed in this paper and the one discussed in [10] and [11], these two sections have been taken from the papers previously cited.

Section 4 reviews the main mathematical methods used to measure the outputs quality. Within this section there are presented the theories developed by R. Färe, S. Grosskopf, F. R. Forsund, K. Hayes, A. Heshmati in [4] and by Chung, Färe and Grasskopf in [2], as well as the additions made by us to these theories and presented in papers [10] and [11].

Section 5 focuses on ways to define the new function proposed in this paper which measures quality based outputs, to present the main features of this function and to discuss the modality it relates to well-known functions.

In section 6, as in section 4, we aim at briefly presenting the existent modalities to evaluate the notion of productivity of non-marketable activities, notion, which intuitively, is meant to measure the "quality" surplus (or deficit) registered from one stage to the other. On this occasion, next to points of view related to the classical theory presented in papers [1, 2, 3, 4, 5], we shall also discuss our opinions presented in papers [10] and [11] with regard to these approaches.

As in section 5, section 7 highlights the new results. Within this section we shall define the formulas proposed by us for the productivity computation of non-marketable activities. The difference between the new and the old formulas is given by the functions used (which differ) to express the performances achieved at different time moments within the activity which needs to be monitored.

The formulas used to evaluate the output quality do not generally employ direct computation modalities, and the formulas proposed by us make no exception in this respect. From this reason, section 8 focuses on proposing a way in order to effectively calculate this formula. The advantages deriving from using formulas to compute the productivity of non-marketable activities proposed by us in this work are discussed in section 9.

As in [10] and [11], the last section of this paper focuses on demonstrating the mathematical properties of the elements that build up the apparatus used to determine the productivity of training activities, as well as the relationships between them. This section can be ignored by non-mathematicians. The reason why we have not given up this section relies on the fact that the theory presented herein plays the role to mathematically fundament the instruments proposed in this paper.

2. Design technology with quality outputs

In accordance with our point of view [6 - 9], any training process can be modelled by applying the abstract notion of dynamic system wherein the inputs are given by the teaching staff and logistic infrastructure of the institution, which is the service provider, while the outputs are given by students' performances within various educational activities, or graduates' professional performances.

In this section we shall present the mathematical apparatus in order to quantify the qualitative performances expected at the end of each training stage. This apparatus, taken from [10], represents a specialization of the technology used in [2, 4] to measure the outputs, so that it (the new technology) becomes capable of hierarchically displaying "purely" qualitative outputs.

Mathematically, the inputs of the training systems will be expressed by means of vector $\mathbf{x} \in \mathbb{R}^{m}_{+}$, whose components represent the logistics of the services providing institution. Explicitly these can be: the number of teachers involved in the program, the number of available seats in a program, the actual number of lecture hours per student, available space per student (measured in m^2), the number of teaching materials used, or their costs, indicators related to medical assistance, or to the administrative support of the institution, etc.

Similarly, the outputs of the training systems will be mathematically expressed in a vectorial mode. In this case, the components of an output individualized by a vector $\mathbf{y} \in \mathbb{R}^n_+$, may be: the average promotability, the average competence acquisition, the average competitiveness degree, etc.

Interdependences between an input signal (logistics based resource) $\mathbf{x} \in \mathbb{R}^m_+$ and the accompanying output signals (as outcomes of the training process reflected through their quality) are modelled in [2 - 4] by means of a set

$$\mathcal{P}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n_+ : \mathbf{x} \text{ can produce } \mathbf{y}\},$$

named technology. This denomination derives from the fact that $\mathcal{P}(\mathbf{x})$ expresses an enterprise's (in this case an education institution's) capability of producing "something" based on certain resources "$\mathbf{x}$", which with regard to the research object may either refer to one of the following, namely, certain didactical facilities, human resources or logistics, such as teaching staff, libraries, laboratories etc., or to particular organizational patterns, or to all of them.

In numerous applications the "technology" we refer to, needs to be time related. In this case, in order to mark the moment in time t we refer to, the generic notation $\mathcal{P}(\mathbf{x})$, assigned to the notion of technology, will be replaced by $\mathcal{P}^t(\mathbf{x})$.

In order to concretely express the notion of "technology", set $\mathcal{P}(\mathbf{x})$, or $\mathcal{P}^t(\mathbf{x})$ which designates it, has to fulfil certain specific properties. For example, the intuitive perception, such as the assertion that if results $\mathbf{y}$ are obtained by using resources $\mathbf{x}$, then results $\mathbf{y}'$ worse than $\mathbf{y}$ are obtained by using the same resources $\mathbf{x}$, can be modelled in two distinct ways:

1) by the condition

$$\forall \mathbf{y} \in \mathcal{P}(\mathbf{x}) \text{ and } \forall \theta \in [0,1] \Rightarrow \theta\mathbf{y} \in \mathcal{P}(\mathbf{x}),$$

considered in [2] as the "weak" form to express the property under discussion; and

2) by the condition[4]

$$\forall \mathbf{y} \in \mathcal{P}(\mathbf{x}) \text{ and } \forall \mathbf{y}' \in \mathbb{R}^n_+,\ \mathbf{y}' \le \mathbf{y} \Rightarrow \mathbf{y}' \in \mathcal{P}(\mathbf{x}),$$

considered in [2] as the "strong" form to express the property under discussion.

3. A practical modality to construct technologies with quality outputs

An important issue consists in shifting from the descriptive form regarding the notion of technology to the practical one. In this section we shall present the

[4] The order relationship we refer to herein is the usual order relationship of any space $\mathbb{R}^n$, namely $\mathbf{y}' = (y'_1,..,y'_n) \le \mathbf{y} = (y_1,..,y_n) \Leftrightarrow y'_k \le y_k,\ k = 1,2,..,n$.

solution offered by Chung et al. in [2]. According to these authors, the technology at a given moment in time t is determined based on K observations[5] of the inputs and outputs $\left(\mathbf{x}^t{}_k, \mathbf{y}^t{}_k\right)$, $k = 1,2,..,K$, regarding certain education systems considered as reference systems. Thus, for a vector of the inputs $\mathbf{x} = \left(x_1,..,x_m\right) \in \mathbb{R}^m{}_+$, technology $\mathcal{P}^t\left(\mathbf{x}\right)$, corresponding to resource $\mathbf{x}$ at a moment in time t, is defined as a set of all the outputs $\mathbf{y} = \left(y_1,..,y_n\right) \in \mathbb{R}^n{}_+$ for which $\exists z_k \geq 0$, $k = 1,2,..,K$, so that $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \leq \mathbf{x}$, and $\sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \geq \mathbf{y}$.

Set $\mathcal{P}^t\left(\mathbf{x}\right)$, thus defined, fulfils more important properties[6] for the theory we are going to present, namely:

1) the property of constant returns to scale, i.e.

$$\mathcal{P}^t\left(\lambda\mathbf{x}\right) = \lambda\mathcal{P}^t\left(\mathbf{x}\right), \forall\lambda > 0;$$

2) the property of strong disposability of inputs, i.e.

$$\mathbf{x} \leq \mathbf{x}' \Rightarrow \mathcal{P}^t\left(\mathbf{x}\right) \subseteq \mathcal{P}^t\left(\mathbf{x}'\right);$$

3) the property of weak disposability of undesirable outputs, i.e.

$$\forall\,\mathbf{y} \in \mathcal{P}^{\mathbf{t}}\left(\mathbf{x}\right) \text{ and } \forall\,\theta \in \left[0,1\right] \Rightarrow \theta\mathbf{y} \in \mathcal{P}^t\left(\mathbf{x}\right);$$

4) the property of free disposability of undesirable outputs, i.e.

$$\forall\,\mathbf{y} \in \mathcal{P}^{\mathbf{t}}\left(\mathbf{x}\right) \text{ and } \forall\,\mathbf{y}' \leq \mathbf{y} \Rightarrow \mathbf{y}' \in \mathcal{P}^{\mathbf{t}}\left(\mathbf{x}\right);$$

5) the set $\mathcal{P}^t\left(\mathbf{x}\right)$ is convex, i.e.

$$\mathbf{u}, \mathbf{v} \in \mathcal{P}^t\left(\mathbf{x}\right) \Rightarrow \left(1-\lambda\right)\mathbf{u} + \lambda\mathbf{v} \in \mathcal{P}^t\left(\mathbf{x}\right), \forall\lambda \in \left[0,1\right].$$

4. Measuring quality outputs. The classical theory

In order to evaluate the quality of the outputs $\mathbf{y}$ corresponding to some resources specified in the components of various vectors $\mathbf{x}$, and some given technologies $\mathcal{P}\left(\mathbf{x}\right)$, several ways can be used:

1) like Färe et al. [4], we can use the Shephard measure defined by

[5] The modality to determine the value of number $K \geq 1$, and the one to choose K observations depends on each researcher, but for the results of the study to become relevant, they need to be significant for the set goal, thus, the use of statistical selection rules becomes necessary.

[6] The justification of the assertions is presented in section 10 of this paper.

$$\mathcal{S}(\mathbf{x},\mathbf{y}) = \inf\left\{\theta > 0 \mid \frac{\mathbf{y}}{\theta} \in \mathcal{P}(\mathbf{x})\right\},$$

2) or, according to [2], we can use the so-called directional measure

$\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ defined by a non-zero fixed vector $\mathbf{a} \in \mathbb{R}^n$, by

$$\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = \sup\{\tau \geq 0 \mid \mathbf{y} + \tau\mathbf{a} \in \mathcal{P}(\mathbf{x})\},$$

3) or, finally, we can consider the indicator

$$d_{\mathbf{a}}(\mathbf{x},\cdot) = \frac{1}{1+\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)}, \quad (1)$$

defined by us in [11].

Observations: 1) *The terms of Shephard measure or distance, respectively Malmquist-Luenberger measure or distance should not be used taking into account the usual meaning of measure or distance, as notions applied in mathematics. In order to avoid this confusion we have adopted the terms of Shephard or Malmquist-Luenberger index or indicator with the meaning of "expressing" the quality attributes of non-marketable activities.*

2) *All three indices $\mathcal{S}(\mathbf{x},\cdot)$, $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ and $d_{\mathbf{a}}(\mathbf{x},\cdot)$, (parameters $\mathbf{x} \in \mathbb{R}^m{}_+$, $\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}$, are to be considered fixed), are well defined on the set $\mathcal{P}(\mathbf{x})$.*

3) *The index $\mathcal{S}(\mathbf{x},\cdot)$ can also be extended outwards the domain $\mathcal{P}(\mathbf{x})$ without any problems, more exactly it can be extended on the set $\mathbb{R}^n{}_+ \setminus \mathcal{P}(\mathbf{x})$. Moreover, in the definition given by us in this paragraph to $\mathcal{S}(\mathbf{x},\cdot)$ one can waive carefree the restriction $\theta > 0$ in favour of the general case $\theta \in \mathbb{R}\setminus\{0\}$, without causing any modification to the application.*

4) *The index $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ cannot always be extended outwards the domain $\mathcal{P}(\mathbf{x})$. For example, in the case in which the domain $\mathcal{P}(\mathbf{x})$ is bounded, then there exists a point $P \in \mathbb{R}^n{}_+ \setminus \mathcal{P}(\mathbf{x})$* and a line d which passes through this point and which does not intersect the domain $\mathcal{P}(\mathbf{x})$. *Let $\mathbf{y}$ be the position vector of the point P and $\mathbf{a}$ a non zero vector parallel with the line d. Under these conditions, the expression $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$ has no sense. Moreover, under the conditions set by our definition ($\tau \geq 0$), the index $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ cannot be extended outwards the*

domain $\mathcal{P}(\mathbf{x})$ *not even for the particular case (but important for applications)* $\mathbf{a}=\mathbf{y}$. *Indeed, if the extremity of the vector* $\mathbf{y}$ *is situated in the region* $\mathbb{R}^n_+ \setminus \mathcal{P}(\mathbf{x})$, *then the expression* $\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})$ *(corresponding to the case* $\mathbf{a}=\mathbf{y}$*) has no sense. This situation can be eliminated if the condition* $\tau \geq 0$ *is dropped out and if in the definition of* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ *the case* $\tau \in \mathbb{R}$ *is accepted. On this occasion we must notice that this aspect does not bother in any way the rightful users of this theory, see the original definition [2]. The case when the indicator* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ *is negative can be interpreted as an indication that the "measured" vector is not fully included in the domain* $\mathcal{P}(\mathbf{x})$.

5) *The passage (in the definition of the indicator* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$*) from the restrictive condition* $\tau \geq 0$ *to the very general one* $\tau \in \mathbb{R}$, *has repercussions on the function* $d_{\mathbf{a}}(\mathbf{x},\cdot)$, *as well. Indeed, in certain cases this function can also have negative values and moreover, one can get to cases of indetermination even for those vectors* $\mathbf{y}$ *for which the function* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ *is well defined. For example, if* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) < -1$, *then* $d_{\mathbf{a}}(\mathbf{x},\mathbf{y}) < 0$. *Also, the function* $d_{\mathbf{a}}(\mathbf{x},\cdot)$ *is not defined for those vectors* $\mathbf{y} \in \mathbb{R}^n_+$ *for which* $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = -1$.

Under the conditions set by the definitions that we have adopted in this paragraph, the functions $\mathcal{S}(\mathbf{x},\cdot)$, $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ and $d_{\mathbf{a}}(\mathbf{x},\cdot)$, considered to be defined only on the set $\mathcal{P}(\mathbf{x})$, satisfy the following properties[7]:

1) The Shephard measure extends the performances of output $\mathbf{y} \in \mathcal{P}(\mathbf{x})$, subject of evaluation, by proportionality up to the superior feasibility limit. Indeed, the extremity of vector $\mathbf{y}/\mathcal{S}(\mathbf{x},\mathbf{y})$ belongs to the frontier of set $\mathcal{P}(\mathbf{x})$.

2) The Shephard measure completely characterizes technology $\mathcal{P}(\mathbf{x})$ which is associated to, in the sense that $\mathbf{y} \in \mathcal{P}(\mathbf{x}) \Leftrightarrow \mathcal{S}(\mathbf{x},\mathbf{y}) \leq 1$.

3) If $\mathcal{S}(\mathbf{x},\mathbf{y}) = 1$, then the extremity of vector $\mathbf{y}$ belongs to the frontier of domain $\mathcal{P}(\mathbf{x})$.

4) $\mathcal{S}(\mathbf{x},\cdot)$ is a first degree homogenous function.

5) $\mathcal{S}(\mathbf{x},\cdot)$ is a concave function.

[7] The demonstration regarding properties 2), 3), 4), 5), 8), 10.1), 10.2) and 10.3) is presented in section 10.

6) Parameter τ from the definition of measure $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ is a scaling parameter.

7) The directional measure $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$, earlier defined, develops, in a certain way, the performances of output $\mathbf{y}\in\mathcal{P}(\mathbf{x})$, which is submitted for evaluation up to the superior limit of feasibility, not in direction of vector $\mathbf{y}$, as in the case of the Shephard measure $\mathcal{S}(\mathbf{x},\cdot)$, but in direction of vector $\mathbf{a}$ established from the very beginning. Indeed, the extremity of vector $\mathbf{y}+\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y})\mathbf{a}$ belongs to the frontier of set $\mathcal{P}(\mathbf{x})$.

8) For $\mathbf{a}=\mathbf{y}$ we obtain

$$\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})=\frac{1}{\mathcal{S}(\mathbf{x},\mathbf{y})}-1\Leftrightarrow\mathcal{S}(\mathbf{x},\mathbf{y})=\frac{1}{1+\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})}. \tag{2}$$

9) The relation obtained at 8) demonstrates the fact that the Shephard measure $\mathcal{S}(\mathbf{x},\cdot)$ is a particular case (case $\mathbf{a}=\mathbf{y}$) of the directional measure $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$.

10) As presented in [11], our opinion is that indicator $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$, previously defined, largely differs from the meaning which Shepard's indicator $\mathcal{S}(\mathbf{x},\cdot)$ bears. In this respect, with regard to the notion of directional measure function we believe that the proposed indicator $d_{\mathbf{a}}(\mathbf{x},\cdot)$ is more appropriate. Indeed, function $d_{\mathbf{a}}(\mathbf{x},\cdot)$, fulfils properties that draw it closer to the Shepard measure, properties which function $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$ does not fulfil. For instance:

10.1) $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})\le 1$, for any $\mathbf{x}\in\mathbb{R}^{m}{}_{+}$, $\mathbf{y}\in\mathcal{P}(\mathbf{x})$ and any $\mathbf{a}\in\mathbb{R}^{n}$, $\mathbf{a}\neq\mathbf{0}$;

10.2) if $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})=1$, then the extremity of vector $\mathbf{y}$ belongs to the frontier of domain $\mathcal{P}(\mathbf{x})$;

10.3) in the special case, when $\mathbf{a}=\mathbf{y}\in\mathcal{P}(\mathbf{x})$, expressions $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})$ and $\mathcal{S}(\mathbf{x},\mathbf{y})$ coincide.

5. Measuring quality outputs. A new point of view

As compared to the indicators defined earlier, within this paper we propose a new way to measure the output quality, namely

$$\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = \frac{\|\mathbf{y}\|}{\|\mathbf{y}\| + \mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \cdot \|\mathbf{a}\|}, \tag{3}$$

where $\|\cdot\|$ represents the usual norm of space $\mathbb{R}^n$.

From among the remarkable properties of this measure, we mention:

1) For $\mathbf{x} \in \mathbb{R}^m{}_+$, $\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}$, fixed, the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ is well defined on the set $\mathcal{P}^*(\mathbf{x}) = \mathcal{P}(\mathbf{x}) \setminus \{\mathbf{0}\}$. We must mention that there are nonetheless cases where the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ is defined even in $\mathbf{y} = \mathbf{0}$. This happens if the vector $\mathbf{a}$ is not fully included in the set $\mathbb{R}^n{}_+ \setminus \mathcal{P}^*(\mathbf{x})$, that is, if $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \neq 0$. However, in all the other cases the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ will have an indetermination in $\mathbf{y} = \mathbf{0}$.

2) Like the other indices of measure of the attributes of quality (efficiency), the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ can also be extended, for certain vectors $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$, outwards the set $\mathcal{P}(\mathbf{x})$. Evidently, the possibilities of extension of this function increase, if in the definition of the function $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$, with whose help the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ has been built, we consider the case $\tau \in \mathbb{R}$ instead of the case $\tau \geq 0$. After accomplishing this extension the function $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$ must be used further on with caution because the new function obtained will have a lot of cases of indetermination, just like the function $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$.

3) In case $\mathbf{a} = \mathbf{y} \in \mathcal{P}(\mathbf{x})$, index $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$ coincides with the Shephard index $\mathcal{S}(\mathbf{x},\mathbf{y})$, as it can easily be observed by using the first of the formulas (2).

4) $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \leq 1$, for any $\mathbf{x} \in \mathbb{R}^m{}_+$, $\mathbf{y} \in \mathcal{P}^*(\mathbf{x})$ and any $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$. Indeed, $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \leq 1 \Leftrightarrow \|\mathbf{y}\| \leq \|\mathbf{y}\| + \mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \cdot \|\mathbf{a}\|$.

5) For any $\mathbf{y} \in \mathcal{P}(\mathbf{x})$ and $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$, if $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = 1$, then the extremity of vector $\mathbf{y}$ belongs to the frontier of domain $\mathcal{P}(\mathbf{x})$. Indeed, $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = 1 \Leftrightarrow \mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) = 0 \Rightarrow \mathbf{y} \in \partial\mathcal{P}(\mathbf{x})$.

6. Productivity index. The classical theory

In this section we shall present known modalities of how to compute the productivity evaluation indices of non-marketable activities from a genuinely qualitative point of view. By following the steps presented in [10], if $t = 1, 2, .., T$, represent T distinct time periods, then ratios

$$CCD(t)=\frac{\mathcal{S}^t\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{S}^t\left(\mathbf{x}^t,\mathbf{y}^t\right)},\ CCD(t+1)=\frac{\mathcal{S}^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{S}^{t+1}\left(\mathbf{x}^t,\mathbf{y}^t\right)},$$

express the efficiency degree regarding the change in the state of the education system at a given moment t, state synthetically described by vector $\left(\mathbf{x}^t,\mathbf{y}^t\right)$, towards the state at a moment $t+1$, state synthetically described by vector $\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)$; the first of these rations is computed by means of the technology existent at moment t, technology previously marked by $\mathcal{P}^t\left(\mathbf{x}\right)$, while the second is computed by means of the technology existent within the education system at moment $t+1$, technology designated by $\mathcal{P}^{t+1}\left(\mathbf{x}\right)$. More exactly, the components of the two ratios have the following meanings

$$\mathcal{S}^t\left(\mathbf{x}^t,\mathbf{y}^t\right)=\inf\left\{\theta>0\,|\,\frac{\mathbf{y}^t}{\theta}\in\mathcal{P}^t\left(\mathbf{x}^t\right)\right\},$$

$$\mathcal{S}^t\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\inf\left\{\theta>0\,|\,\frac{\mathbf{y}^{t+1}}{\theta}\in\mathcal{P}^t\left(\mathbf{x}^{t+1}\right)\right\},$$

$$\mathcal{S}^{t+1}\left(\mathbf{x}^t,\mathbf{y}^t\right)=\inf\left\{\theta>0\,|\,\frac{\mathbf{y}^t}{\theta}\in\mathcal{P}^{t+1}\left(\mathbf{x}^t\right)\right\},$$

$$\mathcal{S}^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\inf\left\{\theta>0\,|\,\frac{\mathbf{y}^{t+1}}{\theta}\in\mathcal{P}^{t+1}\left(\mathbf{x}^{t+1}\right)\right\}.$$

The two ratios earlier defined and denoted by $CCD(t)$ and $CCD(t+1)$ were introduced by D. Caves, L. Christensen, W. E. Diewert in [1] and are two different modalities of evaluating the productivity systems they apply to (such as education institutions, in this case).

By taking these considerations into account, the Malmquist productivity index $M(t,t+1)$ is defined as a geometric mean of $CCD(t)$ and $CCD(t+1)$, namely

$$M(t,t+1)=\sqrt{CCD(t)CCD(t+1)}.$$

Observation: *The tendency to use the productivity Malmquist index instead of the indices introduced by Caves, Christensen, and Diewert, has to do with its capacity to better emphasize the way inputs' quality attributes are reflected by outputs' quality attributes.*

Analogously, by following the steps presented in [11], indicators

$$CCD_{\mathbf{a}}(t)=\frac{d_{\mathbf{a}}^{\,t}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{d_{\mathbf{a}}^{\,t}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)}=\frac{1+\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)}{1+\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)},$$

$$CCD_{\mathbf{a}}(t+1)=\frac{d^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{d^{t+1}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)}=\frac{1+\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)}{1+\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)},$$

where, as earlier, $t=1,2,..,T$, represent T distinct time periods, express, relatively to a non-zero, given vector $\mathbf{a}\in\mathbb{R}^{n}$, the efficiency degree of the change regarding the existent situation within the educational system at a moment t, situation synthesized by vector $\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)$, in regard to the existent situation at moment $t+1$, situation synthesized by vector $\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)$; the first from these indicators is computed using technology $\mathcal{P}^{t}(\mathbf{x})$ existent at moment t, while the second is computed using technology $\mathcal{P}^{t+1}(\mathbf{x})$ within the educational system at moment $t+1$. More precisely, the components $\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)$, $\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)$, $\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)$, $\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)$, which are used to compute indicators $CCD_{\mathbf{a}}(t)$ and $CCD_{\mathbf{a}}(t+1)$, have the following meaning

$$\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)=\sup\left\{\tau\geq 0\,|\,\mathbf{y}^{t}+\tau\mathbf{a}\in\mathcal{P}^{t}\left(\mathbf{x}^{t}\right)\right\},$$

$$\mathcal{D}^{t}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\sup\left\{\tau\geq 0\,|\,\mathbf{y}^{t+1}+\tau\mathbf{a}\in\mathcal{P}^{t}\left(\mathbf{x}^{t+1}\right)\right\},$$

$$\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)=\sup\left\{\tau\geq 0\,|\,\mathbf{y}^{t}+\tau\mathbf{a}\in\mathcal{P}^{t+1}\left(\mathbf{x}^{t}\right)\right\},$$

$$\mathcal{D}^{t+1}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\sup\left\{\tau\geq 0\,|\,\mathbf{y}^{t+1}+\tau\mathbf{a}\in\mathcal{P}^{t+1}\left(\mathbf{x}^{t+1}\right)\right\}.$$

The two indicators earlier defined and denoted by $CCD_{\mathbf{a}}(t)$ and $CCD_{\mathbf{a}}(t+1)$ represent the development of the indices introduced by D. Caves, L. Christensen and W. E. Diewert in [1]. These indices provide two new different modalities of evaluating the productivity of non-marketable systems such as the systems existent within educational or professional training institutions.

By taking these considerations into account, the directional Malmquist - Luenberger productivity index $ML_{\mathbf{a}}(t,\,t+1)$ is defined as a geometric mean of $CCD_{\mathbf{a}}(t)$ and $CCD_{\mathbf{a}}(t+1)$, namely

$$ML_{\mathbf{a}}(t,t+1)=\sqrt{CCD_{\mathbf{a}}(t)\,CCD_{\mathbf{a}}(t+1)}\,.$$

Observation: *As earlier, the preference to use the directional productivity Malmquist - Luenberger index instead of the directional indices* $CCD_{\mathbf{a}}(t)$ and $CCD_{\mathbf{a}}(t+1)$*, has to do with its capacity to better emphasize the way inputs' quality attributes are reflected by outputs' quality attributes.*

7. Productivity index. A new point of view

Instead of measures $\mathcal{S}(\mathbf{x},\cdot)$ and $d_{\mathbf{a}}(\mathbf{x},\cdot)$, this time we shall use measure $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\cdot)$, defined by us in section 5. As a result we shall obtain a new index to measure the productivity of quality attributes. Thus, if $t=1,2,..,T$, represent T distinct time periods, then ratios

$$\widetilde{CCD}_{\mathbf{a}}(t)=\frac{\mathcal{M}_{\mathbf{a}}^{t}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{M}_{\mathbf{a}}^{t}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},\ \widetilde{CCD}_{\mathbf{a}}(t+1)=\frac{\mathcal{M}_{\mathbf{a}}^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{M}_{\mathbf{a}}^{t+1}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},$$

express, relatively to a non-zero, given vector $\mathbf{a}\in\mathbb{R}^n$, the efficiency degree of the change regarding the existent situation within the educational system at a moment t, situation synthesized by vector $\left(\mathbf{x}^t,\mathbf{y}^t\right)$, in regard to the existent situation at moment $t+1$, situation synthesized by vector $\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)$; the first from these ratios is computed using technology $\mathcal{P}^t(\mathbf{x})$ existent at moment t, while the second is computed using technology $\mathcal{P}^{t+1}(\mathbf{x})$ within the educational system at moment $t+1$. More precisely, the components of the two ratios have the following meaning

$$\mathcal{M}^{t}{}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)=\frac{\left\|\mathbf{y}^{t}\right\|}{\left\|\mathbf{y}^{t}\right\|+\mathcal{D}^{t}{}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)\cdot\left\|\mathbf{a}\right\|},$$

$$\mathcal{M}^{t}{}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\frac{\left\|\mathbf{y}^{t+1}\right\|}{\left\|\mathbf{y}^{t+1}\right\|+\mathcal{D}^{t}{}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)\cdot\left\|\mathbf{a}\right\|},$$

$$\mathcal{M}^{t+1}{}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)=\frac{\left\|\mathbf{y}^{t}\right\|}{\left\|\mathbf{y}^{t}\right\|+\mathcal{D}^{t+1}{}_{\mathbf{a}}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)\cdot\left\|\mathbf{a}\right\|},$$

$$\mathcal{M}^{t+1}{}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)=\frac{\left\|\mathbf{y}^{t+1}\right\|}{\left\|\mathbf{y}^{t+1}\right\|+\mathcal{D}^{t+1}{}_{\mathbf{a}}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)\cdot\left\|\mathbf{a}\right\|}.$$

The two ratios earlier defined and denoted by $\widetilde{CCD}_{\mathbf{a}}(t)$ and $\widetilde{CCD}_{\mathbf{a}}(t+1)$ represent the correspondent of indices $CCD_{\mathbf{a}}(t)$ and $CCD_{\mathbf{a}}(t+1)$, mentioned in the previous section. Indices $\widetilde{CCD}_{\mathbf{a}}(t)$ and $\widetilde{CCD}_{\mathbf{a}}(t+1)$ provide two different modalities of evaluating the productivity of activities which deliver qualitative and not quantitative results, such as those achieved in education institutions.

By using the new productivity evaluation indices of non-marketable activities, the alternative to the directional Malmquist - Luenberger productivity index $ML_{\mathbf{a}}(t, t+1)$ which we propose in this paper, is defined as a geometric mean of $\widetilde{CCD}_{\mathbf{a}}(t)$ and $\widetilde{CCD}_{\mathbf{a}}(t+1)$, namely

$$\widetilde{ML}_{\mathbf{a}}(t,t+1) = \sqrt{\widetilde{CCD}_{\mathbf{a}}(t)\widetilde{CCD}_{\mathbf{a}}(t+1)}\,.$$

Observation: *From among indicators* $\widetilde{CCD}_{\mathbf{a}}(t)$, $\widetilde{CCD}_{\mathbf{a}}(t+1)$ *and* $\widetilde{ML}_{\mathbf{a}}(t, t+1)$, *which mainly express the same thing, the last is preferred over the others due its capacity to better emphasize the way inputs' quality attributes are reflected by outputs' quality attributes.*

8. An effective modality of computing index $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$

It is known that index $\mathcal{M}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$, $\mathbf{x} \in \mathbb{R}^m{}_+$, $\mathbf{y} \in \mathcal{P}(\mathbf{x})$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0}$, is defined by formula (3), but in order to actually compute this formula we need to know the value of indicator $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$. To do this, we observe that the problem related to evaluating the directional measure $\mathcal{D}^t{}_{\mathbf{a}}(\mathbf{x},\mathbf{y})$, (defined in section 4) regarding the outputs of a technology $\mathcal{P}^t(\mathbf{x})$ constructed on the basis of the doublets $(\mathbf{x}^t{}_k, \mathbf{y}^t{}_k)$, $k = 1,2,..,K$, (see the section 3), can be reduced, by adapting the method presented in [2] and [4], to a convex programming problem as it follows

$$\mathcal{D}^t{}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) =$$
$$= \sup\left\{\tau > 0 \mid \exists\, z_k \geq 0,\ k = 1, 2,..,K, \text{ so that } \sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \leq \mathbf{x}, \text{ and } \sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \geq \mathbf{y} + \tau\mathbf{a}\right\}.$$

9. Comments

The motivation to find some effective means to evaluate educational - professional performances is nurtured by the intense development of scientific and technological issues. This aspect has been confirmed through numerous research works elaborated in the field. In this paper, we aim at improving the progress reached within the training - learning process measured from the perspective of qualitative outputs proposed by us in [10] and [11].

A similar way to evaluate instructive - educational performances can be found in [4], but contrary to this classical point of view, where the vector which expresses the outputs of the system studied contains, apart from information regarding the quality of the educational process, information regarding quantitative aspects, as well, such as the number of student participants or the time spent in the classroom, in our research the vector which expresses the outputs of the system studied only contains information regarding the quality of the educational process.

While developing the knowledge regarding the non-marketable processes, authors like Y. H. Chung, R. Färe, S. Grosskopf [2] have felt the need to replace the traditional Shephard measure of the output quality with a measure providing practitioners with more degrees of freedom. Unfortunately, the so-called "directional measure" proposed by them, does not possess the property of preserving all Shepard measure’s properties, fact which is a real loss for the theory. In this respect, the measure functions (1) and (3), proposed by us (the first in [11] and the second in the present paper) as working alternatives aim at improving the actual state of fact. Indeed, besides the fact that these measure functions are directional, they also fulfil conditions specific to the Shephard measure: see the set of conditions 10.1) - 10.3) within section 4, fulfilled by the first of these functions and the set of conditions 3) -5) within section 5, fulfilled by the second one.

It is natural to expect that the special properties of the measure functions (1) and (3) of the output quality induce properties special to the productivity indices of non-marketable activities which are defined based on them. Given this, if we restrain the analysis only to the implications of the measure (3), introduced in this paper, and since the technologies defined in section 3 satisfy constant returns to scale (i.e. $\mathcal{P}^t(\lambda\mathbf{x}) = \lambda\mathcal{P}^t(\mathbf{x}), \forall\lambda > 0$ - property 1) from the same section, then indices

$$\widetilde{CCD}_{\mathrm{y}}(t) = \frac{\mathcal{M}_{\mathrm{y}}^{t}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{M}_{\mathrm{y}}^{t}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},\ \widetilde{CCD}_{\mathrm{y}}(t+1) = \frac{\mathcal{M}_{\mathrm{y}}^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{M}_{\mathrm{y}}^{t+1}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},$$

give a measure of a total factor productivity change in terms of average products. Indeed, in this particular case (when $\mathbf{a} = \mathbf{y}$) we have

$$\widetilde{CCD}_{\mathrm{y}}(t) = \frac{\mathcal{S}^{t}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{S}^{t}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},\ \widetilde{CCD}_{\mathrm{y}}(t+1) = \frac{\mathcal{S}^{t+1}\left(\mathbf{x}^{t+1},\mathbf{y}^{t+1}\right)}{\mathcal{S}^{t+1}\left(\mathbf{x}^{t},\mathbf{y}^{t}\right)},$$

which perfectly match the conditions of the theory presented in [4].

The study undertaken by us in this paper provides managers a method to express the productivity of education or academic institutions by exclusively approaching the quality of the knowledge and skills acquisition process, facilitating in this way, the design of policies through a clear target definition.

10 Appendix

In this last section we shall demonstrate the mathematical properties of set $\mathcal{P}^t(\mathbf{x})$ defined in section 3, as well as properties of measures $\mathcal{S}(\mathbf{x},\cdot)$, $\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)$, and $d_{\mathbf{a}}(\mathbf{x},\cdot)$ defined in section 4.

Proposition: *Technology* $\mathcal{P}^t(\mathbf{x})$ *verifies relation*

$$\mathcal{P}^t(\lambda\mathbf{x}) = \lambda\mathcal{P}^t(\mathbf{x}), \forall\lambda > 0,$$

which expresses the property of constant return to scale.

Proof: For any $\lambda > 0$, we have

$$\mathcal{P}^t(\lambda\mathbf{x}) =$$

$$= \left\{\mathbf{y}' \mid \exists z'_k \geq 0,\ k = 1, 2,.., K, \text{ such that } \sum_{k=1}^{K} z'_k\,\mathbf{x}^t_k \leq \lambda\mathbf{x}, \text{ and } \sum_{k=1}^{K} z'_k\,\mathbf{y}^t_k \geq \mathbf{y}'\right\} =$$

$$= \left\{\mathbf{y}' \mid \exists z'_k \geq 0, k = 1, 2,.., K, \text{such that } \sum_{k=1}^{K} \frac{z'_k}{\lambda}\mathbf{x}^t_k \leq \mathbf{x}, \text{ and } \sum_{k=1}^{K} z'_k\,\mathbf{y}^t_k \geq \mathbf{y}'\right\} =$$

$$= \left\{\mathbf{y}' \mid \exists z_k \geq 0,\ k = 1, 2,.., K, \text{such that } \sum_{k=1}^{K} z_k\mathbf{x}^t_k \leq \mathbf{x}, \text{ and } \sum_{k=1}^{K} \lambda z_k\mathbf{y}^t_k \geq \mathbf{y}'\right\} =$$

$$= \left\{\mathbf{y}' \mid \exists z_k \geq 0,\ k = 1, 2,.., K, \text{such that } \sum_{k=1}^{K} z_k\mathbf{x}^t_k \leq \mathbf{x}, \text{ and } \sum_{k=1}^{K} z_k\mathbf{y}^t_k \geq \frac{1}{\lambda}\mathbf{y}'\right\} =$$

$$= \left\{\lambda\mathbf{y} \mid \exists z_k \geq 0,\ k = 1,2,.., K,\ \sum_{k=1}^{K} z_k\mathbf{x}^t_k \leq \mathbf{x}, \text{ and } \sum_{k=1}^{K} z_k\mathbf{y}^t_k \geq \mathbf{y}\right\} = \lambda\mathcal{P}^t(\mathbf{x}).$$

Proposition: *For any* $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^m_{+,}$ $\mathbf{x} \leq \mathbf{x}'$, *relation* $\mathcal{P}^t(\mathbf{x}) \subseteq \mathcal{P}^t(\mathbf{x}')$ *takes place, relation which expresses the property of strong disposability of inputs.*

Proof: $\mathbf{y} \in \mathcal{P}^t(\mathbf{x}) \Leftrightarrow \sum_{k=1}^{K} z_k\mathbf{y}^t_k \geq \mathbf{y}$, for some values $z_k \geq 0,\ k = 1, 2,.., K$, for which $\sum_{k=1}^{K} z_k\mathbf{x}^t_k \leq \mathbf{x}$. If $\mathbf{x} \leq \mathbf{x}'$, then these values $z_k \geq 0,\ k = 1, 2,.., K$, for which

$\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}$, will verify inequality $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}'$, so, $\mathbf{y} \in \mathcal{P}^t(\mathbf{x}')$, and consequently $\mathcal{P}^t(\mathbf{x}) \subseteq \mathcal{P}^t(\mathbf{x}')$.

Observation: *The result above indicates a strong impact of the inputs upon the outputs (results obtained).*

Proposition: *If* $\mathbf{y} \in \mathcal{P}^t(\mathbf{x})$*, then* $\forall\, 0 \le \theta \le 1 \Rightarrow \theta\mathbf{y} \in \mathcal{P}^t(\mathbf{x})$.

Proof: $\mathbf{y} \in \mathcal{P}^t(\mathbf{x}) \Leftrightarrow \sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \ge \mathbf{y}$ for a set of values $z_k \ge 0,\ k = 1, 2,.., K$ for which $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}$. In such case, for the same set of values $z_k \ge 0,\ k = 1, 2,.., K$, the following inequalities $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}$, and $\sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \ge \mathbf{y} \ge \theta\mathbf{y}$ are satisfied, as well, if $\theta \in [0, 1]$. Thus, $\theta\mathbf{y} \in \mathcal{P}^t(\mathbf{x})$.

Proposition: *If* $\mathbf{y} \in \mathcal{P}^t(\mathbf{x})$*, then* $\forall\, \mathbf{y}' \in \mathbb{R}^n{}_+,\ \mathbf{y}' \le \mathbf{y} \Rightarrow \mathbf{y}' \in \mathcal{P}^t(\mathbf{x})$.

Proof: $\mathbf{y} \in \mathcal{P}^t(\mathbf{x}) \Leftrightarrow \sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \ge \mathbf{y}$ for a set of values $z_k \ge 0,\ k = 1, 2,.., K$ for which $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}$. Since for the same set of values $z_k \ge 0,\ k = 1, 2,.., K$, inequalities $\sum_{k=1}^{K} z_k \mathbf{x}^t{}_k \le \mathbf{x}$, and $\sum_{k=1}^{K} z_k \mathbf{y}^t{}_k \ge \mathbf{y} \ge \mathbf{y}'$, are satisfied, we deduce that $\mathbf{y}' \in \mathcal{P}^t(\mathbf{x})$.

Proposition: *Set* $\mathcal{P}^t(\mathbf{x})$ *is convex.*

Proof: Let $\mathbf{u}, \mathbf{v} \in \mathcal{P}^t(\mathbf{x})$. Let us remember that this thing is equivalent to the existence of a set of values $\alpha_k \ge 0$, and $\beta_k \ge 0$, $k = 1, 2,.., K$, for which inequalities

$$\sum_{k=1}^{K} \alpha_k \mathbf{x}^t{}_k \le \mathbf{x},\ \sum_{k=1}^{K} \beta_k \mathbf{x}^t{}_k \le \mathbf{x},\ \sum_{k=1}^{K} \alpha_k \mathbf{y}^t{}_k \ge \mathbf{u}, \sum_{k=1}^{K} \beta_k \mathbf{y}^t{}_k \ge \mathbf{v},$$

are satisfied. In these conditions, for any $\lambda \in [0,1]$ we have

$$\sum_{k=1}^{K} \left[(1-\lambda)\alpha_k + \lambda\beta_k\right] \mathbf{x}^t{}_k \le \mathbf{x},$$

and

$$\sum_{k=1}^{K}\left[(1-\lambda)\alpha_k+\lambda\beta_k\right]\mathbf{y}^t{}_k \geq (1-\lambda)\mathbf{u}+\lambda\mathbf{v}.$$

Then

$$(1-\lambda)\mathbf{u}+\lambda\mathbf{v}\in\mathcal{P}^t(\mathbf{x}),\forall\lambda\in[0,1].$$

Proposition: $\mathbf{y}\in\mathcal{P}^t(\mathbf{x})\Rightarrow\mathcal{S}(\mathbf{x},\mathbf{y})\leq 1$.

Proof: We suppose that $\mathbf{y}\in\mathcal{P}^t(\mathbf{x})$. Because for $\theta=1$, vector $\frac{\mathbf{y}}{\theta}\in\mathcal{P}^t(\mathbf{x})$, we deduce

$$\mathcal{S}(\mathbf{x},\mathbf{y})=\inf\left\{\theta>0\,|\,\frac{\mathbf{y}}{\theta}\in\mathcal{P}(\mathbf{x})\right\}\leq 1.$$

Proposition: $\mathcal{S}(\mathbf{x},\mathbf{y})\leq 1\Rightarrow\mathbf{y}\in\mathcal{P}^t(\mathbf{x})$.

Proof: $\mathcal{S}(\mathbf{x},\mathbf{y})\leq 1\Leftrightarrow\exists\theta\in(0,1]$ so that $\frac{\mathbf{y}}{\theta}\in$ $\mathcal{P}^t(\mathbf{x})\Leftrightarrow$ $\exists z_k\geq 0,\ k=1,2,..,K$, so that

$$\sum_{k=1}^{K}z_k\mathbf{x}^t{}_k\leq\mathbf{x},\ \sum_{k=1}^{K}z_k\mathbf{y}^t{}_k\geq\frac{\mathbf{y}}{\theta}.$$

Since $\theta\in(0,1]$, we have $\sum_{k=1}^{K}\theta z_k\mathbf{x}^t{}_k\leq\sum_{k=1}^{K}z_k\mathbf{x}^t{}_k\leq\mathbf{x}$. Moreover, $\sum_{k=1}^{K}z_k\mathbf{y}^t{}_k\geq\frac{\mathbf{y}}{\theta}$ $\Leftrightarrow$ $\Leftrightarrow\sum_{k=1}^{K}\theta z_k\mathbf{y}^t{}_k\geq\mathbf{y}$. Thus, by denoting $\tau_k=\theta z_k,\quad k=1,2,..,K$, we obtain a set of values $\tau_k\geq 0,\ k=1,\ 2,..,K$, so that $\sum_{k=1}^{K}\tau_k\mathbf{x}^t{}_k\leq\mathbf{x}$ and $\sum_{k=1}^{K}\tau_k\mathbf{y}^t{}_k\geq\mathbf{y}$. Thus, $\mathbf{y}\in\mathcal{P}^t(\mathbf{x})$.

Proposition: If $\mathcal{S}(\mathbf{x},\mathbf{y})=1$, then the extremity of vector $\mathbf{y}$ belongs to the frontier of domain $\mathcal{P}(\mathbf{x})$.

Proof: This result is a direct consequence of the way in which the Shephard measure function $\mathcal{S}(\mathbf{x},\cdot)$ is defined.

Proposition: $\mathcal{S}(\mathbf{x},\cdot)$ *is a first order homogenous function, i.e.*

$$\mathcal{S}(\mathbf{x},\alpha\mathbf{y})=\alpha\mathcal{S}(\mathbf{x},\mathbf{y}),\forall\alpha>0.$$

Proof: For any $\alpha>0$, we have

$$\mathcal{S}(\mathbf{x},\alpha\mathbf{y}) = \inf\{\theta > 0 \,|\, \alpha\mathbf{y}/\theta \in \mathcal{P}(\mathbf{x})\} = \inf_{\tau>0}\{\alpha\tau \,|\, \mathbf{y}/\tau \in \mathcal{P}(\mathbf{x})\} =$$
$$= \alpha \inf_{\tau>0}\{\tau \,|\, \mathbf{y}/\tau \in \mathcal{P}(\mathbf{x})\} = \alpha\mathcal{S}(\mathbf{x},\mathbf{y}).$$

Proposition: $\mathcal{S}(\mathbf{x},\cdot)$ *is a concave function.*

Proof: Let $\mathbf{u}$ and $\mathbf{v}$ from $\mathbb{R}^n{}_+$. If $\mathcal{S}(\mathbf{x},\mathbf{u}) = \infty$ or $\mathcal{S}(\mathbf{x},\mathbf{v}) = \infty$, then

$$\mathcal{S}(\mathbf{x},\,(1-\lambda)\mathbf{u}+\lambda\mathbf{v}) \le (1-\lambda)\mathcal{S}(\mathbf{x},\mathbf{u}) + \lambda\mathcal{S}(\mathbf{x},\mathbf{v}), \forall\,\lambda \in (0,1)\ .$$

We now suppose that $\mathcal{S}(\mathbf{x},\mathbf{u}) < \infty$ and $\mathcal{S}(\mathbf{x},\mathbf{v}) < \infty$. For any fixed $\lambda \in [0,1]$, we consider sequences $(\alpha_n)_{n\ge1}$ and $(\beta_n)_{n\ge1}$ with properties

$$\alpha_n > 0,\ \ \forall\, n \ge 1,\ \ (1-\lambda)\frac{\mathbf{u}}{\alpha_n} \in \mathcal{P}^t(\mathbf{x}),\ \lim \alpha_n = \mathcal{S}(\mathbf{x},\,(1-\lambda)\mathbf{u}),$$

respectively,

$$\beta_n > 0,\ \ \forall\, n \ge 1,\ \ \lambda\frac{\mathbf{v}}{\beta_n} \in \mathcal{P}^t(\mathbf{x}),\ \lim \beta_n = \mathcal{S}(\mathbf{x},\,\lambda\mathbf{v}).$$

Let $\theta_n = \sup\{\alpha_n, \beta_n\}, n \ge 1$. Since $(1-\lambda)\frac{\mathbf{u}}{\theta_n} \le (1-\lambda)\frac{\mathbf{u}}{\alpha_n}$, we deduce that

$$(1-\lambda)\frac{\mathbf{u}}{\theta_n} \in \mathcal{P}^t(\mathbf{x}).$$

Analogously, since $\lambda\frac{\mathbf{v}}{\theta_n} \le \lambda\frac{\mathbf{u}}{\beta_n}$, we deduce that

$$\lambda\frac{\mathbf{v}}{\theta_n} \in \mathcal{P}^t(\mathbf{x}).$$

Since $\mathcal{P}^t(\mathbf{x})$ is a convex set, it results that

$$(1-\lambda)\frac{\mathbf{u}}{\theta_n} + \lambda\frac{\mathbf{v}}{\theta_n} \in \mathcal{P}^t(\mathbf{x}).$$

Under these conditions

$$\mathcal{S}(\,\mathbf{x},\,(1-\lambda)\mathbf{u}+\lambda\mathbf{v}) = \ \inf\left\{\theta > 0 \,|\, (1-\lambda)\frac{\mathbf{u}}{\theta} + \lambda\frac{\mathbf{v}}{\theta} \in \mathcal{P}(\mathbf{x})\right\} \le \theta_n \le \alpha_n + \beta_n, \forall\, n \ge 1.$$

By passing the limit according to $n \to \infty$ within the inequalities above, we obtain

$$\mathcal{S}(\,\mathbf{x},\,(1-\lambda)\mathbf{u}+\lambda\mathbf{v}) \le \mathcal{S}(\,\mathbf{x},\,(1-\lambda)\mathbf{u}) +$$
$$+\ \mathcal{S}(\,\mathbf{x},\,\lambda\mathbf{v}) \ = \ (1-\lambda)\mathcal{S}(\,\mathbf{x},\,\mathbf{u}) + \lambda\mathcal{S}(\,\mathbf{x},\,\mathbf{v}),$$

(the last equality is obtained by using the previous proposition). Consequently, $\mathcal{S}(\mathbf{x},\cdot)$ is a concave function.

Proposition: *For* $\mathbf{a}=\mathbf{y}\in\mathcal{P}(\mathbf{x})$*, the following relations take place*

$$\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})=\frac{1}{\mathcal{S}(\mathbf{x},\mathbf{y})}-1 \Leftrightarrow \mathcal{S}(\mathbf{x},\mathbf{y})=\frac{1}{1+\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})}.$$

Proof: $\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})=\sup\{\tau\ge 0 \mid \mathcal{S}(\mathbf{x},\mathbf{y}+\tau\mathbf{y})\le 1\}=\sup\{\tau\ge 0 \mid (1+\tau)\mathcal{S}(\mathbf{x},\mathbf{y})\le 1\}=$

$$=\sup\left\{\tau\ge 0 \mid \tau\le\frac{1}{\mathcal{S}(\mathbf{x},\mathbf{y})}-1\right\}=\frac{1}{\mathcal{S}(\mathbf{x},\mathbf{y})}-1.$$

Proposition: *The function* $d_{\mathbf{a}}(\mathbf{x},\cdot)=\dfrac{1}{1+\mathcal{D}_{\mathbf{a}}(\mathbf{x},\cdot)}$*, fulfils the following properties:*

1) $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})\le 1$*, for any* $\mathbf{x}\in\mathbb{R}^m_+$, $\mathbf{y}\in\mathcal{P}(\mathbf{x})$ *and any* $\mathbf{a}\in\mathbb{R}^n_+$, $\mathbf{a}\ne\mathbf{0}$;

2) *if* $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})=1$*, then the extremity of vector* $\mathbf{y}\in\mathcal{P}(\mathbf{x})$ *belongs to the frontier of domain* $\mathcal{P}(\mathbf{x})$;

3) *in the special case, when* $\mathbf{a}=\mathbf{y}\in\mathcal{P}(\mathbf{x})$*, expressions* $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})$ *and* $\mathcal{S}(\mathbf{x},\mathbf{y})$ *coincide.*

Proof: 1) $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})\le 1 \Leftrightarrow 1\le 1+\mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y}) \Leftrightarrow \mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y})\ge 0$, $\forall\mathbf{x}\in\mathbb{R}^m_+$, $\mathbf{y}\in\mathcal{P}(\mathbf{x})$, $\mathbf{a}\in\mathbb{R}^n_+$, $\mathbf{a}\ne\mathbf{0}$;

2) $d_{\mathbf{a}}(\mathbf{x},\mathbf{y})=1 \Leftrightarrow \mathcal{D}_{\mathbf{a}}(\mathbf{x},\mathbf{y})=0 \Rightarrow \mathbf{y}\in\partial\mathcal{P}(\mathbf{x})$;

3) For $\mathbf{a}=\mathbf{y}\in\mathcal{P}(\mathbf{x})$ we have $d_{\mathbf{y}}(\mathbf{x},\mathbf{y})=\dfrac{1}{1+\mathcal{D}_{\mathbf{y}}(\mathbf{x},\mathbf{y})}=\mathcal{S}(\mathbf{x},\mathbf{y})$, in conformity with the formula (2).

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