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OPTIMALITY CONDITIONS FOR A FAMILY OF CURVES

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Fie mulţimea deschisă D în $\mathbb{R}^p$ şi a un punct în D. Fie $\Gamma(a)$ familia curbelor parametrizate ce trec prin punctul a. Această lucrare introduce condiţiile pe care trebuie să le satisfacă $\Gamma(a)$ pentru ca două probleme de extrem să devină echivalente: problema de extrem local şi problema de extrem local restricţionată de familia $\Gamma(a)$, pentru o funcţie arbitrară $f\colon D \to \mathbb{R}$. La final, sunt enunţate două probleme deschise.

Let be given D an open set in $\mathbb{R}^p$, and a a point in D. Also, let $\Gamma(a)$ be a family of parametrized curves passing through the point a. This work introduces a set of conditions to be satisfied by $\Gamma(a)$ in order that two extremum problems become equivalent: the local extremum problem and the extremum problem constrained by family $\Gamma(a)$, for an arbitrary function $f\colon D \to \mathbb{R}$. Finally, two open problems are stated.

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1. Introduction and preliminaries

Let us consider the extremum problem

$$\min f(x), \text{ subject to } x \in M,$$

where M is a subset of $\mathbb{R}^p$. If M is an open set, then the extremum problem is called *unconstrained*. Otherwise, the extremum problem is called *constrained*. In several recent works [1], [2], [4], [8]÷[11], it was shown that the above type of problem is related to extremum problems constrained by a family of parametrized curves. This work develop further this relationship from a different perspective: we investigate the conditions that a family of parametrized curves has to satisfy such that a local extremum problem be equivalent to an extremum problem constrained by this family of parametrized curves.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $\alpha : I \to \mathbb{R}^p$ of class C^m, $m \geq 1$, is called *parametrized* curve of class C^m. We shall say that the curve α:

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1) *passes through* the point $a \in \mathbb{R}^p$ if there exists $t_0 \in I$ such that $\alpha(t_0) = a$;

2) is *regular* at the point $a = \alpha(t_0)$ if $\alpha'(t_0) \neq 0$;

3) *has a tangent* at the point $a = \alpha(t_0)$ if there exists $k = \overline{1,m}$ such that $\alpha^{(k)}(t_0) \neq 0$.

Throughout this work, we shall refer to a function $f : D \to \mathbb{R}$, where D is an open subset in $\mathbb{R}^p$.

Definition 1.2. Let $f : D \to \mathbb{R}$, let $a \in D$, and $\alpha : I \to D$ be a parametrized curve passing through a. We say that a is *minimum point for f constrained by α* if for any $t_0 \in I$, with $\alpha(t_0) = a$, it follows

$$f(a) = f(\alpha(t_0)) \leq f(\alpha(t)), \ \forall t \in [t_0, t_0 + \varepsilon) \in I.$$

This Definition is more general than that used in works [1], [2] and [3], where the above inequality is valid for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset I$.

Definition 1.3. Let $\Gamma(a)$ be a family of parametrized curves passing through the point $a \in \mathbb{R}^p$. We say that a is *minimum point of f constrained by the family $\Gamma(a)$* if a is minimum point of f constrained by each curve of the family $\Gamma(a)$.

It is obvious that any local minimum point of f is also a minimum point of f constrained by $\Gamma(a)$.

2. **Curves subordinate to a certain sequence**

Let a be a point in $\mathbb{R}^p$ and $S(a)$ be a family of sequences with elements from $\mathbb{R}^p$, convergent to a.

Definition 2.1. A parametrized curve α passing though a ($\alpha(t_0) = a$) is called *subordinate to the sequence* $(x_n) \in S(a)$ if there exist a subsequence (x_{n_k}) and a decreasing sequence of real numbers (t_k), $t_k \to t_0$, such that $\alpha(t_k) = x_{n_k}, \ \forall k \in \mathbb{N}^*$.

Let $(x_n) \in S(a)$ and α a parametrized curve having a tangent at the point a and subordinate to the sequence (x_n). Then, the direction of the tangent of α at a is one of the limit points of the sequence $\dfrac{x_n - a}{\|x_n - a\|}$. Indeed, let us assume that α is of C^m-class and has a tangent at the point $a = \alpha(t_0)$, that is there exists $k = \overline{1,m}$ such that $\alpha^{(k)}(t_0) \neq 0$. It follows $\lim\limits_{t \to t_0} \dfrac{\alpha(t) - \alpha(t_0)}{(t - t_0)^k} = \alpha^{(k)}(t_0)$. For $t > t_0$ we get $\lim\limits_{t \to t_0} \dfrac{\alpha(t) - \alpha(t_0)}{\|\alpha(t) - \alpha(t_0)\|} = \dfrac{\alpha^{(k)}(t_0)}{\left\|\alpha^{(k)}(t_0)\right\|}$.

Definition 2.2. Let $\Gamma(a)$ be a family of parametrized curves passing through the point a in $\mathbb{R}^p$. The family $\Gamma(a)$ is called *$S(a)$-subordinate* if for each $(x_n) \in S(a)$ there exists $\alpha \in \Gamma(a)$ subordinate to the sequence (x_n).

Theorem 2.1. *Let $f : D \to \mathbb{R}$ and a be a point in D. Consider $C(a)$ the family of all sequences of distinct elements from the open set D converging to a. Assume that $\Gamma(a)$ is $C(a)$-subordinate. Then, a is local minimum point for f if and only if a is minimum for f constrained by $\Gamma(a)$.*

Proof. Let us assume that a is a minimum point for f constrained by the family $\Gamma(a)$. By reductio ad absurdum, suppose that a is not a local minimum point for the function f. Then, there exists a sequence of distinct elements (x_n) from D such that $x_n \to a$ and $f(x_n) < f(a), \forall n \in \mathbb{N}$. Since the family $\Gamma(a)$ is $C(a)$-subordinate, we can find a sequence (x_{n_k}), a parametrized curve $\alpha \in \Gamma(a)$ $(\alpha(t_0) = a)$ and a decreasing sequence of real numbers (t_k) where $t_k \to t_0$, so that $\alpha(t_k) = x_{n_k}$, for all k in $\mathbb{N}^*$. Therefore $f(\alpha(t_k)) < f(a)$, $\forall k \in \mathbb{N}^*$, which contradicts the hypothesis that a is a minimum constrained by $\Gamma(a)$. □

Let $g = (g^1, ..., g^s) : D \to \mathbb{R}^s$ be a C^1-class function. We set a in D such that $g(a) = 0$. Let $C_g(a)$ be the family of all sequences (x_n) of distinct points from D which satisfy the relations $g(x_n) \geq 0$ and $x_n \to a$. Let $\Gamma_g(a)$ be a family of parametrized curves α passing through the point a, with the property that if $\alpha(t_0) = a$, then $g(\alpha(t)) \geq 0$, for all $t \in [t_0, t_0 + \varepsilon)$.

Theorem 2.2. *Let $f : D \to \mathbb{R}$ and a be a point in D. Assume that the family $\Gamma_g(a)$ is $C_g(a)$-subordinate. Then, a is local minimum for f constrained by $g \geq 0$ if and only if a is minimum point for f constrained by the family $\Gamma_g(a)$.*

The proof is similar to those in Theorem 2.1.

3. Families of parametrized curves

We shall prove that there exist families of parametrized curves $C(a)$-subordinate.

Lemma 3.1. *Let a, b, c, and d be real numbers with $a < b$ and $m \in \mathbb{N}^*$. There exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ of class C^∞ strictly monotonic on $[a, b]$ such that:*

$$\varphi(a) = c, \ \varphi(b) = d, \ \varphi^{(i)}(a) = \varphi^{(i)}(b) = 0, \ i = \overline{1, m} \tag{1}$$

$$\left|\varphi^{(i)}(x)\right| \leq k_i \frac{|d - c|}{(b - a)^i}, \ i = \overline{1, m}, \tag{2}$$

where k_i are constants that do not depend on a, b, c, and d.

Proof. We shall prove that the interpolation Hermite polynomial φ for the data in (1) is appropriate for our purpose. From (1), we have

$$\varphi'(x) = A(x - a)^m (x - b)^m, \ A \in \mathbb{R}.$$

We shall show that φ satisfies conditions (2). First of all, φ is strictly monotonic on $[a, b]$. Now

$$\varphi(x) = A \int_a^x (t - a)^m (t - b)^m \, dt + B.$$

From $\varphi(a) = c$ and $\varphi(b) = d$ we get

$$B = c, \ A = \frac{d - c}{\displaystyle\int_a^b (x - a)^m (x - b)^m \, dx}.$$

Calculating the integral, we obtain

$$A = k\frac{d-c}{(b-a)^{2m+1}}, \text{ where } k = (-1)^m \frac{(2m+1)!}{(m!)^2}.$$

From $|(x-a)(x-b)| \le (b-a)^2, \ \forall x \in [a,b]$, it follows

$$\left|\varphi'(x)\right| \le |A|(b-a)^{2m} = k_1\frac{|d-c|}{b-a}, \text{ where } k_1 = \frac{(2m+1)!}{(m!)^2}.$$

By Leibniz's law we obtain

$$\left|\varphi^{(i)}(x)\right| \le k_i\frac{|d-c|}{(b-a)^i}, \ i = \overline{2,m},$$

k_i being constants not depending on a, b, c, and d. □

Remark 3.1. Keeping in mind the value of A in the above, if $c < d$, then the function φ is strictly increasing.

Lemma 3.2. [1] *Let (x_n) be a sequence of real numbers such that:*

1) $x_n \neq 0$, $x_n \neq x_{n+1}$, *for all* $n \in \mathbb{N}$;
2) *there exists* $\lambda > 0$ *with* $\left|\frac{x_n}{x_{n+1}} - 1\right| \ge \lambda$, *for all* $n \in \mathbb{N}$.

If (y_n) is a sequence of real numbers such that

3) *there exists* $\lim_{n\to\infty}\frac{y_n}{x_n} = r$,

then the sequence $\frac{y_{n+1}-y_n}{x_{n+1}-x_n}$ *is convergent to* r.

Lemma 3.3. *If (x_n) is a sequence of positive real numbers and $x_{n+1} \le \frac{1}{2^m}x_n$, $\forall n \in \mathbb{N}$, where $m \in \mathbb{N}^*$, then there exists $\mu > 0$ such that*

$$\frac{x_n - x_{n+1}}{\left(x_n^{1/m} - x_{n+1}^{1/m}\right)^m} \le \mu, \ \forall n \in \mathbb{N}.$$

Proof. Denote $t_n = x_n^{1/m}$. We have

$$\frac{x_n - x_{n+1}}{\left(x_n^{1/m} - x_{n+1}^{1/m}\right)^m} = \frac{t_n^m - t_{n+1}^n}{(t_n - t_{n+1})^m} = \frac{t_n^{m-1} + t_n^{m-2}t_{n+1} + \cdots + t_{n+1}^{m-1}}{(t_n - t_{n+1})^m}$$

$$\le \frac{t_n^{m-1}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{m-1}\right)}{t_n^{m-1}\left(1-\frac{1}{2}\right)^m} = \frac{1-\left(\frac{1}{2}\right)^m}{\left(1-\frac{1}{2}\right)^m},$$

and the statement is proved. □

Lemma 3.4. *Let (x_n) and (y_n) be two sequences of real numbers such that (x_n) is strictly monotonic, $x_n \to 0$, $y_n \to 0$ and $\frac{y_n}{x_n} \to 0$. Then, for any $m \in \mathbb{N}^*$ there exist* *two functions* $f, g \in C^m(\mathbb{R})$, *two subsequences* (x_{n_k}) *and* (y_{n_k}) *and a sequence of real numbers* (t_k) *such that* $t_k \to 0$ *strictly monotonic and*

$$f(t_k) = x_{n_k},\ g(t_k) = y_{n_k},\ \forall k \in N,\ f^{(i)}(0) = 0,\ \forall i = \overline{1, m-1},$$

$$g^{(i)}(0) = 0,\ \forall i = \overline{1, m},\ f^{(m)}(0) \neq 0.$$

Moreover, f *does not depend on* (y_n), *and, if* $y_n > 0, \forall n \in \mathbb{N}$, *then* f *is increasing.*

Proof. Let us assume, for instance, that $x_n > 0$, $\forall n \in \mathbb{N}$. Replacing (x_n) by one of its subsequences, we can assume $x_{n+1} \leq \frac{1}{2^m} x_n$, $\forall n \in \mathbb{N}$. Let $t_n = x_n^{1/m}$ be a strictly decreasing sequence. Clearly, the function $f(x) = x^m$ satisfies the hypotheses. We proceed by constructing the function g. From Lemma 3.2 it follows that $\frac{y_{n+1} - y_n}{x_{n+1} - x_n} \to 0$. Hence

$$\frac{y_{n+1} - y_n}{(t_{n+1} - t_n)^m} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \frac{x_{n+1} - x_n}{(t_{n+1} - t_n)^m}.$$

Now, we can apply Lemma 3.3: $\frac{x_{n+1} - x_n}{(t_{n+1} - t_n)^m}$ is bounded, therefore

$$\frac{y_{n+1} - y_n}{(t_{n+1} - t_n)^m} \to 0. \tag{3}$$

Then, for any $n \in \mathbb{N}$ we consider for $a = t_{n+1},\ b = t_{n+1},\ c = y_{n+1}$ and $d = y_n$, in Lemma 3.1 and we obtain a function $\varphi_n \in C^\infty(\mathbb{R})$ such that:

$$\varphi_n(t_n) = y_n,\ \varphi_n(t_{n+1}) = y_{n+1},\ \varphi_n^{(i)}(t_n) = \varphi_n^{(i)}(t_{n+1}) = 0,\ \forall i = \overline{1, m}$$

and

$$\left|\varphi_n^{(i)}(x)\right| \leq k_i \frac{|y_n - y_{n+1}|}{(t_n - t_{n+1})^i},\ i = \overline{1, m} \tag{4}$$

where k_i are constants that do not depend on t_n and y_n.
Let

$$g(x) = \begin{cases} 0, & \text{if } \ x \leq 0 \\ \varphi_n(x), & \text{if } \ x \in [t_{n+1}, t_n] \\ y_0, & \text{if } \ x > t_0. \end{cases}$$

It is clear that g is a C^∞-class function on $\mathbb{R} \setminus 0$. From (3) and (4) we conclude that g is a C^m-class function on $\mathbb{R}$ and $g^{(i)}(0) = 0,\ i = \overline{1, m}$. Furthermore, according to Remark 3.1, if $y_n > 0$ for all $n \in \mathbb{N}$, then g is increasing. □

Theorem 3.1 in the following is a refinement of a result in [3], since in the conclusion the sequence (t_k) is strictly decreasing.

Theorem 3.1. *Let* (x_n) *be a sequence of distinct points of* $\mathbb{R}^p$, *convergent to* $a \in \mathbb{R}^p$. *Then, for any* $m \in \mathbb{N}^*$, *exist a subsequence* (x_{n_k}), *a* C^m*-class parametrized curve* α, *with* $\alpha(0) = a$, *having a tangent at* a, *and a strictly decreasing sequence* (t_k) *of real numbers with* $t_k \to 0$ *such that* $\alpha(t_k) = x_{n_k}$ *for all* $k \in \mathbb{N}$.

Proof. By a translation, we can suppose $a = (0, ..., 0) \in \mathbb{R}^p$. Clearly $u_n = \dfrac{x_n}{\|x_n\|}$ is bounded. Considering it as a subsequence, we can assume without loss of generality that $u_n \to u \in \mathbb{R}^p$. By a rotation, we can assume $u = (1, 0, ..., 0)$. Consequently, if $x_n = \left(x_n^1, ..., x_n^p\right)$, it follows

$$\frac{x_n^1}{|x_n^1| \sqrt{1 + \left(\dfrac{x_n^2}{x_n^1}\right)^2 + ... + \left(\dfrac{x_n^p}{x_n^1}\right)^2}} \to 1.$$

Hence $x_n > 0$ for a large enough value of n and $\dfrac{x_n^i}{x_n^1} \to 0$, for all $i = \overline{2,p}$. Obviously, there exists a subsequence (x_{n_k}) such that $x_{n_k}^1 > 0$ and $x_{n_{k+1}}^1 < \dfrac{1}{2^m} x_{n_k}^1$, for all $k \in \mathbb{N}$. Applying Lemma 3.4 to the pair of sequences $x_{n_k}^1$ and $x_{n_k}^i$, $i = \overline{2,p}$, we can find non decreasing functions $\varphi_i : \mathbb{R} {\to} \mathbb{R}^p$, $i = \overline{1,p}$, of class C^m and a strictly decreasing sequence (t_k) of real numbers such that $t_k \to 0$, $\varphi_i(t_k) = x_{n_k}^i$, $i = \overline{1,p}$, $\varphi_i^{(j)}(0) = 0$, $i = \overline{1,p}$, $j = \overline{1, m-1}$, $\varphi_i^{(m)}(0)$, $i = \overline{2,p}$ and $\varphi_1^{(m)}(0) \neq 0$. Then, the parametrized curve $\alpha(t) = (\varphi_1(t), ..., \varphi_p(t))$, $t \in \mathbb{R}$, has the required properties. □

Denote $(\mathbb{R}_q^p)^+ = \left\{x = \left(x^1, ..., x^p\right) \mid x^{q+1} \geq 0, ..., x^p \geq 0\right\}$.

Theorem 3.2. *Let (x_n) be a sequence of distinct points of $(\mathbb{R}_q^p)^+$ convergent to the point $0 \in (\mathbb{R}_q^p)^+$. Then, for any $m \in \mathbb{N}^*$, there exist a subsequence (x_{n_k}), a C^m-class parametrized curve α with $\alpha(0) = 0$, having a tangent at 0, and a strictly decreasing sequence (t_k) of real numbers, with $t_k \to 0$, such that $\alpha(t_k) = x_{n_k}$, for all $k \in \mathbb{N}$ and $\alpha(t) \in (\mathbb{R}_q^p)^+$, for all $t \in [0, \varepsilon)$.*

Proof. It is enough to prove the following statement: if the sequence (x_n) has the property $g(x_n^p) \geq 0$, for all $n \in \mathbb{N}$, then the parametrized curve $\alpha(t) = \left(x^1(t), ..., x^p(t)\right)$ given by Theorem 3.1 has the same property, for $t \in [0, \varepsilon)$. We can suppose that $x_n^p > 0$; if (x_n) contains a subsequence (x_{n_k}) with $x_{n_k} = 0$, then we can define $x^p(t) = 0$. Let us consider $u_n = x_n / \|x_n\|$, which being bounded, can be assumed to be convergent to a unit vector $u = \left(u^1, ..., u^p\right)$. We have $u^p \geq 0$.

The case $u^p = 0$. By a rotation in the subspace $x^p = 0$ we can assume $u = (1, 0, ..., 0)$. Then, by following the proof of Theorem 3.1 we have $u_n^p \to 0$ and $x_n^1 / \|x_n\| \to 1$, hence $x_n^1 > 0$. By the same Theorem, we get $\alpha(t) = (\varphi^1(t), ..., \varphi^p(t))$, where φ^i are nondecreasing functions and therefore, the parametrized curve α is the required one.

The case $u^p > 0$. By a rotation we can suppose that $u = (1, 0, ..., 0)$. By this rotation the halfspace $x^p > 0$ becomes the halfspace $h\left(x^1, ..., x^p\right) > 0$, where $h\left(x^1, ..., x^p\right) = \sum_{i=1}^{p} c_i x^i$. We have , in this case, $c_1 = h(u) > 0$.

By a change of parameter, the parametrized curve α as in Theorem 3.1 has the properties $\alpha^{(k)}(0) = 0$, $k = \overline{1, m-1}$ and $\alpha^{(m)}(0) = u$. Let $\psi(t) = h(\alpha(t))$. Then

$\psi^{(k)}(0) = 0,\ k = \overline{1, m-1}$ and $\psi^{(m)}(0) = c_1 > 0$. It results $h(\alpha(t)) > 0,\ \forall t \in (0, \varepsilon)$ and so α has the required properties. □

Corollary 3.1. *Consider D be an open set in $\mathbb{R}^p$. Let $g^i \colon D \to \mathbb{R}$, $i = \overline{q+1, p}$, be C^1-class functions with* rank $\left[\frac{\partial g^i}{\partial x^j}(a)\right] = p - q$. *Let a be a point in D such that $g^i(a) = 0$, $i = \overline{q+1, p}$, and a sequence $(x_n) \subset D$ of distinct points, $x_n \to a$, with the property $g^i(x_n) \geq 0$, $i = \overline{q+1, p}$, $n \in \mathbb{N}$. Then, for any m in $\mathbb{N}^*$ there exist a subsequence (x_{n_k}), a strictly decreasing sequence $t_k \to 0$ of numbers and a C^m-class parametrized curve α passing through a, which has a tangent at the point a, such that $\alpha(0) = a$, $\alpha(t_k) = x_{n_k}$ and $g^i(\alpha(t)) \geq 0$, $i = \overline{q+1, p}$, $n \in \mathbb{N}$, for all $t \in [0, \varepsilon)$.*

Proof. Consider the change of variable $y = G(x)$,

$$\begin{aligned} y^i &= x^i, \quad \text{for } i \leq q \text{ or } i \geq q+1 \text{ and } g^i(a) > 0 \\ y^i &= g^i(x), \quad \text{for } i \geq q+1 \text{ and } g^i(a) = 0, \end{aligned}$$

which is a diffeomorphism. Now we apply Theorem 3.2. □

4. Optimal families of curves

Consider D an open set in $\mathbb{R}^p$ and a point in D. Let $g = (g^1, ..., g^s) \colon D \to \mathbb{R}^s$ be a C^1-class vector function such that rank $\left[\frac{\partial g^i}{\partial x^j}(a)\right] = s$ and $g(a) \geq 0$. Let $C(a)$ be the family of all sequences of distinct element from D convergent to a and $C_g(a)$ be the family of all sequences (x_n) of distinct elements of D such that $g(x_n) \geq 0$ and $x_n \to a$.

Definition 4.1. Let $\Gamma(a)$ a family of parametrized curves passing through a. The family $\Gamma(a)$ is called *optimal* if for any function $f : D \to \mathbb{R}$ having a as minimum point constrained by $\Gamma(a)$, it follows that a is also a local minimum point for f.

Definition 4.2. Let $\Gamma_g(a)$ be a family of parametrized curves α passing through the point a, with the property: if $\alpha(t_0) = a$, then $g(\alpha(t)) \geq 0$, $\forall t \in [t_0, t_0 + \varepsilon)$. The family $\Gamma_g(a)$ is called *optimal* if, given a function $f : D \to \mathbb{R}$ for which a is a minimum point constrained by $\Gamma_g(a)$, a is also a minimum point for the function f constrained by $g \geq 0$.

The following two corollaries are consequences of Theorem 2.1 and Theorem 2.2.

Corollary 4.1. *If $\Gamma(a)$ is $C(a)$-subordinate family, then $\Gamma(a)$ is an optimal family.*

Corollary 4.2. *If $\Gamma_g(a)$ is $C_g(a)$-subordinate family, then $\Gamma_g(a)$ is an optimal family.*

For each m in $\mathbb{N}^*$ we denote by $\Gamma^m(a)$ the family of all C^m parametrized curves passing through the point a having a tangent at a. We denote by $\Gamma_g^m(a)$ the family of all parametrized curves $\alpha \in \Gamma^m(a)$, with $\alpha(t_0) = a$, such that $g(\alpha(t)) \geq 0$, for all $t \in [t_0, t_0 + \varepsilon)$.

From Theorem 3.1 and Corollary 3.1 we obtain:

Theorem 4.1. *The family* $\Gamma^m(a)$ *is an optimal family.*

Theorem 4.2. *The family* $\Gamma_g^m(a)$ *is an optimal family.*

The above theorems can be rephrased as follows:

Theorem 4.3. *Let* $f : D \to \mathbb{R}$ *and* $a \in D$*. Then,* a *is local minimum point for* f *if and only if* a *is minimum point constrained by* $\Gamma^m(a)$.

Theorem 4.3 is more general than Theorem 3.1 in [3], since it uses the notion of minimum point constrained by a family of parametrized curves in a more general sense.

Theorem 4.4. *Let* $f : D \to \mathbb{R}$ *and* a *be point in* D*. Let* $g = (g^1, ..., g^s) : D \to \mathbb{R}^s$ *of* C^1*-class with* rank $\left[\dfrac{\partial g^i}{\partial x^j}(a)\right] = s$ *such that* $g(a) = 0$*. Then,* a *is minimum point for the function* f *constrained by* $g \geq 0$ *if and only if* a *is minimum point constrained by* $\Gamma_g^m(a)$.

It is worth noting that no restriction is placed on the function f in the above two theorems. Theorem 4.3 answers to one of the questions asked in [7], [12]. Theorem 4.4 generalizes the results of [3], [6], [13]÷[16].

Some remarks hold true.

1) The properties of a family of parametrized curves to be optimal or $C(a)$-subordinate also hold for all families that include it. Therefore is useful to find families of curves with these properties with as few elements as possible.

2) The optimality property of a family is not preserved for all its subfamilies. In $\mathbb{R}^2$ let us consider the family $\Gamma^m(0)$, with $m \geq 2$. We have established that this family is optimal. Let us define the subfamily $\Phi^m(0)$ consisting in all parametrized curves $\alpha \in \Gamma^m(0)$ $(\alpha(0) = 0)$ for which $\min\{i \mid \alpha^{(i)}(0) \neq 0\} \leq m-1$. We shall prove first that $\Phi^m(0)$ is not $C(0)$-subordinate family. The sequence $\left(x_n = \dfrac{1}{n},\ y_n = 3x_n^{(m+1)/m}\right)$ converges to 0. On one hand,

$$\frac{y_n}{x_n} = 3\frac{1}{n^{1/m}} \to 0 \tag{5}$$

and on the other hand,

$$\frac{y_n}{x_n^{(k+1)/k}} > n^{1/k(k+1)},\ k \leq m-1. \tag{6}$$

By reductio ad absurdum, we assume that $\Phi^m(0)$ is a $C(0)$-subordinate family. Then, there exist a subsequence $\left(x_{n_p}, y_{n_p}\right)$, a parametrized curve $\alpha \in \Phi(0)$ $(\alpha(0) = 0)$ and a decreasing sequence $t_p \to 0$ such that $\alpha(t_p) = \left(x_{n_p}, y_{n_p}\right)$, $\forall p \in \mathbb{N}^*$. If $\alpha(t) = (x(t), y(t))$, it is obvious that

$$x(t) = t^k(a + tf(t)), \quad y(t) = t^k(b + tg(t)),$$

where $a^2 + b^2 > 0,\ k \leq m-1$ and f, g are continuous functions.

Suppose that $a \neq 0$. Therefore, $\dfrac{y_{n_p}}{x_{n_p}} = \dfrac{b + t_{n_p} g\left(t_{n_p}\right)}{a + t_{n_p} f\left(t_{n_p}\right)}$. Using (5) we get that $b = 0$. Hence,

$$\frac{y_{n_p}}{x_{n_p}^{(k+1)/k}} = \frac{g\left(t_{n_p}\right)}{\left(a + t_{n_p} f\left(t_{n_p}\right)\right)^{(k+1)/k}} \to g\left(0\right).$$

However, from (6), it follows that $\dfrac{y_{n_p}}{x_{n_p}^{(k+1)/k}} \to \infty$, which is a contradiction.

If $a = 0$, then

$$\frac{x_{n_p}}{y_{n_p}} = \frac{t_{n_p} f\left(t_{n_p}\right)}{b + t_{n_p} g\left(t_{n_p}\right)} \to 0,$$

which contradicts (5).

Let us prove now that the family $\Phi^m(0)$ is not optimal. In this respect, it is enough to consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \left(y^m - x^{m+1}\right)\left(y^m - 3^m x^{m+1}\right),$$

which is of C^∞-class. The critical point $(0,0)$ is not a local minimum for the function f. Following the above ideas, it can be proven that $(0,0)$ is a minimum point for f constrained by the family $\Phi^m(0)$.

5. Conclusions and further development

Inspired and motivated by the ongoing research in this area, [1]÷[16], we introduced and studied optimality conditions for a family of curves. Using essentially the techniques of Oltin Dogaru and his research collaborators, our results propose the conditions that a family of parametrized curves needs to satisfy such that a local extremum problem be equivalent to an extremum problem constrained by this family of parametrized curves.

Open problem 5.1. Do there exist minimal elements with respect to inclusion in the class of all optimal families or in the class of all $C\,(a)$-subordinate families?

Open problem 5.2. Do there exist optimal families of curves which are not $C\,(a)$-subordinate?

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REFERENCES

[1] *O. Dogaru, I. Ţevy and C. Udrişte*, Extrema constrained by a family of curves and local extrema, J. Optim. Theory Appl., **97**(1998), No. 3, 605-621.

[2] *O. Dogaru and I. Ţevy*, Extrema Constrained by a Family of Curves, Proc. Workshop on Global Analysis, Differ. Geom. and Lie Algebras, 1996, Ed. Gr. Tsagas, Geometry Balkan Press, 1999, 185-195.

[3] *O. Dogaru and V. Dogaru*, Extrema constrained by C^k curves, Balkan J. Geom. Appl., **4**(1999), No. 1, 45-52.

[4] *O. Dogaru*, Construction of a function using its values along C^1 curves, Note Mat., **27**(2007), No. 1, 131-137.

[5] *O. Dogaru and M. Postolache*, A survey on constrained extrema, J. Adv. Math. Stud., **4**(2011), No. 1, 11-42.

[6] *O. Dogaru, C. Udrişte and C. Stamin*, From curves to extrema, continuity and convexity, Geometry Balkan Press 2007, Proc. 4th Int. Coll. Math. Engng. Num. Phys., Oct. 6-8, 2006 Bucharest, pp. 58-62.

[7] *M. Postolache and I. Ţevy*, Open problems risen by Constantin Udrişte and his research collaborators, J. Adv. Math. Stud., **3**(2010), No. 1, 93-102.

[8] *C. Udrişte and O. Dogaru*, Extrema with nonholonomic constraints, Buletinul Institutului Politehnic Bucureşti, Seria Energetică, **50**(1988), 3-8.

[9] *C. Udrişte and O. Dogaru*, Convex nonholonomic hypersurfaces, in *Math. Heritage of C. F. Gauss* (G. Rassias (Ed.)), World Scientific, 1991, pp. 769-784.

[10] *C. Udrişte, O. Dogaru and I. Ţevy*, Sufficient conditions for extremum on differentiable manifolds, Sci. Bull. P.I.B., Electrical Engineering, **53**(1991), No. 3-4, 341-344.

[11] *C. Udrişte, O. Dogaru and I. Ţevy*, Extremum points associated with Pfaff forms, Tensor, N.S., **54**(1993), 115-121.

[12] *C. Udrişte, O. Dogaru and I. Ţevy*, Open problems in extrema theory, Sci. Bull. P.U.B., Series A, **55**(1993), No. 3-4, 273-277.

[13] *C. Udrişte, O. Dogaru and I. Ţevy*, Extrema constrained by a Pfaff system, in *Fundamental open problems in science at the end of millenium*, vol. I-III, (T. Gill, K. Liu and E. Trell (Eds.)), Hadronic Press, Palm Harbor, 1999, pp. 559-573.

[14] *C. Udrişte, O. Dogaru and I. Ţevy*, Extrema with Nonholonomic Constraints, Monographs and Textbooks 4, Geometry Balkan Press, 2002.

[15] *C. Udrişte, O. Dogaru, M. Ferrara and I. Ţevy*, Pfaff inequalities and semi-curves in optimum problems, in Recent Advances in Optimization (G. P. Crespi, A. Guerragio, E. Miglierina and M. Rocca (Eds.)), DATANOVA, 2003, pp. 191-202.

[16] *C. Udrişte, O. Dogaru, M. Ferrara and I. Ţevy*, Extrema with constraints on points and/or velocities, Balkan J. Geom. Appl., **8**(2003), No. 1, 115-123.